Chapter 2

Random Variable

| CLO2 | Define single random variables in terms of their PDF and CDF, and calculate moments such as the mean and variance. |
1. Introduction

- In Chapter 1, we introduced the concept of event to describe the characteristics of outcomes of an experiment.
- Events allowed us more flexibility in determining the proprieties of the experiments better than considering the outcomes themselves.
- In this chapter, we introduce the concept of random variable, which allows us to define events in a more consistent way.
- In this chapter, we present some important operations that can be performed on a random variable.
- Particularly, this chapter will focus on the concept of expectation and variance.

2. The random variable concept

- A random variable $X$ is defined as a real function that maps the elements of sample space $S$ to real numbers (function that maps all elements of the sample space into points on the real line).

$$X: S \rightarrow \mathbb{R}$$

- A random variable is denoted by a capital letter (such as: $X, Y, Z$) and any particular value of the random variable by a lowercase letter (such as: $x, y, z$).
- We assign to $s$ (every element of $S$) a real number $X(s)$ according to some rule and call $X(s)$ a random variable.

**Example 2.1:**

An experiment consists of flipping a coin and rolling a die.

Let the random variable $X$ chosen such that:

- A coin head ($H$) corresponds to positive values of $X$ equal to the die number
- A coin tail ($T$) corresponds to negative values of $X$ equal to twice the die number.
Plot the mapping of $S$ into $X$.

**Solution 2.1:**

The random variable $X$ maps the samples space of 12 elements into 12 values of $X$ from -12 to 6 as shown in Figure 1.

![Diagram of mapping from S to X](image)

**Figure 1. A random variable mapping of a sample space.**

- **Discrete random variable:** If a random variable $X$ can take only a particular finite or counting infinite set of values $x_1, x_2, ..., x_N$, then $X$ is said to be a discrete random variable.

- **Continuous random variable:** A continuous random variable is one having a continuous range of values.

### 3. Distribution function

- If we define $P(X \leq x)$ as the probability of the event $X \leq x$ then the **cumulative probability distribution function** $F_X(x)$ or often called **distribution function** of $X$ is defined as:
\[ F_X(x) = P(X \leq x) \text{ for } -\infty < x < \infty \]  

(1)

- The argument \( x \) is any real number ranging from \(-\infty\) to \(\infty\).

- **Properties:**
  1) \( F_X(-\infty) = 0 \)
  2) \( F_X(\infty) = 1 \)
      (since \( F_X \) is a probability, the value of the distribution function is always between 0 and 1).
  3) \( 0 \leq F_X(x) \leq 1 \)
  4) \( F_X(x_1) \leq F_X(x_2) \text{ if } x_1 < x_2 \) (event \( \{X \leq x_1\} \) is contained in the event \( \{X \leq x_2\} \), monotonically increasing function)
  5) \( P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1) \)
  6) \( F_X(x^+) = F_X(x) \), where \( x^+ = x + \epsilon \) and \( \epsilon \to 0 \) (Continuous from the right)

- For a discrete random variable \( X \), the distribution function \( F_X(x) \) must have a "stairstep form" such as shown in Figure 2.
The random variable

The amplitude of a step equals to the probability of occurrence of the value $X$ where the step occurs, we can write:

$$F_X(x) = \sum_{i=1}^{N} P(x_i) \cdot u(x - x_i)$$

4. Density function

- The probability density function (pdf), denoted by $f_X(x)$ is defined as the derivative of the distribution function:

$$f_X(x) = \frac{dF_X(x)}{dx}$$

- $f_X(x)$ is often called the density function of the random variable $X$. 

Figure 2. Example of a distribution function of a discrete random variable.
• For a discrete random variable, this density function is given by:

\[ f_X(x) = \sum_{i=1}^{N} P(x_i) \delta(x - x_i) \]  

\( \delta \) Unit impulse function: \( \delta(x) = \begin{cases} 1 & x = 0 \\ 0 & \text{otherwise} \end{cases} \)

- Properties:

  ✓ \( f_X(x) \geq 0 \) for all \( x \)
  ✓ \( F_X(x) = \int_{-\infty}^{x} f_X(\theta)d\theta \)
  ✓ \( \int_{-\infty}^{\infty} f_X(x)dx = F_X(\infty) - F_X(-\infty) = 1 \)
  ✓ \( P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(\theta)d\theta \)

**Example 2.2:**

Let \( X \) be a random variable with discrete values in the set \{-1, -0.5, 0.7, 1.5, 3\}. The corresponding probabilities are assumed to be \{0.1, 0.2, 0.1, 0.4, 0.2\}.

a) Plot \( F_X(x) \), and \( f_X(x) \)

b) Find \( P(x < -1) \), \( P(-1 < x \leq -0.5) \)

**Solution 2.2:**

a)
b) \( P(X<-1) = 0 \) because there are no sample space points in the set \( \{ X<-1 \} \). Only when \( X=-1 \) do we obtain one outcome and we have immediate jump in probability of 0.1 in \( F_X(x) \). For \(-1<x<-0.5\) there are no additional space points so \( F_X(x) \) remains constant at the value 0.1.

\[
P(-1 < X \leq -0.5) = F_X(-0.5) - F_X(-1) = 0.3 - 0.1 = 0.2
\]

**Example 3:**

Find the constant \( c \) such that the function:

\[
f_X(x) = \begin{cases} 
  c \cdot x & 0 \leq x \leq 3 \\
  0 & \text{otherwise} 
\end{cases}
\]

is a valid probability density function (pdf)

Compute \( P(1 < x \leq 2) \)
Find the cumulative distribution function $F_X(x)$

**Solution:**

---

### 5. Examples of distributions

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### The Gaussian distribution

- The Gaussian or normal distribution is one of the important distributions as it describes many phenomena.

- A random variable $X$ is called Gaussian or normal if its density function has the form:

  $$f_X(x) = \frac{1}{\sqrt{2\pi \sigma_x^2}} e^{-\frac{(x-\mu)^2}{2\sigma_x^2}}$$  \hspace{1cm} (5)

$\sigma_x > 0$ and $\mu$ are, respectively the mean and the standard deviation of $X$ which measures the width of the function.
Figure 3. Gaussian density function

Figure 4. Gaussian density function with $a = 0$ and different values of $\sigma_x$
• The distribution function is:

\[
F_X(x) = \frac{1}{\sqrt{2\pi}\sigma_x^2} \int_{-\infty}^{x} e^{-\frac{(\theta-a)^2}{2\sigma_x^2}} d\theta
\]  

(5)

This integral has no closed form solution and must be solved by numerical methods.

• To make the results of \( F_X(x) \) available for any values of \( x, a, \sigma_x \), we define a standard normal distribution with mean \( a = 0 \) and standard deviation \( \sigma_x = 1 \), denoted \( N(0,1) \):

\[
f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}
\]  

(6)

\[
F(x) = \frac{1}{\sqrt{2\rho}} \int_{-\infty}^{x} e^{-\frac{\beta^2}{2}} d\beta
\]  

(7)

• Then, we use the following relation:

\[
F_Z(z) = F_X \left( \frac{x-a}{\sigma_x} \right)
\]  

(8)

• To extract the corresponding values from an integration table developed for \( N(0,1) \).
Example 4:
Find the probability of the event \( \{X \leq 5.5\} \) for a Gaussian random variable with \( \mu = 3 \) and \( \sigma_x = 2 \)

Solution:

\[
P(X \leq 5.5) = F_Z(5.5) = F_X\left(\frac{5.5 - 3}{2}\right) = F_X(1.25)
\]

Using the table, we have: \( P\{X \leq 5.5\} = F_X(1.25) = 0.8944 \)

Example 5:
In example 4, find \( P\{X > 5.5\} \)

Solution:

\[
P\{X > 5.5\} = 1 - P\{X \leq 5.5\} = 1 - F(1.25) = 0.1056
\]

6. Other distributions and density examples

The Binomial distribution

- The binomial density can be applied to the Bernoulli trial experiment which has two possible outcomes on a given trial.
- The density function \( f_X(x) \) is given by:

\[
f_X(x) = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} \delta(x - k)
\]

Where \( \binom{N}{k} = \frac{N!}{(N-k)! k!} \) and \( \delta(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases} \)
• Note that this is a discrete r.v.
• The Binomial distribution $F_X(x)$ is:

$$F_X(x) = f_X^x \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} \delta(x - k)$$

$$= \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} u(x - k)$$

The Uniform distribution

• The density and distribution functions of the uniform distribution are given by:

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{elsewhere} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x < a \\ \frac{(x-a)}{(b-a)} & a \leq x < b \\ 1 & x \geq b \end{cases}$$

The Exponential distribution

• The density and distribution functions of the exponential distribution are given by:
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\[ f_X(x) = \begin{cases} \frac{1}{b} e^{-\frac{(x-a)}{b}} & x \geq a \\ 0 & x < a \end{cases} \]  

(13)

\[ F_X(x) = \begin{cases} 1 - e^{-\frac{(x-a)}{b}} & x \geq a \\ 0 & x < a \end{cases} \]  

(14)

where \( b > 0 \)

7. Expectation

- Expectation is an important concept in probability and statistics. It is called also expected value, or mean value or statistical average of a random variable.
- The expected value of a random variable \( X \) is denoted by \( E[X] \) or \( \bar{X} \)
- If \( X \) is a continuous random variable with probability density function \( f_X(x) \), then:
  \[ E[X] = \int_{-\infty}^{+\infty} x f_X(x) dx \]  

(15)

- If \( X \) is a discrete random variable having values \( x_1, x_2, ..., x_N \), that occurs with probabilities \( P(x_i) \), we have
  \[ f_X(x) = \sum_{i=1}^{N} P(x_i) \delta(x - x_i) \]  

(16)

Then the expected value \( E[X] \) will be given by:
\[ E[X] = \sum_{i=1}^{N} x_i P(x_i) \quad \text{(17)} \]

**Example 3.1:** Find \( E[x] \) for the exponential r.v.:

\[
    f_X(x) = \begin{cases} 
        \frac{1}{b} e^{-\frac{(x-a)}{b}} & x \geq a \\
        0 & x < a 
    \end{cases}
\]

\[
    \text{Solu: } E[X] = \int_{-\infty}^{\infty} x f(x) \, dx = \int_{a}^{\infty} x e^{-\frac{x-a}{b}} \, dx = \frac{a/b}{b} \int_{a}^{\infty} x e^{-x/b} \, dx
\]

From integration table we have: \( \int xe^{cx} \, dx = e^{cx} \left[ \frac{x}{c} - \frac{1}{c^2} \right] \)

Here \( c = -\frac{1}{b} \Rightarrow E[X] = \frac{e^{a/b}}{b} \left[ e^{-\frac{a}{b}}(-\infty) - e^{-\frac{a}{b}}(-ab - b^2) \right] \)

\[
    = \frac{e^{a/b}}{b} \left[ e^{-a/b} - e^{-a/b}(-ab - b^2) \right] = \frac{e^{a/b} - e^{-a/b}(-ab + b^2)}{b} = a+b
\]

**Example 3.2:** Find the expected value of the points on the top face of tossing a fair die experiment?

\( X = \{1, 2, 3, 4, 5, 6\} \) and \( P(x_i) = \frac{1}{6} \) for \( i = 1, \ldots, 6 \) since the die is fair.

So, \( E[X] = \sum_{i=1}^{6} x_i P(x_i) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = 3.5 \)

### 7.1 Expected value of a function of a random variable

- Let be \( X \) a random variable then the function \( g(X) \) is also a random variable, and its expected value \( E[g(X)] \) is given by

\[ E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) \, dx \quad \text{(18)} \]
• If $X$ is a discrete random variable then

$$E[g(X)] = \sum_{i=1}^{N} g(x_i)P(x_i)$$  

(19)

**Example 3.3:** A random voltage has $f_X(x) = \begin{cases} \frac{2}{5}xe^{-\frac{x^2}{5}} & x \geq 0 \\ 0 & x < 0 \end{cases}$

The voltage is applied to a device that generates a voltage $Y = g(x) = X^2$, which is equal to the power in 1Ω resistor. Find the average power in $X$?

**Sol:** Power in $X = E[g(x)] = E[X^2] = \int_{0}^{\infty} x^2 \frac{2}{5}e^{-\frac{x^2}{5}}dx = \frac{2}{5} \int_{0}^{\infty} x^3 e^{-\frac{x^2}{5}}dx$

Let $\beta = \frac{x^2}{5}$, $d\beta = \frac{2}{5}dx$ and $\int xe^{\beta}dx = e^{\beta}x\bigg|_{0}^{\infty} - \frac{1}{e^\beta}$

Power in $X = \int_{0}^{\infty} x^2 e^{-\frac{x^2}{5}} \cdot \frac{2}{5}dx = \int_{0}^{\infty} 5\beta e^{-\beta}d\beta = 5[e^{-\beta}(\frac{\beta}{1}-\frac{1}{e^\beta})]\bigg|_{0}^{\infty} = 5[0 - (0 - 1)] = 5$ Watts

**8. Moments**

• An immediate application of the expected value of a function $g(\cdot)$ of a random variable $X$ is in calculating moments.

• Two types of moments are of particular interest, those about the origin and those about the mean.

**8.1 Moments about the origin**

• The function $g(X) = X^n, n = 0, 1, 2, ...$ gives the moments of the random variable $X$.

• Let us denote the $n^{th}$ moment about the origin by $m_n$ then:

$$m_n = E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)dx$$  

(20)
8.2 Moments about the mean (Central moments)

- Moments about the mean value of $X$ are called central moments and are given the symbol $\mu_n$.
- They are defined as the expected value of the function

$$
ge(X) = (X - E[X])^n, n = 0, 1, \ldots$$

Which is

$$
\mu_n = E[(X - E(X))^n] = \int_{-\infty}^{\infty} (x - E[X])^n f_X(x) \, dx \quad (22)
$$

Notes:

- $u_0 = 1$, the area of $f_X(x)$

$$
u_1 = \int_{-\infty}^{\infty} x f_X(x) \, dx - E[X] \int_{-\infty}^{\infty} f_X(x) \, dx = 0
$$

8.2.1 Variance

The variance is an important statistic and it measures the spread of $f_X(x)$ about the mean.

- The square root of the variance $\sigma_X$, is called the standard deviation.
- The variance is given by:

$$
\sigma_X^2 = u_2 = E \left[ (X - E(X))^2 \right] = \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x) \, dx \quad (23)
$$

We have:

$$
\sigma_X^2 = E[X^2] - E[X]^2 \quad (24)
$$
This means that the variance can be determined by the knowledge of the first and second moments.

**Example 3.4:** \( f_X(x) = \begin{cases} \frac{1}{b} e^{-\frac{(x-a)}{b}} & x \geq a \\ 0 & x < a \end{cases} \)

Find \( \sigma_x^2 \)?

**Solu:** \( \sigma_x^2 = \int_a^\infty (x - \bar{x})^2 \frac{1}{b} e^{-\frac{(x-a)}{b}} \, dx \)

Let \( \beta = x - \bar{x}, d\beta = dx \), \( x = a \Rightarrow \beta = a - \bar{x} \)

Then, \( \sigma_x^2 = \int_{a-\bar{x}}^\infty \beta^2 \frac{1}{b} e^{-\frac{(x-x+\bar{x}-a)}{b}} \, d\beta \)

\[ = \frac{e^{-\frac{(x-a)}{b}}}{b} \int_{a-\bar{x}}^\infty \beta^2 e^{-\frac{\beta}{b}} \, d\beta \]

From table: \( \int x^2 e^{cx} \, dx = e^{cx} \left( \frac{x^2}{c} - \frac{2x}{c^2} + \frac{2}{c^3} \right) \)

\[ \Rightarrow \sigma_x^2 = e^{-\frac{(x-a)}{b}} \left[ \frac{-\beta}{b} \left( -b\beta^2 - 2b^2 - 2b^3 \right) \right]_{a-\bar{x}}^\infty \]

\[ = e^{-\frac{(x-a)}{b}} \left[ 0 - e^{-\frac{(x-a)}{b}} \left( -b(a - \bar{x})^2 - 2b^2(a - \bar{x}) - 2b^3 \right) \right] \]

\[ = (a - \bar{x})^2 - 2b(a - \bar{x}) - 2b^2 \]

since \( \bar{X} = E[X] = a + b \) (see example 3.1)

\( \sigma_x^2 = (a - a - b)^2 - 2b(a - a - b) - 2b^2 = b^2 + 2b^2 - 2b^2 = b^2 \)

**Another solution:** Use \( \sigma_x^2 = E[X^2] - \bar{X}^2 \)
8.2.2 Skew

- The skew or third central moment is a measure of asymmetry of the density function about the mean.

\[ u_3 = E \left[ (X - E[X])^3 \right] = \int_{-\infty}^{\infty} (x - E[X])^3 f_X(x) \, dx \]

(25)

\[ u_3 = 0 \quad \text{If the density is symmetric about the mean} \]

Example 3.5. Compute the skew of a density function uniformly distributed in the interval [-1, 1].

**Solution:**

\[ f_X(x) = \begin{cases} \frac{1}{2} & \text{for } -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \]

\[ E[X] = \int_{-\infty}^{+\infty} x f_X(x) \, dx = \int_{-1}^{+1} x \cdot \frac{1}{2} \, dx = \frac{1}{2} \left[ \frac{x^2}{2} \right]_{-1}^{1} = 0 \]

\[ u_3 = E \left[ (X - E[X])^3 \right] = \int_{-\infty}^{\infty} (x - E[X])^3 f_X(x) \, dx = \int_{-1}^{1} (x)^3 \frac{1}{2} \, dx = \frac{1}{2} \left[ \frac{x^4}{4} \right]_{-1}^{1} = 0 \]

9. Functions that give moments

- The moments of a random variable \( X \) can be determined using two different functions: Characteristic function and the moment generating function.
9.1 Characteristic function

- The characteristic function of a random variable $X$ is defined by:

$$\Phi_X(\omega) = E[e^{j\omega X}]$$  \hspace{1cm} (26)

- $j = \sqrt{-1}$ and $-\infty < \omega < +\infty$

- $\Phi_X(\omega)$ can be seen as the Fourier transform (with the sign of $\omega$ reversed) of $f_X(x)$:

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} f_X(x)e^{j\omega x} \, dx$$  \hspace{1cm} (27)

If $\Phi_X(\omega)$ is known then density function $f_X(x)$ and the moments of $X$ can be computed.

- The density function is given by:

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_X(\omega) e^{-j\omega x} \, d\omega$$  \hspace{1cm} (28)

- The moments are determined as follows:

$$m_n = (-j)^n \frac{d^n \Phi_X(\omega)}{d\omega^n} \bigg|_{\omega=0}$$  \hspace{1cm} (29)

- Note that $|\Phi_X(\omega)| \leq \Phi_X(0) = 1$

Differentiate $n$ times with respect to $\omega$ and set $\omega = 0$ in the derivative.
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**Example 3.6:** Let \( f_X(x) = \begin{cases} \frac{1}{b} e^{-\frac{(x-a)}{b}} & x \geq a \\ 0 & x < a \end{cases} \)

Evaluate the characteristic function and first moment.

**Solu:**

\[
\phi_X(w) = \int_{a}^{\infty} e^{-\frac{(x-a)}{b}} e^{jwx} dx
\]

\[
= \frac{e^{\frac{a}{b}}}{b} \int_{a}^{\infty} e^{\left(-\frac{x}{b} + jw\right)x} dx = \frac{e^{\frac{a}{b}}}{b(-\frac{1}{b} + jw)} e^{\left(-\frac{1}{b} + jw\right)a} |_{a}^{\infty}
\]

\[
= \frac{e^{\frac{a}{b}(-\frac{1}{b} + jw)} - e^{\frac{a}{b}(-\frac{1}{b} + jw)}}{-1 + jbw} = \frac{e^{jaw}}{1 - jbw}
\]

Then the moments are obtained from the moment generating function using the following expression:

\[
m_n = \left. \frac{d^n M_X(v)}{dv^n} \right|_{v=0}
\]

**9.2 Moment generating function**

- The moment generating function is given by:

\[
M_X(v) = E[e^{vX}] = \int_{-\infty}^{\infty} f_X(x) e^{vx} dx
\]

Where \( v \) is a real number: \(-\infty < v < \infty\)

- Then the moments are obtained from the moment generating function using the following expression:

\[
m_n = \left. \frac{d^n M_X(v)}{dv^n} \right|_{v=0}
\]

Compared to the characteristic function, the moment generating function may not exist for all random variables.
Example 3.7: Compute $M_X(v)$ and $m_1$ for the exponential r.v.

Solu: $M_X(v) = \int_a^\infty \frac{1}{b} e^{(x-a)/b} e^{vx} dx$

$$= \frac{e^{av}}{1 - bv}$$

$$m_1 = \left. \frac{ae^{av}(1 - bv) + e^{av}b}{(1 - bv)} \right|_{v=0} = a + b$$

10 Transformation of a random variable

- A random variable $X$ can be transformed into another r.v. $Y$ by:
  $$Y = T(X)$$

- Given $f_X(x)$ and $F_X(x)$, we want to find $f_Y(y)$, and $F_Y(y)$.
- We assume that the transformation $T$ is continuous and differentiable.

10.1 Monotonic transformation

- A transformation $T$ is said to be monotonically increasing $T(x_1) < T(x_2)$ for any $x_1 < x_2$.
- $T$ is said monotonically decreasing if $T(x_1) > T(x_2)$ for any $x_1 < x_2$. 
10.1.1 Monotonic increasing transformation

- In this case, for particular values \( x_0 \) and \( y_0 \) shown in figure 1, we have:

\[
y_0 = T(x_0)
\]

and

\[
x_0 = T^{-1}(y_0)
\]

- Due to the one-to-one correspondence between \( X \) and \( Y \), we can write:

\[
F_Y(y_0) = P\{Y \leq y_0\} = P\{X \leq T^{-1}(y_0)\} = F_X(x_0)
\]

\[
F_Y(y_0) = \int_{-\infty}^{y_0} f_Y(y)dy = \int_{-\infty}^{x_0} f_X(x)dx
\]

- Differentiating both sides with respect to \( y_0 \) and using the expression \( x_0 = T^{-1}(y_0) \), we obtain:

\[
f_Y(y_0) = f_X[T^{-1}(y_0)] \frac{d}{dy_0} T^{-1}(y_0)
\]
• This result could be applied to any \( y_0 \), then we have:

\[
f_Y(y) = f_X[T^{-1}(y)] \frac{dT^{-1}(y)}{dy}
\]  

(38)

• Or in compact form:

\[
f_Y(y) = f_X(x) \frac{dx}{dy} \bigg|_{x = T^{-1}(y)}
\]  

(39)

### 10.1.2 Monotonic decreasing transformation

![Monotonic decreasing transformation](image)

Figure 6. Monotonic decreasing transformation

• From Figure 2, we have

\[
F_Y(y_0) = P\{Y \leq y_0\} = P\{X \geq x_0\} = 1 - F_X(x_0)
\]  

(40)

\[
F_Y(y_0) = \int_{-\infty}^{y_0} f_Y(y)dy = 1 - \int_{-\infty}^{x_0} f_X(x)dx
\]  

(41)

• Again Differentiating with respect to \( y_0 \), we obtain:

\[
f_Y(y_0) = -f_Y[T^{-1}(y_0)] \frac{dT^{-1}(y_0)}{dy_0}
\]  

(42)
As the slope of \( T^{-1}(y_0) \) is negative, we conclude that for both types of monotonic transformation, we have:

\[
f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right| \quad \text{and} \quad x = T^{-1}(y)
\]  

(43)

**Example 3.8:** Let \( Y=aX+b \). Find \( f_Y(y) \) given that \( f_X(x) \) is Gaussian r.v. with mean \( a_x \) and standard deviation \( \sigma_x \).

**Solution:**

\[
Y = aX + b \implies X = \frac{y - b}{a} \quad \text{and} \quad \frac{dx}{dy} = \frac{1}{a}
\]

\[
\Rightarrow f_Y(y) = f_X \left( \frac{y - b}{a} \right) \left| \frac{1}{a} \right|
\]

When \( f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}} \)

Then \( f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{|y-b-a_x|^2}{2\sigma_x^2}} \left| \frac{1}{a} \right| \)

\[
= \frac{1}{|a|\sqrt{2\pi}\sigma_x} e^{-\frac{|y-(b+aa_x)|^2}{2a^2\sigma_x^2}}
\]

Y is also Gaussian with mean and variance:

\[ a_Y = aa_x + b \quad \text{and} \quad \sigma_Y^2 = a^2\sigma_x^2 \]

a. **Non-monotonic transformation**

*In general, a transformation could be non monotonic as shown in figure 3*
In this case, more than one interval of values of $X$ that correspond to the event $P(Y \leq y_0)$

For example, the event represented in figure 7 corresponds to the event \{\begin{align*}
X \leq x_1 \text{ and } x_2 \leq X \leq x_3
\end{align*}\}.

In general for non-monotonic transformation:

$$f_Y(y) = \sum_{j=1}^{N} \frac{f_X(x_j)}{\left| \frac{dT(x)}{dx} \right|_{x=x_j}}$$ \hspace{1cm} (44)

Where $x_j, j = 1, 2, \ldots, N$ are the real solutions of the equation $T(x) = y$
**Example 3.9**: Let $Y = T(X) = cX^2 ; c > 0$.

Given $f_X(x)$, find $f_Y(y)$?

Soln: $Y = cX^2 \Rightarrow x_1 = \sqrt{\frac{y}{c}}$, $x_2 = -\sqrt{\frac{y}{c}}$

$y' = 2cx$

$$f_Y(y) = \frac{f_X\left(\sqrt{\frac{y}{c}}\right) + f_X\left(-\sqrt{\frac{y}{c}}\right)}{2c\sqrt{\frac{y}{c}}}$$

$$f_Y(y) = \frac{f_X\left(\sqrt{\frac{y}{c}}\right) + f_X\left(-\sqrt{\frac{y}{c}}\right)}{2\sqrt{cy}}, \quad y \geq 0$$