

The Vector Spaces

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Introduction to Vector Spaces

Definition

We say that a non empty set \mathbb{E} is a vector space on \mathbb{R} if:

- 1 (Closure for the sum operation)
 $\forall u, v \in \mathbb{E}, u + v \in \mathbb{E},$
- 2 (Associativity of the sum operation)
 $u + (v + w) = (u + v) + w, \text{ for all } u, v, w \in \mathbb{E}$
- 3 (The identity element) There is $0 \in \mathbb{E}$ called the identity element of the sum operation such that
 $u + 0 = 0 + u = u \forall u \in \mathbb{E}.$
- 4 For all $u \in \mathbb{E}$, there is $v \in \mathbb{E}$ such that $u + v = v + u = 0$.
The vector v is called the symmetric of u and written $-u$.

- 1 (Commutativity of the sum operation)
 $u + v = v + u$ for all $u, v \in \mathbb{E}$.
- 2 (The closure of the exterior operation) $\forall \alpha \in \mathbb{R}$ and $u \in \mathbb{E}$,
 $\alpha u \in \mathbb{E}$,
- 3 If $u, v \in \mathbb{E}$ and $\alpha \in \mathbb{R}$, then $\alpha(u + v) = \alpha u + \alpha v$.
- 4 If $u \in \mathbb{E}$ and $\alpha, \beta \in \mathbb{R}$, then $(\alpha + \beta)u = \alpha u + \beta u$,
- 5 If $u \in \mathbb{E}$ and $\alpha, \beta \in \mathbb{R}$, then $(\alpha \cdot \beta)u = \alpha(\beta u)$,
- 6 If $u \in \mathbb{E}$, then $1 \cdot u = u$.

Examples

- 1 \mathbb{R}^n is a vector space .
- 2 The set $\{(x, y, 2x + 3y); x, y \in \mathbb{R}\}$ is a vector space .
- 3 The set of polynomials $\mathcal{P} = \mathbb{R}[X]$ is a vector space .
Also the set of polynomials of degree less than n , $\mathcal{P}_n = \mathbb{R}_n[X]$ is a vector space .

The Vector Sub-Spaces

Definition

Let V be a vector space and F a subset of V . We say that F is sub-space of V if F is vector space with the same operations on the vector space V .

Theorem

Let V be a vector space and F a subset of V .

F is a sub-space of V if and only if

- 1 $0 \in F$,
- 2 If $u, v \in F$, then $u + v \in F$,
- 3 If $u \in F$, $\alpha \in \mathbb{R}$, then $\alpha u \in F$.

Examples

- 1 The set $F = \left\{ \begin{pmatrix} a & b \\ 0 & 2a - b \end{pmatrix}; a, b \in \mathbb{R} \right\}$ is a sub-space of $V = M_2(\mathbb{R})$.
- 2 Let $A \in M_{m,n}(\mathbb{R})$ be a matrix and $F = \{X \in \mathbb{R}^n; AX = 0\}$. F is sub-space of $V = \mathbb{R}^n$. (F is the set of solutions of the homogeneous system $AX = 0$).
- 3 The set $F = \{(x, x + 1); x \in \mathbb{R}\}$ is not a sub-space of \mathbb{R}^2 since $(0, 0) \notin F$.

Example

The set $W = \{A \in M_n / A = -A^T\}$ is a sub-space of $M_n(\mathbb{R})$.

Solution If $A, B \in W$ and $\lambda \in \mathbb{R}$

$$(A + B)^T = A^T + B^T = -A - B$$

and

$$(\lambda A)^T = \lambda A^T = -\lambda A.$$

Then W is a sub-space of M_n .

Example

The set $E = \{(x, y) \in \mathbb{R}^2; xy = 0\}$ is not a sub-space since $(1, 0) \in E$ and $(0, 1) \in E$ but $(1, 0) + (0, 1) = (1, 1) \notin E$.

Definition

Let V be a vector space and let v_1, \dots, v_n be a finite vectors in V . We say that a vector $w \in V$ is a linear combination of the vectors v_1, \dots, v_n if there is $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ such that

$$w = \alpha_1 v_1 + \dots + \alpha_n v_n.$$

Example

The vector $(4, 1, 1)$ is a linear combination of the vectors $(1, 0, 2), (2, -1, 3), (0, -1, 1)$ because

$$(4, 1, 1) = -2(1, 0, 2) + 3(2, -1, 3) - 4(0, -1, 1).$$

Theorem

Let A be the matrix of type (m, n) and let $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ be the matrix of type $(n, 1)$. If C_1, \dots, C_n are the columns of the matrix A , then

$$AX = x_1 C_1 + \dots + x_n C_n.$$

Corollary

Let A be a matrix of type (m, n) .

The linear system $AX = B$ is consistent if and only if the vector B is a linear combination of the columns of the matrix A .

Definition

Let $S = \{v_1, \dots, v_n\}$ be a set of vectors in a vector space V . We say that the vector in V is generated (or spanned) by the set S if any vector in V is a linear combination of the vectors v_1, \dots, v_n . (We say also that S is a spanning set of V).

Theorem

Let $S = \{v_1, \dots, v_k\} \subset \mathbb{R}^n$ and A a matrix of type (n, k) and C_1, \dots, C_k its columns.

The set S spans the vector space \mathbb{R}^n if and only if the system $AX = B$ is consistent for all $B \in \mathbb{R}^n$.

Determine whether the vectors $v_1 = (1, 1, 4)$, $v_2 = (2, 1, 3)$, and $v_3 = (4, 3, 5)$ span \mathbb{R}^3 .

We solve the following linear system $AX = B$, where

$$A = \begin{pmatrix} 1 & -2 & 4 \\ -1 & 1 & -3 \\ 4 & 3 & 5 \end{pmatrix}, B = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ for arbitrary } a, b, c \in \mathbb{R}.$$

A reduced of the augmented matrix is given by:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -a - 2b \\ 0 & 1 & -1 & -a - b \\ 0 & 0 & 0 & 7a + 11b + c \end{array} \right]. \text{ This system has a solution only when}$$

$7a + 11b + c = 0$. Thus, three vectors do not span \mathbb{R}^3 .

Theorem

Let $S = \{v_1, \dots, v_n\}$ be a set of vectors in a vector space V , then

- 1 the set W of linear combinations of the vectors of S is a linear sub-space in V .
- 2 W is the smallest sub-space of V which contains S .
This sub-space is called the sub-space generated (or spanned) by the set S and is denoted by $\langle S \rangle$ or $\text{Vect}(S)$.

Example

Let in \mathbb{R}^4 the following vectors: $e_1 = (1, 2, 3, 4)$ and $e_2 = (1, -2, 3, -4)$.

Is there a and b in \mathbb{R} such that the vector $(a, 1, b, 1)$ is a vector of the sub-space generated by the vectors e_1, e_2 .

Is there a and b in \mathbb{R} such that the vector $(a, 1, 1, b)$ is a vector of the sub-space generated by the vectors e_1, e_2 .

Solution

The vector $(a, 1, b, 1) \in \langle e_1, e_2 \rangle$ if and only if the linear system

$$AX = B \text{ is consistent with } A = \begin{pmatrix} 1 & 1 \\ 2 & -2 \\ 3 & 3 \\ 4 & -4 \end{pmatrix} \text{ and } B = \begin{pmatrix} a \\ 1 \\ b \\ 1 \end{pmatrix}.$$

The system is not consistent because the second and the fourth equations can not be true in the same time. $((2a - 2b = 1, 4a - 4b = 1))$

The vector $(a, 1, 1, b) \in \langle e_1, e_2 \rangle$ if and only if the linear system

$$AX = B \text{ is consistent with } A = \begin{pmatrix} 1 & 1 \\ 2 & -2 \\ 3 & 3 \\ 4 & -4 \end{pmatrix} \text{ and } B = \begin{pmatrix} a \\ 1 \\ 1 \\ b \end{pmatrix}.$$

The system has a unique solution and in this case $a = \frac{1}{3}$ and $b = 2$.

Example

Let E be the vector sub-space of \mathbb{R}^3 generated by the vectors $(2, 3, -1)$ and $(1, -1, -2)$ and let F be the sub-space of \mathbb{R}^3 generated by the vectors:
 $(3, 7, 0)$ and $(5, 0, -7)$.

The sub-spaces E and F are equal.

$$\begin{cases} 2x + y = a \\ 3x - y = b \\ -x - 2y = c \end{cases}$$

This system is equivalent with the following system

$$\begin{cases} x + 2y = -c \\ -3y = a + 2c \\ -7y = b + 3c \end{cases}$$

This system is consistent if and only if $7a - 3b + 5c = 0$.

We remark that the vectors $(2, 3, -1)$ and $(1, -1, -2)$ are solutions of the system, then $F \subset E$.

With the same method, the vectors $(2, 3, -1)$ and $(1, -1, -2)$ are in the sub-space F . This proves that $E = F$.

Example

Is there $a, b \in \mathbb{R}$ such that the vector $v = (-2, a, b, 5)$ is in the sub-space of \mathbb{R}^4 which is generated by the following vectors $u = (1, -1, 1, 2)$ and $v = (-1, 2, 3, 1)$.

Solution

The vector $v = (-2, a, b, 5)$ is in the sub-space of \mathbb{R}^4 which is generated by the following vectors $u = (1, -1, 1, 2)$ and $v = (-1, 2, 3, 1)$ if the following linear system is consistent $AX = B$ with $A =$

$$\begin{pmatrix} 1 & -1 \\ -1 & 2 \\ 1 & 3 \\ 2 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} -2 \\ a \\ b \\ 5 \end{pmatrix}.$$

This system is consistent if and only if $3 = a - 2 = \frac{b+2}{4}$.
Then $a = 5$ and $b = 10$.

Definition

We say that the set of vectors v_1, \dots, v_n in a vector space V are linearly independent if the following equation has 0 as unique solution $a_1v_1 + \dots + a_nv_n = 0$.

Example

The vectors $u = (1, 1, -2)$, $v = (1, -1, 2)$ and $w = (3, 0, 2)$ are linearly independent in \mathbb{R}^3 .

$$xu + yv + zw = (0, 0, 0) \iff \begin{cases} x + y + 3z & = 0 \\ x - y & = 0 \\ -2x + 2y + 2z & = 0 \end{cases}$$

This system has the unique 0 solution.

The matrix of this system is $\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 0 \\ -2 & 2 & 2 \end{pmatrix}$ and its determinant is
-4.

Definition

We say that the vectors v_1, \dots, v_n in a vector space V are linearly dependent if they are not linearly independent.

Example

The vectors $u = (0, 1, -2, 1)$, $v = (1, 0, 2, -1)$ and $w = (3, 2, 2, -1)$ are linearly dependent in \mathbb{R}^4 .

$$xu + yv + zw = (0, 0, 0, 0) \iff \begin{cases} y + 3z & = 0 \\ x + 2z & = 0 \\ -2x + 2y + 2z & = 0 \\ x - y - z & = 0 \end{cases}$$

This system has infinite solutions.

The extended matrix of this system is $\left[\begin{array}{ccc|c} 0 & 1 & 3 & 0 \\ 1 & 0 & 2 & 0 \\ -2 & 2 & 2 & 0 \\ 1 & -1 & -1 & 0 \end{array} \right]$ and the

reduced row form of this matrix is : $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$.

Theorem

Let $S = \{v_1, \dots, v_n\}$ be a set of vectors in a vector space V , with $n \geq 2$.

The set S is linearly dependant if and only if there is a vector of S which is a linear combination of the rest of vectors.

Theorem

Let $S = \{v_1, \dots, v_n\} \subset \mathbb{R}^m$ and A the matrix of type (m, n) such that its columns are the vectors of S .

The set S is linearly independent if and only if the homogeneous system $AX = 0$ has 0 as unique solution.

Examples

- 1 If A is a matrix of type (m, n) with $m < n$. Then the homogeneous system $AX = 0$ has an infinite solutions.
- 2 If $S = \{v_1, \dots, v_n\} \subset \mathbb{R}^m$ with $m < n$, then the set S is linearly dependant.

The Basis and the Dimension

Definition

Let $S = \{v_1, \dots, v_n\}$ be a set of vectors in a vector space V . We say that S is a basis of the vector space V if :

- 1 The set S generates the vector space V
- 2 The set S is linearly independent.

Theorem

If $S = \{v_1, \dots, v_n\}$ is a basis of the vector space V .

Any vector $v \in V$ can be written uniquely as a linear combination of vectors in the basis S .

Remark

Let $S = \{e_1, \dots, e_n\}$ be the set of the vectors in the vector space \mathbb{R}^n , where

$$e_1 = (1, 0, \dots, 0), \dots, e_k = (0, \dots, 0, 1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1).$$

The set S is a basis of \mathbb{R}^n and is called the natural basis of \mathbb{R}^n .

Exercise

Prove that $S = \{1, X, \dots, X^n\}$ is a basis of the vector space \mathcal{P}_n .

Example

Let $v_1 = (\lambda, 1, 1)$, $v_2 = (1, \lambda, 1)$ and $v_3 = (1, 1, \lambda)$.

Find the values of $\lambda \in \mathbb{R}$ such that $\{v_1, v_2, v_3\}$ is a basis of the vector space \mathbb{R}^3 .

Solution

The set $\{v_1, v_2, v_3\}$ is linearly independent if 0 the unique solution of the equation

$$xv_1 + yv_2 + zv_3 = 0.$$

This is equivalent that the following matrix has an inverse :

$$A = \begin{pmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{pmatrix}.$$

Then $\lambda \notin \{-2, 1\}$.

The set $\{v_1, v_2, v_3\}$ generates the vector space \mathbb{R}^n because the linear system $AX = B$ is consistent for all $B \in \mathbb{R}^n$ since the matrix A has an inverse .

Theorem

Let $S = \{v_1, \dots, v_n\}$ be a basis of the vector space V and let $T = \{u_1, \dots, u_m\}$ be a set of vectors.
If $m > n$, then T is linearly dependant .

Corollary

If $S = \{v_1, \dots, v_n\}$ and $T = \{u_1, \dots, u_m\}$ are basis of the vector space V , then $m = n$.

Definition

If $S = \{v_1, \dots, v_n\}$ is a basis of the vector space V then the number of vectors n of S is called the dimension of the vector space V and denoted by: $\dim V = n$.

Theorem

Let V is a vector space of dimension n . If $S = \{v_1, \dots, v_n\}$ in V .
Then
 S is linearly independent if and only if S generates the vector space V and this is equivalent also with S is a basis of V .

Theorem

If $S = \{v_1, \dots, v_n\}$ generates the vector space V , then it contains a basis of the vector space V .

Remark

If $S = \{v_1, \dots, v_m\} \subset \mathbb{R}^n$ is a set of vectors and F the vector sub-space generated by S . We have the following two algorithms to construct a basis of F .

First Algorithm

- 1 Construct the matrix A such that its rows are the vectors of S
- 2 The non zeros rows of any row echelon form of the matrix A are a basis of the vector space $F = \langle S \rangle$.

Second Algorithm

- 1 Construct the matrix A such that its columns are the vectors of S
- 2 Take any row echelon form C of the matrix A .
- 3 Let C_{k_1}, \dots, C_{k_p} be the columns which contain a leading number and $k_1 < \dots < k_p$. Then $\{v_{k_1}, \dots, v_{k_p}\}$ is a basis of the vector space $F = \langle S \rangle$.

Theorem

- 1 If $S = \{v_1, \dots, v_n\}$ is a set of vectors and generates the vector space V , then S contains a basis of the vector space V .
- 2 If $S = \{v_1, \dots, v_n\}$ is a set of linearly independent vectors in the vector space V , then there is a basis T of V which contains the set S .

Example

Let W be the sub-space of \mathbb{R}^5 generated by the set of following vectors:

$$v_1 = (1, 0, 2, -1, 2), \quad v_2 = (2, 0, 4, -2, 4), \quad v_3 = (1, 2, -1, 2, 0), \\ v_4 = (1, 4, -4, 5, -2).$$

- 1 Find a basis of the sub-space W in $\{v_1, v_2, v_3, v_4\}$.
- 2 Find a basis of the sub-space W in \mathbb{R}^5 and contains $\{v_1, v_3\}$.

Solution

① Let matrix $A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 2 & 4 \\ 2 & 4 & -1 & -4 \\ -1 & -2 & 2 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}$ with columns the components of the vectors v_1, v_2, v_3, v_4 .

The reduced row form the matrix A is $\begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Then $\{v_1, v_3\}$ is basis of the sub-space W .

- 2 If $e_1 = (1, 0, 0, 0, 0)$, $e_2 = (0, 1, 0, 0, 0)$, $e_3 = (0, 0, 1, 0, 0)$.
Then $\{v_1, v_3, e_1, e_2, e_3\}$ is basis of \mathbb{R}^5 and contains $\{v_1, v_3\}$.

Example

Let $W = \{(x, y, z, t) \in \mathbb{R}^4; 2x + y + z = 0, x - y + z = 0\}$

- 1 Prove that W is sub-space of \mathbb{R}^4
- 2 Find basis of the sub-space W .

Solution

① $u = (x, y, z, t) \in W \iff AX = 0$, where

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}.$$

Since the set of solutions of an homogeneous linear system is a vector sub-space, then W is vector sub-space of \mathbb{R}^4 .

$$\begin{aligned} \textcircled{2} \quad AX = 0 &\iff \begin{cases} 2x + y + z = 0 \\ x - y + z = 0 \end{cases} \iff \begin{cases} x = -2y \\ z = 3y \end{cases} \\ &\iff X = y \begin{pmatrix} -2 \\ 1 \\ 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

Then $\{(-2, 1, 3, 0), (0, 0, 0, 1)\}$ is basis of the vector sub-space W .

Example

In the vector space $V = \mathbb{R}^3$, give a set S of vectors in V such that S generates the vector space V and not linearly independent.

Solution

We can take $S = \{(1, 0, 0)\}$ and $T = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}$.

Coordinate System and Change of Basis

Definition

If $S = \{v_1, \dots, v_n\}$ is a basis of the vector space V and if $v \in V$ such that

$$v = x_1 v_1 + \dots + x_n v_n$$

then (x_1, \dots, x_n) are called the system of coordinates of the vector v with respect to the basis S . We denote

$$[v]_S = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

and called the vector of coordinates of the vector v with respect to the basis S .

Theorem

If $B = \{v_1, \dots, v_n\}$ and $C = \{u_1, \dots, u_n\}$ are two basis of the vector space V . We define the matrix ${}_C P_B$ of type n such that its columns are $[v_1]_C, \dots, [v_n]_C$. This matrix ${}_C P_B$ has an inverse and

$$[v]_C = {}_C P_B [v]_B$$

for all $v \in V$.

The matrix ${}_C P_B$ is called the change of basis matrix from the basis B to the basis C .

Exercise

Let $B = \{v_1 = (0, 1, 1), v_2 = (1, 0, -2), v_3 = (1, 1, 0)\}$ be a basis of the vector space \mathbb{R}^3 and let $C = \{u_1 = (1, 0, 0), u_2 = (0, 1, 0), u_3 = (0, 0, 1)\}$ be the standard basis of the vector space \mathbb{R}^3 .

1 Find the following matrix ${}_C P_B$ and ${}_B P_C$.

2 Find $[v]_B$ if $[v]_C = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

Exercise

$$\textcircled{1} \quad {}_C P_B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & -2 & 0 \end{pmatrix} \quad {}_B P_C = \begin{pmatrix} -2 & 2 & -1 \\ -1 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}.$$

$$\textcircled{2} \quad [v]_B = {}_B P_C [v]_C = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Example

Prove that in \mathbb{R}^3 , the vectors $u = (1, 0, 1)$, $v = (-1, -1, 2)$ and $w = (-2, 1, -2)$ form a basis and find the coordinate system of the vector $X = (x, y, z)$ in this basis.

Solution

The matrix which columns the vectors $u = (1, 0, 1)$, $v = (-1, -1, 2)$

and $w = (-2, 1, -2)$ is $A = \begin{pmatrix} 1 & -1 & -2 \\ 0 & -1 & 1 \\ 1 & 2 & -2 \end{pmatrix}$.

Since $|A| = -3$, then $u = (1, 0, 1)$, $v = (-1, -1, 2)$ and $w = (-2, 1, -2)$ is a basis of the vector space \mathbb{R}^3 .

If $X = au + bv + cw$ then $\begin{pmatrix} a \\ b \end{pmatrix} = A^{-1}X = \begin{pmatrix} 2y + z \\ \frac{-x+z}{3} \\ -x+3y+z \end{pmatrix}$.

Example

Prove that the system of vectors $S = \{(1, 1, 1), (-1, 1, 0), (1, 0, -1)\}$ is a basis of the vector space \mathbb{R}^3 .

Find the coordinates of the following vectors $(1, 0, 0)$, $(1, 0, 1)$ and $(0, 0, 1)$ in this basis.

Solution:

$$\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = -3 \neq 0.$$

Then S is a basis of the vector space \mathbb{R}^3 .

$$(1, 0, 0) = \frac{1}{3}(1, 1, 1) - \frac{1}{3}(-1, 1, 0) + \frac{1}{3}(1, 0, -1).$$

Then coordinates in the basis S is $(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3})$.

Solution

$$(0, 0, 1) = \frac{1}{3}(1, 1, 1) - \frac{1}{3}(-1, 1, 0) - \frac{2}{3}(1, 0, -1).$$

Then coordinates in the basis S is $(\frac{1}{3}, -\frac{1}{3}, -\frac{2}{3})$.

$$(1, 0, 1) = (1, 0, 0) + (0, 0, 1).$$

Then coordinates in the basis S is $(\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3})$.

Definition

Let A be a matrix of type (m, n) .

The vector sub-space of \mathbb{R}^n spanned by the rows of the matrix A is called the row vector space of the matrix A and denoted by: $\text{row}(A)$.

The vector sub-space of \mathbb{R}^m spanned by the columns of the matrix A is called the column vector space of the matrix A and denoted by: $\text{col}(A)$.

Theorem

Let A be a matrix of type (m, n) . If B is any matrix which is a result of some row operations on the matrix A , then $\text{row}(A) = \text{row}(B)$.

Theorem

Let A be a matrix of type (m, n) and if B any row echelon form of the matrix A . Then the set of non zero rows of the matrix B are linearly independent.

Definition

Let A be a matrix of type (m, n) .

The dimension of the vector space $\text{row}(A)$ is called the rank of the A .

$$\text{rank}(A) = \dim(\text{row}(A)).$$

Remark

Let A be a matrix of type (m, n) .

The rank of the matrix A is the numbers of leading numbers in any row echelon form of the matrix A .

Theorem

Let A be a matrix of type (m, n) , then

$$\text{rank}(A) = \dim(\text{row}(A)) = \dim(\text{col}(A)).$$

Corollary

Let A be a matrix of type (m, n) , then

$$\text{rank}(A) = \text{rank}(A^T).$$

Corollary

If A is a matrix of type (m, n) and P is any invertible matrix of type m and Q an invertible matrix of type n , then

$$\text{rank}(A) = \text{rank}(PAQ).$$

Proof

There E_1, \dots, E_p elementary matrix of order m such that $P = E_1 \dots E_p$.

We know that if E is a elementary matrix which corresponds to an elementary row operation R , then EA is the result of the elementary row operation R on the matrix A . Then

$$\text{rank}(A) = \text{rank}(PA).$$

Also $\text{rank}(PAQ) = \text{rank}(PAQ)^T = \text{rank}(Q^T A^T P^T) = \text{rank}(A^T P^T) = \text{rank}(PA) = \text{rank}(A)$.

Theorem

If A is a matrix of type (m, n) . We have the equivalence of the following statements:

- 1 The homogeneous system $AX = 0$ has 0 as unique solution.
- 2 The columns of the matrix A are linearly independent .
- 3 $\text{rank}(A) = n$.
- 4 The matrix $A^T A$ has an inverse.

Theorem

Let A be a matrix of type (m, n) . We have the equivalence of the following statements

- 1 The system $AX = B$ is consistent for all $B \in \mathbb{R}^m$.
- 2 The columns of the matrix A generates the vector space \mathbb{R}^m .
- 3 $\text{rank}(A) = m$.
- 4 The matrix AA^T has an inverse.

Definition

Let A be a matrix of type (m, n) . The vector sub-space

$$\{X \in \mathbb{R}^n; AX = 0\}$$

is called the nullspace of the matrix A and denoted by: $N(A)$. Its dimension is denoted by $\text{nullity}(A)$.

Also the vector sub-space

$$\{AX; X \in \mathbb{R}^n\}$$

is called the image of the matrix A and denoted by: $\text{Im}(A)$.

Theorem

Let A be a matrix of type (m, n) . Then $\text{Im}(A) = \text{col}(A)$.

Rank-Nullity Theorem

For any matrix A of type (m, n) ,

$$\text{nullity}(A) + \text{rank}(A) = n.$$

Example

Let the matrix $A = \begin{pmatrix} 1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 3 \\ 2 & 3 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$

- 1 Find a basis of the vector space $N(A)$.
- 2 Find a basis of the vector space $\text{Col}(A)$.
- 3 Find the rank of the matrix A .

Solution

The reduced row form the matrix A is
$$\begin{pmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

- ① $(-3, 2, 1, 0), (-5, 3, 0, 1)$ is basis of the vector space $N(A)$..

- ② $(0, 1, 2, 1), (-1, 2, 3, 1)$ is a basis of the vector space $\text{Col}(A)$.
- ③ The rank of the matrix A is 2.

Example

Let $e_1 = (0, 1, -2, 1)$, $e_2 = (1, 0, 2, -1)$, $e_3 = (3, 2, 2, -1)$, $e_4 = (0, 0, 1, 0)$ and $e_5 = (0, 0, 0, 1)$ vectors in \mathbb{R}^4 .

Is the following statements are true?

- 1 $\text{Vect}\{e_1, e_2, e_3\} = \text{Vect}\{(1, 1, 0, 0), (-1, 1, -4, 2)\}$.
- 2 $(1, 1, 0, 0) \in \text{Vect}\{e_1, e_2\} \cap \text{Vect}\{e_2, e_3, e_4\}$.
- 3 $\text{Vect}\{e_1, e_2\} + \text{Vect}\{e_2, e_3, e_4\} = \mathbb{R}^4$.

Solution

- ① Let the matrix A which rows are the vectors e_1, e_2, e_3 .
The vector space $\text{Vect}\{e_1, e_2, e_3\}$ is the row vector space of the matrix A .

The reduced row form of the matrix A is

$$A_1 = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $\dim \text{Vect}\{e_1, e_2, e_3\} = 2$.

We have $\text{Vect}\{e_1, e_2, e_3\} = \text{Vect}\{(1, 1, 0, 0), (-1, 1, -4, 2)\}$ if and only if the rank of the following matrix B is 2

$$B = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -2 & 1 \\ -1 & 1 & -4 & 2 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

The reduced row form of the matrix B is $A_2 = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$

Then

$$\text{Vect}\{e_1, e_2, e_3\} = \text{Vect}\{(1, 1, 0, 0), (-1, 1, -4, 2)\}.$$

- ② $(1, 1, 0, 0) = e_1 + e_2$, $2(1, 1, 0, 0) = e_3 - e_2$.
Then $(1, 1, 0, 0) \in \text{Vect}\{e_1, e_2\} \cap \text{Vect}\{e_2, e_3, e_4\}$.
- ③ $(1, 1, 0, 0) \in \text{Vect}\{e_1, e_2\} \cap \text{Vect}\{e_2, e_3, e_4\}$ and
 $e_2 \in \text{Vect}\{e_1, e_2\} \cap \text{Vect}\{e_2, e_3, e_4\}$.
Then $\dim \text{Vect}\{e_1, e_2\} \cap \text{Vect}\{e_2, e_3, e_4\} = 2$ and

$$\dim \text{Vect}\{e_1, e_2\} + \text{Vect}\{e_2, e_3, e_4\} \leq 3$$

Then $\text{Vect}\{e_1, e_2\} + \text{Vect}\{e_2, e_3, e_4\} \neq \mathbb{R}^4$.

Example

Let in \mathbb{R}^3 the vectors, $u_1 = (1, 2, 1)$, $u_2 = (1, 3, 2)$, $u_3 = (1, 1, 0)$
and $u_4 = (3, 8, 5)$.

Let $F = \text{Vect}(u_1, u_2)$ and $G = \text{Vect}(u_3, u_4)$.

Prove that $F = G$.

Solution

As the vectors u_1, u_2 are linearly independent and also the vectors u_3, u_4 are linearly independent, then $\dim E = \dim F = 2$.

$F = G$ if and only if the rank of the following matrix is 2, $A =$

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 1 & 0 \\ 3 & 8 & 5 \end{pmatrix}.$$

The reduced row form of this matrix is $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$

Then $F = G$.