324 Stat
Lecture Notes

(1) Probability

(Chapter 2 of the book pg 35-71)

Definitions:

Sample Space:

Is the set of all possible outcomes of a statistical experiment, which is denoted by the symbol S •

Notes:

-Each outcome in a sample space is called an element or a member or a sample point.

- S is called the **sure event**

EX (1):

Consider the experiment of tossing a die. If we are interested in the number that shows on the top face, the sample space would be:

S= {1, 2, 3, 4, 5, 6}

If we are interested only in whether the number is even or odd, the sample space is simply:

S= {even, odd}

EX (2):

The sample space of tossing a coin **3** times:

S= {HHH, HHT, THH, HTH, HTT, TTH, THT, TTT}.

Note:

- you can use the tree diagram on pg 37 to find S in this example

EX (3):

S= {X|X+2=0} = {-2}.

See Ex 2.3 pg 37

Events:

An event A is a subset of a sample space ●

For instance, when a die is tossed, then the sample space is {1,2,3,4,5,6}.
If the event **A** is concerned with the elements that divisible by **3** such that

A={3,6}, then $A \subset S$.

EX (4):

The set which contains no elements at all is called the null set or the impossible event denoted by **Φ** .

Ex

B= {X|X is an even factor of 7}, then

B= **Φ= { }**

Note:

$$\Phi \subset S \, , S \subset S$$

Complement of an event A:

The complement of an event **A** with respect to **S** is the set of all elements of **S** that not in **A** denoted by

A' or A^C or $\overline{A}$.

See Ex 2.6 pg 39

EX (5):

S= {1, 2, 3, 4, 5, 6}.

Let **B** the event that has the number which greater than **3**

such that **B** = {4, 5, 6}, then B' = {1, 2, 3}.

Intersection:

The intersection of two events **A** and **B** denoted by

$$A \cap B$$

is the event containing all elements that are common to **A** and **B**.

EX (6):

Let M= {a, e, i, o, u ,r , s} and N={r, s, t},

then $M \cap N =$ {r, s}

Let **S**= {1,2,3,4,5,6}, **A**={1,2,3},**B**={3,6},

Then $A \cap B$ = {3}.

See Ex 2.7 pg 39

Mutually Exclusive (Disjoint) Events:

Two events **A** and **B** are mutually exclusive (disjoint)

if and only if $A \cap B = \Phi$,

that is if **A** and **B** have no elements in common.

See Ex 2.8 pg 40

EX (7):

S= {1, 2, 3, 4, 5, 6},

A= {1, 3, 5}, **B**= {2, 4, 6},

then $A \cap B = \Phi$

A and B are mutually exclusive or disjoint.

Union:

The union of the two events **A** and **B** denoted by

$$A \cup B$$

which is the event containing all the elements

that belong to **A** or **B** or both●

that is $A \cup B$ occurs if at least one of A or B occurs.

EX (8):

Let $M = \{X \mid 3 < X < 9\}$ $N = \{y \mid 5 < y < 12\}$

Then $M \cup N = \{z \mid 3 < z < 12\}$.

EX (9):

Let A= {a,b,c}, B={b,c,d,e}

then $A \cup B = \{a, b, c, d, e\}$

See Ex 2.11 and 2.12 pg 40

1.7 Definition:

If A contains B, then B is called a subset of A, that is $B \subset A$.

EX (10):

A= {1, 2, 3, 4}, **B**= {2, 3}. $\because B \subset A$

$$\therefore A \cap B = B, A \cup B = A, (A \cap B)^C = B^C, (A \cup B)^C = (A)^C$$

2.3 Counting Sampl Points (pg 44):

If an operation can be performed in **(n_1)** ways and if for each of these a second operation can be performed in **(n_2)** ways, then the two operations can be performed together in **($n_1 n_2$)** ways.

EX (11):

How many sample points are in the sample space when a pair of dice is thrown once?

Solution:

The first die can land in any one of $\mathbf{n_1}$=6 ways. For each of these 6 ways the second die can also land in $\mathbf{n_2}$=6 ways. Therefore, the pair of dice can land in $\mathbf{n_1 n_2} = (6)\times(6) = \mathbf{36}$ possible ways.

EX (12):

How many lunches consisting of a soup, sandwich, dessert and a drink are possible if we can select from 4 soups, 3 kinds of sandwiches, 5 desserts and 4 drinks?

Solution:

Since $\mathbf{n_1}$=4, $\mathbf{n_2}$=3, $\mathbf{n_3}$=5, $\mathbf{n_4}$=4, then $(n_1)\ (n_2)\ (n_3)\ (n_4) = (4)\ (3)\ (5)\ (4) = 240$ different ways to choose lunch.

See Ex 2.15 and 2.16 pg 46

EX (13):

How many even three – digit numbers can be formed from the digits 1,2,5,6 and 9 if each digit can be used only once (without replacement)

3	4	2

$(n_1)(n_2)(n_3)= (2)\ (4)\ (3) =24$ even three – digit numbers.

Ex 2.17 pg 46

How many even four digit number can be formed from the digits 0,1,2,5,6 ,9 if each digit can be used only once?

3	4	5	1

or

4	3	4	2

↓ if the unit position is 0

↓ if the unit position is not 0

$(n_1)(n_2)(n_3)(n_4)+(n_1)(n_2)(n_3)(n_4)=$

$(1\times5\times4\times3)+(2\times4\times4\times3)=60+96=165$

Definition:

A permutation is an arrangement of all or part of a set of objects ■

- The number of permutations of **n** distinct objects is **n!**.

EX (14):

How many words can be obtained using the three letters: a, b, c

Solution:

n=3, then n! =3! =6

In general n distinct objects can be arranged in

n! =n (n-1) (n-2)…3x2x1 ways.

By definition **1! =0! =1**

Permutations:

The number of permutations of **n** distinct objects taken **r** at a time is given by:

$$p_r^n = \frac{n!}{(n-r)!} \quad where \;\; n \geq r \qquad (1)$$

Where:

1. the order is important.
2. the chosen is without replacement

EX (15):

A president, treasurer and secretary all different are to be chosen from a club consisting of **10** – people. How many different choices of officers are possible?

Solution:

$$p_3^{10} = \frac{10!}{7!} = 720$$

■ See Ex 2.18 pg 48

EX (16):

How many ways can a local chapter of the American Chemical Society Schedules 3 speakers for 3 different meetings if they are available on any of 5 possible dates?

Solution:

The total number of possible schedules is:

$$p_3^5 = \frac{5!}{2!} = \frac{120}{2} = 60$$

Theorem:

The number of permutations of **n** distinct objects arranged in

a. a row is **n!**

b. a circle is **(n-1)!**

EX (17):

How many ways can **3** Arabic books, **2** Math books and **1** chemistry book arranged:

1. in a book shelf ?
2. in a rounded table?

Solution:

The number of books is **n=6** books, then:

1. **n!**=6!=720

2. (n-1)!=5!=120

Exercise:

In how many ways can 4 people sit in:

(a) a row.

(b) in a circle

(answer: 4!=24, 3!=6)

Theorem:

The number of distinct permutations of **n** things of which **n_1** are of one kind, n_2 of a second kind,…, **n_K** of a k^{th} kind is

$$\frac{n!}{n_1!\,n_2!\ldots n_K!} \qquad (2)$$

where $n = n_1 + n_2 + \ldots + n_K$

EX (18):

How many different ways can **3** red, **4** yellow and **2** blue bulbs be arranged in a string of Christmas tree light with **9** sockets?

Solution:

The total number of distinct arrangements is:

$$\frac{9!}{3!\ 4!\ 2!} = 1260$$

See Ex 2.20 pg 49

Exercise:

How many different ways can we arrange the letters in the word statistics?

Combinations:

The number of combinations of **n** distinct objects taken **r** at time is given by:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \qquad where\ n \geq r \qquad (3)$$

where:

1. the order is not important
2. the chosen is without replacement

Note:

1. $$\binom{n}{0} = 1$$

2. $$\binom{n}{1} = n$$

3. $$\binom{n}{n-1} = n$$

Ex (19):

In how many ways can 5 starting positions on a basketball team be filled with 8 men who can play any of the positions?

Solution:

$$\binom{8}{5} = \frac{n!}{r!(n-r)!} = \frac{8!}{5!3!} = 56$$

Theorem:

The number of ways of partitioning a set of n objects into r cells with n_1 elements in the first cell, n_2 elements in the second , and so forth is

$$\binom{n}{n_1,\ n_2,\ \ldots,n_k} = \frac{n!}{n_1!\ n_2!\ \ldots n_k!}$$

Where $n_{1}+ n_{2}+\ldots+ n_{k}= n$

Ex 2.21 pg 50

In how many ways can 7 graduate students be assigned to 1 triple and 2 double hotel rooms during a conference?

Solution

$$\binom{7}{3,\quad 2,\quad 2} = \frac{7!}{3!\,2!\,2!} = 210$$

Note

$$\binom{n}{r,\quad n-r} = \binom{n}{r} = \binom{n}{n-r}$$

See Ex 2022 pg 50

1.9 Probability of an Event:

The probability of an event **A**, P (A) is the sum of the weights of all sample points in **A.**

$$P(A)=\frac{n(A)}{n(S)}=\frac{number\ of\ ways\ event\ A\ occures}{total\ outcomes\ in\ S}$$

Therefore:

1. $P(\Phi)=\frac{n(\Phi)}{n(S)}=0$

2. $P(S)=\frac{n(S)}{n(S)}=1$

3. $0\le P(A)\le 1,$

EX 2.24 pg 53:

A coin is tossed twice, what is the probability that at least one head occurs?

Solution:

S = {HH, HT, TH, TT},

n (A) = 3, n(S) = 4

4w=1, w=1/4,

A= {HH, HT, TH},

P (A) = n (A)/n(S) = 3/4

Ex 2.26 pg 54:

A die is loaded in such a way that an even number is twice as likely to occur as an odd number. Let **A** be the event tha even number turns up and **B** be the event that a number divisible by **3** occurs. Find $P(A \cap B)$ and $P(A \cup B)$.

Solution:

Let A: even number $\rightarrow$ A= {2, 4, 6},

B: number divisible by 3 $\rightarrow$ B= {3, 6},

That is P(1)=P(3)=P(5)= w , P(2)=P(4)=P(6)=2w. Then

3w+6w=9w=1, w =1/9.

$A \cap B = \{6\}$, $A \cup B = \{2,4,6,3\}$, $P(A \cap B)$ =2/9 , $P(A \cup B)$ =7/9

See Ex 2.25 pg 53

Theorem:

If an experiment can result in any one of **N** different equally likely outcomes and if exactly **n** of these outcomes correspond to event **A**, then the probability of an event **A** is:

$$P(A) = \frac{n}{N} = \frac{n(A)}{n(S)} \qquad (4)$$

where n(A) is the number of outcomes that satisfy the event A and n(S) is the total outcomes in the sample space S.

EX (25):

A mixture of candies has **6** mints, **4** toffees and **3** chocolates. If a person makes a random selection of one of these candies, find the probability of getting:

(a) a mint .

(b) a toffee or a chocolate.

Solution:

n (S)=13, n(M)=6, n(T)=4, n(C)=3

Let **M**, **T** and **C** represent the events that the person selects respectively a mint, a toffee or chocolate candy:

$$P(M) = \frac{n(M)}{n(S)} = \frac{6}{13} \text{ , } P(T) = \frac{n(T)}{n(S)} = \frac{4}{13} \text{ , } P(C) = \frac{n(C)}{n(S)} = \frac{3}{13}$$

a. P (getting a mint) = $P(M) = \frac{6}{13}$

b. P(getting a toffee or a chocolate) =

$$P(T \ or \ C) = P(T \cup C) = P(T) + P(C) = \frac{4}{13} + \frac{3}{13} = \frac{7}{13}$$

See Ex 2.27 pg 54

2.5 Additive Rule:

Theorem:

If **A** and **B** are any two events, then:

$$P(A \bigcup B) = P(A) + P(B) - P(A \bigcap B) \qquad (5)$$

Corollary:

If **A** and **B** are mutually exclusive, then:

$$P(A \bigcup B) = P(A) + P(B) \qquad (6)$$

Since $A \bigcap B = \Phi$, P (Φ) =0

Corollary:

If $A_1, A_2 \ldots A_n$ are mutually exclusive (disjoint), then

$$P(A_1 \cup A_2 \cup \ldots A_n) = P(A_1) + P(A_2) + \ldots + P(A_n) = P(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) \qquad (7)$$

Corollary:

$$P(A_1 \cup A_2 \cup \ldots A_n) = P(A_1) + P(A_2) + \ldots + P(A_n) = P(S) = 1 \qquad (8)$$

EX (26):

The probability that Paula passes mathematics is 2/3 and the probability that she passes English is 4/9. If the probability of passing both courses is 1/4, what is the probability that Paula will pass at least one of these courses?

Solution:

Let M be the event "passing mathematics" and E be the event "passing English", then

$$P(M) = 2/3 \ , \ P(E) = 4/9 \ , \ P(M \cap E) = 1/4$$

$$P(M \cup E) = P(M) + P(E) - P(M \cap E) = \frac{2}{3} + \frac{4}{9} - \frac{1}{4} = 31/36$$

Ex 2.30 pg 57

What is the probability of getting a total of **7**or **11**when a pair of dice are tossed?

Solution:

Let A be the event that **7**occures and B the event **11** comes up, then:

A={(1,6),(6,1),(3,4),(4,3),(2,5),(5,2)}

B= {(5,6),(6,5)}

$\because A \cap B = \Phi, P(A) = 6/36, P(B) = 2/36, P(A \cap B) = 0$,

$\therefore P(A \cup B) = P(A) + P(B) = 6/36 + 2/36 = 8/36$

See Ex2.29 & 2.31 pg 57-58

Theorem:

If A and A' are complementary events, then

$$P(A) + P(A') = 1$$

$$\Rightarrow P(A') = 1 - P(A) \qquad (9)$$

EX (2.23 pg 58):

If the probabilities that an automobile mechanic will service **3,4,5,6,7,8** cars on any given working day are respectively **0.12,0.19,0.28,0.24,0.1** and **0.07**, what is the probability that will service at least **5** cars on his next day at work?

Solution:

Let E be the event that at least 5 cars are serviced,

then

$$P(E) = P(X \geq 5) = P(5) + P(6) + P(7) + P(8),$$

$$P(E) = 1 - P(E')$$

$$P(E') = P(X = 3 \; or \; 4) = .12 + .19 = .31,$$

$$P(E) = 1 - P(E') = 1 - .31 = .69$$

Conditional Probability:

The conditional probability of occurring an event B when knowing that an event A is happened, that is B given **A** denoted by P (B|A) which is defined by:

$$P(B|A)=\frac{P(B\cap A)}{P(A)}=\frac{n(B\cap A)}{n(A)}=\frac{n(B\cap A)/n(S)}{n(A)/n(S)} \quad if \quad P(A)>0$$

$$P(A|B)=\frac{P(A\cap B)}{P(B)}=\frac{n(A\cap B)}{n(B)}=\frac{n(A\cap B)/n(S)}{n(B)/n(S)} \quad if \quad P(B)>0 \qquad (9)$$

EX (29):

	(E)Employed	(E^C) Unemployed	Total
(M) Male		40	
(M^C) Female	140		400
Total	600	300	900

Complete the table, then answer the questions:

What is the probability of:

1. getting a male
2. getting a male given he is an employed
3. getting an unemployed female
4. getting an employed or male

Solution:

1. $P(M)=\frac{n(M)}{n(S)}=\frac{500}{900}=.555$

2. $P(M \mid E)=\frac{P(M \cap E)}{P(E)}=\frac{n(M \cap E)}{n(E)}=\frac{460/900}{600/900}=460/600=.766$

3. $P(E^C \cap M^C)=260/900=0.289$

4. $$\begin{aligned} P(E \cup M) &= P(E)+P(M)-P(E \cap M) \\ &= 600/900+500/900-460/900 \\ &= 640/900=0.711 \end{aligned}$$

See Ex 2.34 pg 63

Independent Events:

Two events **A** and **B** are independent if and only if:

P (B|A) = P (B) and P (A|B) = P (A)

Theorem:

Two events **A** and **B** are independent if and only if:

$$P(A \cap B) = P(A)P(B)$$

EX (30):

Let A and B are independent events as follows:

$P(A) = 0.5$, $P(B) = 0.6$, find $P(A|B)$, $P(B|A)$.

Solution:

$\because$ A and B are independent

$\therefore P(A|B) = P(A) = 0.5$, $P(B|A) = P(B) = 0.6$

Multiplicative Rules:

Theorem:

If in experiment the events **A** and **B** can both occur**,**

then

$$P(A\cap B) = P(A)P(B \mid A) \quad or \quad P(A\cap B) = P(B)P(A \mid B)$$

EX 2.37 pg 66

One bag contains **4** white balls and **3** black balls and a second bag contains **3** white balls and **5** black balls. One ball is drawn from the first bag and placed unseen in the second bag. What is the probability that a ball now drawn from the second bag is black?

Solution:

Let $B_1, B_2\ and\ W_1$ represent respectively the drawing of a black ball from bag **1**, a black ball from bag **2** and a white ball from bag **1**.

We are interested in the union of $B_1 \cap B_2$ and $W_1 \cap B_2$.

$$P[(B_1 \cap B_2) \cup P(W_1 \cap B_2)] = P(B_1 \cap B_2) + P(W_1 \cap B_2)$$

$$= P(B_1)P(B_2 \mid B_1) + P(W_1)P(B_2 \mid W_1) = (3/7)(6/9) + (4/7)(5/9) = 38/63$$

<u>EX (32):</u>

If A and B are independent events, $P(A)=0.4, P(A \cup B)=0.6$,

find $P(B)$.

Solution:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= P(A) + P(B) - P(A)P(B)$$

$$0.6 \quad = 0.4 + P(B) - 0.4P(A)$$

$$0.2 \quad = 0.6P(B) \Rightarrow P(B) = 0.2/0.6 = 0.33$$

Theorem:

If in an experiment the events $A_1 \ldots A_K$ can occur,

then:

$$P(A_1 \cap \ldots \cap A_K) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_2 \cap A_1)\ldots P(A_K \mid A_1 \cap \ldots \cap A_{K-1})$$

If $A_1 \ldots A_K$ are independent, then

$$P(A_1 \cap \ldots \cap A_K) = P(A_1)P(A_2)P(A_3)\ldots P(A_K)$$

EX (33):

A coin is biased so that a head is twice as likely to occur as a tail. If the coin is tossed **3** times, what is the probability of getting **(2)** tails and **(1)** head?

Solution:

H : T

2w : w

$2w+w=3w=1 \Rightarrow w=1/3$

$P(H)=2/3,\ P(T)=1/3$

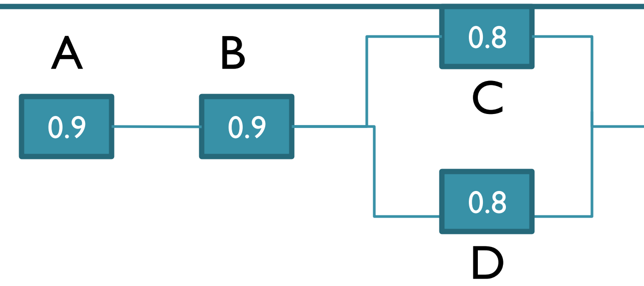
S= {HHH, HHT, THH, HTH, HTT, TTH, THT, TTT}

$P(H) = 2/3,\ P(T) = 1/3$

A = {TTH, HTT, THT}

$P(TTH) = (1/3)(2/3)(1/3) = 2/27$

$P(A) = 3(2/27) = 6/27$



Ex 2.39 pg 67

Find the probability that

a) The entire system works? b) the component C does not work given that the entire system work?

If the 4 components work independently

P (entire system works) = P $[A \cap B \cap (C \cup D)]$=P(A) P(B) P(CUD)=
(0.9) (0.9)[(0.8) + (0.8) - (0.8) (0.8)]= 0.7779

Or P (entire system works)= P $[A \cap B \cap (C \cup D)]$=P(A) P(B) $[1-P(\bar{C} \cap \bar{D})]$)

b) P (C does not work| the entire system work)=P$(A \cap B \cap \bar{C} \cap D)$ / P (entire system works)

= (0.9) (0.9)[1-(0.8)] (0.8) / 0.7776 =0.1667

EX (21): (Reading)

A box contains **4** white balls, **2** red balls and **3** green balls. Two balls are drawn without replacement, find the probability that:

1. the two balls are white
2. the two balls are red
3. the two balls are green
4. one ball is red and one ball is white
5. the two balls are not green
6. the two balls are the same colour

Solution:

1. $$P(two\,balls\,are\,white) = \frac{\binom{4}{2}\binom{5}{0}}{\binom{9}{2}} = \frac{(6)(1)}{(36)} = 0.1667$$

2. $$P(two\,balls\,are\,red) = \frac{\binom{2}{2}\binom{7}{0}}{\binom{9}{2}} = \frac{(1)}{(36)} = 0.028$$

3. $P(\text{two balls are green}) = \dfrac{\binom{3}{2}\binom{6}{0}}{\binom{9}{2}} = \dfrac{(3)(1)}{(36)} = 0.083$

4. $P(\text{one ball is red and one ball is white}) = \dfrac{\binom{4}{1}\binom{2}{1}\binom{3}{0}}{\binom{9}{2}} = \dfrac{(4)(2)(1)}{(36)} = 0.222$

5. $$P(\text{the two balls are not green}) = \frac{\binom{2}{2}\binom{4}{0}\binom{3}{0}}{\binom{9}{2}} + \frac{\binom{4}{2}\binom{2}{0}\binom{3}{0}}{\binom{9}{2}} = 0.028 + 0.167 = 0.195$$

6. $$P(\text{the two balls are the same colour}) = \frac{\binom{4}{2}\binom{5}{0}}{\binom{9}{2}} + \frac{\binom{2}{2}\binom{7}{0}}{\binom{9}{2}} + \frac{\binom{3}{2}\binom{6}{0}}{\binom{9}{2}}$$

$$= 0.25 + 0.028 + 0.083 = 0.361$$