1.5 Elementary Row operations:

Elementary row operations are steps for solving the linear system of equations:

I. Interchange two rows
II. Multiply a row with non zero real number
III. Add a multiple of one row to another row

Note: *Elementary row operations produce same results when operated either on a system or on its augmented matrix form.*

1.6 Methods for solving System of Linear equations

1. **Gaussian Elimination Method**

2. **Gauss – Jorden Elimination Method**

1.7 Gaussian Elimination Method

STEP 1. by using elementary row operations

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & A_{12} & A_{13} & B_1 \\ 0 & 1 & A_{23} & B_2 \\ 0 & 0 & 1 & B_3 \end{bmatrix}$$

STEP 2. Find solution by back – substitutions.

Example:3. Solve the system of linear equations by Gaussion- elimination method

$$x-2y-z=3$$
$$3x-6y-5z=3$$
$$2x-y+z=0$$

Solution: Augmented matrix is

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 3 & -6 & -5 & 3 \\ 2 & -1 & 1 & 0 \end{bmatrix}$$

STEP 1. Creating 0 in the first below first entry by performing row operations
$-3R_1+R_2 \Rightarrow R_2,\quad -2R_1+R_3 \Rightarrow R_3$

$$\approx \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 0 & -2 & -6 \\ 0 & 3 & 3 & -6 \end{bmatrix} \approx \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 3 & 3 & -6 \\ 0 & 0 & -2 & -6 \end{bmatrix} R_2 \Leftrightarrow R_3$$

Creating 1 in second entry of the second row and in third entry of the third row by performing row operations $\frac{1}{3}R_2 \Rightarrow R_2, -\frac{1}{2}R_3 \Rightarrow R_3$

$$\approx \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Equivalent system of equations form is:

$$x-2y-z=3$$
$$y+z=-2$$
$$z=3$$

STEP 2. <u>**Back Substitution**</u>

$$z=3$$
$$y=-z-2=-3-2=-5$$
$$x=2y+z+3=-10+3+3=-4$$

Solution is

$$x=-4\ ,\ y=-5,\ z=3$$

Gauss Elimination method

Example 3.1. Solve the system of linear equations by Gauss Elimination method

$$\begin{aligned} 4x_1 + 5x_2 \quad &= 2 \\ 11x_1 + x_2 + 2x_3 &= 3 \\ x_1 + 5x_2 + 2x_3 &= 1 \end{aligned}$$

Solution:

Step I Elimination

It is to eliminate the values below the leading entries to zero

$$\begin{bmatrix} 4 & 5 & 0 & 2 \\ 11 & 1 & 2 & 3 \\ 1 & 5 & 2 & 1 \end{bmatrix} \cong \begin{bmatrix} 1 & 5 & 2 & 1 \\ 11 & 1 & 2 & 3 \\ 4 & 5 & 0 & 2 \end{bmatrix} R_{1\leftrightarrow}R_3 \cong \begin{bmatrix} 1 & 5 & 2 & 1 \\ 0 & -54 & -20 & -8 \\ 0 & -15 & -8 & -2 \end{bmatrix}$$

$$-11R_1 + R_2, -4R_1 + R_3 \cong \begin{bmatrix} 1 & 5 & 2 & 1 \\ 0 & 1 & \frac{10}{27} & \frac{4}{27} \\ 0 & -15 & -8 & -2 \end{bmatrix} \frac{-R_2}{54} \cong$$

$$\cong -15R_2 + R_3 \begin{bmatrix} 1 & 5 & 2 & 1 \\ 0 & 1 & \frac{10}{27} & \frac{4}{27} \\ 0 & 0 & \frac{-22}{9} & \frac{2}{9} \end{bmatrix} \frac{-9R_3}{22} \cong \begin{bmatrix} 1 & 5 & 2 & 1 \\ 0 & 1 & \frac{10}{27} & \frac{4}{27} \\ 0 & 0 & 1 & \frac{-1}{11} \end{bmatrix}$$

Step II. Backward substitution

Writing the equation form of the augmented matrix and then substituting value of x_3

$$\begin{aligned} x_1 + 5x_2 + 2x_3 &= 1 \\ x_2 + \frac{10}{27}x_3 &= \frac{4}{27} \\ x_3 &= \frac{-1}{11} \end{aligned}$$

OR

$$x_3 = \frac{-1}{11}$$

$$x_2 + \frac{10}{27}\left(\frac{-1}{11}\right) = \frac{4}{27}, \quad x_2 = \frac{4}{27}$$

$$x_1 = -5x_2 - 2x_3 + 1$$

$$x_1 = -5\frac{4}{27} - 2\left(\frac{-1}{11}\right) + 1 = \frac{3}{11}$$

Solution is $x_1 = \frac{3}{11}, \quad x_2 = \frac{4}{27}, \quad x_3 = \frac{-1}{11}$

Example 3.2. Find values of *x, y, and z* by solving the system of equations by Gauss Elimination method

$$\frac{1}{x}+\frac{8}{y}+\frac{2}{z}=7$$
$$\frac{2}{x}+\frac{4}{y}-\frac{4}{z}=3$$
$$\frac{2}{x}+\frac{1}{y}+\frac{1}{z}=2$$

Solution:

Step I *is to eliminate the values below the leading entries to zero of the Augmented matrix* [A:b]

$$[A:b]\cong\begin{bmatrix}1&8&2&7\\2&4&-4&3\\2&1&1&2\end{bmatrix}$$

$$\xrightarrow{-2R_1+R_2,-2R_1+R_3}\begin{bmatrix}1&8&2&7\\0&-12&-8&-11\\0&-15&-3&-12\end{bmatrix}\xrightarrow{-\frac{1}{12}R_2}\begin{bmatrix}1&8&2&7\\0&1&\frac{2}{3}&\frac{11}{12}\\0&-15&-3&-12\end{bmatrix}$$

$$\xrightarrow{15R_2+R_3}\begin{bmatrix}1&8&2&7\\0&1&\frac{2}{3}&\frac{11}{12}\\0&0&7&\frac{7}{4}\end{bmatrix}\xrightarrow{\frac{1}{7}R_3}\begin{bmatrix}1&8&2&7\\0&1&\frac{2}{3}&\frac{11}{12}\\0&0&1&\frac{1}{4}\end{bmatrix}$$

<u>Step II</u>. Backward substitution

Writing the equation form of the augmented matrix and then substituting value of x_3

$$\frac{1}{x}+\frac{8}{y}+\frac{2}{z}=7$$
$$\frac{1}{y}+\frac{2}{3z}=\frac{11}{12}$$
$$\frac{1}{z}=\frac{1}{4}$$

OR

$$\frac{1}{z}=\frac{1}{4}\quad\Rightarrow\quad z=4$$
$$\frac{1}{y}=-\frac{2}{3z}+\frac{11}{12}=-\frac{1}{6}+\frac{11}{12}=\frac{3}{4}\quad\Rightarrow\quad y=\frac{4}{3}$$
$$\frac{1}{x}=-\frac{8}{y}-\frac{2}{z}+7=-8\left(\frac{3}{4}\right)-2\left(\frac{1}{4}\right)+7=\frac{1}{2}\quad\Rightarrow\quad x=2$$

Solution is $x=2,\ y=\frac{4}{3}$ *and* $z=4$

Example 4. Suppose that points (-2,-1), (-1,2), (1,2) lie on parabola

$$y = a + bx + cx^2,$$

(i) Determine a linear system of equations in three unknown a, b and c,
(ii) Find the equation of parabola by solving the system of linear equation.

Solution:

(i) The system of linear equations can be obtained by substituting these points in the equation of parabola as these lie on the parabola.

Through point $(-2,-1)$ $a - 2b + 4c = -1$
through point $(-1,2)$ $a - b + c = 2$
through point $(1,2)$ $a + b + c = 2$

The system of linear equations is

$$\begin{aligned} a - 2b + 4c &= -1 \\ a - b + c &= 2 \\ a + b + c &= 2 \end{aligned}$$

(ii)

STEP: I.

Matrix form of the system is:

$$\begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

Augmented matrix form is:

$$\begin{bmatrix} 1 & -2 & 4 & -1 \\ 1 & -1 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Creating 0 in the first below first entry by performing row operations
$-R_1+R_2$ **and** $-R_1+R_3$

$$\approx \begin{bmatrix} 1 & -2 & 4 & -1 \\ 0 & 1 & -3 & 3 \\ 0 & 3 & -3 & 3 \end{bmatrix}$$

Creating 0 in second entry of the third row by performing row operations
$-3R_2+R_3$

$$\approx \begin{bmatrix} 1 & -2 & 4 & -1 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 6 & -6 \end{bmatrix}$$

STEP: II.

We are using Gauss Elimination method , so we write the equation of the matrix

$$a \ -2b \ + \ 4c = -1$$

$$-b \ -3c \ = \ 3$$

$$6c \ = -6$$

Solving by backward substitution

$$6c \ = -6 \quad \Rightarrow c = -1$$

$$-b \ = 3c \ + \ 3 = -3+3 = 0 \qquad \Rightarrow b = 0$$

$$a \ = 2b \ - \ 4c - 1 = 0 + 4 - 1 = 3 \ \Rightarrow a = 3$$

Solution of the system *is* $a = 3, \ b \ = 0$ and $c = -1$

Equation of parabola is $y = 3 - x^2$

1.8 Gauss – Jorden Elimination Method

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & B_1 \\ 0 & 1 & 0 & B_2 \\ 0 & 0 & 1 & B_3 \end{bmatrix}$$

Example.4. Solve the system of linear equations by Gauss - Jorden elimination method

$$x_1 + x_2 + 2x_3 = 8$$
$$-x_1 - 2x_2 + 3x_3 = 1$$
$$3x_1 - 7x_2 + 4x_3 = 10$$

Solution: Augmented matrix is

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix}$$

$$\approx \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{bmatrix} \quad R_1+R_2,\ -3R_1+R_3$$

$$\approx \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{bmatrix} \quad -R_2,\ 10R_2+R_3$$

$$\approx \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad -R_3/52$$

$$\approx \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad -2R_3+R_1,\ 5R_3+R_2$$

$$\approx \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad -R_2+R_1$$

Equivalent system of equations form is:

$x_1 = 3$
$x_2 = 1$
$x_3 = 2$ is the solution of the system.

1.9 Row Echelon Form

A form of a matrix, which satisfies following conditions, is row echelon form

i. '1' (leading entry) must be in the beginning of each row,
ii. '1' must be on the right of the above leading entry,
iii. Below the leading entry all values must be zero,
iv. A row containing all zero values must be in the bottom.

Examples:

$$\text{(i)} \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad \text{(ii)} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{(iii)} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

1.10 Reduced Row Echelon Form

A form of a matrix, which satisfies following conditions, is row echelon form

i. '1' (leading entry) must be in the beginning of each row,
ii. '1' must be on the right of the above leading entry,
iii. All entries in the column containing leading entry must be zero,
iv. A row containing all zero values must be in the bottom.

Examples

$$\text{(i)} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad \text{(ii)} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{(iii)} \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note: 1. Gaussian Elimination method is reducing the given Augmented matrix to Row echelon form and backward substitution.
Note: 2. Gauss- Jordan Elimination method is reducing the given Augmented matrix to Reduced Row echelon form.

Example:5. Use Gauss – Jorden method to solve the system of linear system

$$\begin{aligned} x - y + 2z - w &= -1 \\ 2x + y - 2z - 2w &= -2 \\ -x + 2y - 4z + w &= 1 \\ 3x \qquad\quad - 3w &= -3 \end{aligned}$$

Solution: Gauss-Jorden method is same as to reduce the augmented matrix to reduced row echelon from.

Augmented matrix is

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix}$$

There is a leading entry '1' in the first row, making all other entries in the first column zero

$$\approx \begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix} \quad (-2R_1+R_2)/3,\ R_1+R_3,\ -3R_1+R_4$$

$$\approx \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2+R_1,\ -R_2+R_3,\ -3R_2+R_4$$

is reduced row echelon form

Equivalent matrix form is

$x - w = -1$

$y - 2z = 0$

there are four variables x, y, w and z in the example, variables appearing as leading entries are called LEADING VARIABLES, and other variables are FREE VARIABLE

x and y are leading variables and w and z are free variables.

Let $z = s$ and $w = t$, where s and t are real numbers ,

$x = -1 + w = -1 + t$

$y = 2z \quad = 2s$

$z = s$

$w = t, \quad t \text{ and } s \in R$

There are infinite many solutions of the given system.

1.11 SYSTEM *WITH NO SOLUTION*

Example: 6 . Solve the system of linear equations

$$\begin{aligned} x - 2y + z - 4u &= 1 \\ x + 3y + 7z + 2u &= 2 \\ x - 12y - 11z - 16u &= 5 \end{aligned}$$

Solution:

Augmented matrix is:

$$\begin{bmatrix} 1 & -2 & 1 & -4 & 1 \\ 1 & 3 & 7 & 2 & 2 \\ 1 & -12 & -11 & -16 & 5 \end{bmatrix}$$

Reducing it to row echelon form (using Gaussian - elimination method)

$$\approx \begin{bmatrix} 1 & -2 & 1 & -4 & 1 \\ 0 & 5 & 6 & 6 & 1 \\ 0 & -10 & -12 & -12 & 4 \end{bmatrix} \quad R_2 - R_1,\ R_3 - R_1$$

$$\approx \begin{bmatrix} 1 & -2 & 1 & -4 & 1 \\ 0 & 5 & 6 & 6 & 1 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix} \quad -R_3 + 2R_2$$

Last equation is

$$0x + 0y + 0z + 0u = 6$$

but $0 \neq 6$

hence there is no solution for the given system of linear equations.

EXERCISE 1.1

SYSYTEM OF LINEAR EQUATIONS

Solve the system of linear equations

(a) Gaussian Elimination (Row Echelon form)

(b) Gauss – Jordon method (Reduced Row Echelon form)

1.
$$x+y+z=7$$
$$-x+y+z=5$$
$$x-y+z=5$$
Ans: $x=1, y=1, z=5$

2.
$$x+2y-z=2$$
$$2x+3y+2z=5$$
$$3x+2y+3z=10$$
Ans: $x=4, y=-1, z=0$

3.
$$3x+y-z=-4$$
$$x+y-2z=-4$$
$$-x+2y-z=1$$
Ans: $x=-1, y=1, z=2$

4.
$$x+2y+3z=6$$
$$2x-3y+2z=14$$
$$3x+y-z=-2$$
Ans: $x=1, y=-2, z=3$

5.
$$x-y+z=4$$
$$4x+2y+2z=8$$
$$3x+2y+2z=2$$
Ans: $x=6, y=-3, z=-5$

6.
$$x+2y+3z=17$$
$$3x+2y+z=11$$
$$x-5y+3z=-5$$
Ans: $x=1, y=2, z=4$

7.

$$x+8y+2z=7$$
$$2x+4y-4z=3$$
$$2x+y+z=2$$

Ans: $x=1/2, y=3/4, z=1/4$

8.

$$x+y-2z=1$$
$$2x-y+z=2$$
$$x-2y-4z=-4$$

Ans: $x=26/21, y=25/21, z=5/7$

Solve the system of linear equations by Gauss – Jordan method

9.

$$4x_1 - x_3 + 2x_4 = -3$$
$$x_2 + x_3 + 3x_4 = 1$$
$$x_1 - 4x_2 + 3x_3 + x_4 = 0$$

10.

$$x_1 + 2x_2 + 4x_3 + x_4 = 3$$
$$2x_1 + 3x_3 = 5$$
$$x_1 + 3x_2 + 2x_4 = 2$$

11.

$$x_1 - 2x_2 + x_3 - 3x_4 = 2$$
$$x_1 + 3x_2 + 7x_3 + 2x_4 = 3$$
$$x_1 - 12x_2 - 11x_3 - 13x_4 = 7$$

12.

$$x_1 - x_2 + 2x_3 + x_4 = -1$$
$$2x_1 + x_2 + x_3 - x_4 = 4$$
$$x_1 + 2x_2 - x_3 - 2x_4 = 5$$
$$x_1 + x_3 = 1$$

13.

$$x+2y+3z+4w=5$$
$$x+3y+5z+7w=11$$
$$x - z - 2w = -6$$

Ans: No solution

14.

$$x+y+2z-5w=3$$
$$2x+y-z+3w=-11$$
$$2x+5y-z-9w=-3$$
$$x-3y+2z+7w=-5$$

Ans: $x=-5-2t, y=2+3t, z=3+2t, w=t$

15.

$$\begin{aligned}2x_1 + x_2 + 6x_3 + x_4 &= 6\\ 6x_1 - 6x_2 + 2x_3 + 12x_4 &= 36\\ 4x_1 + 3x_2 + 3x_3 - 3x_4 &= -1\\ 2x_1 + 2x_2 - x_3 + x_4 &= 10\end{aligned}$$

Ans: $x_1 = 2, x_2 = 1, x_3 = -1, x_4 = 3$

16.

$$\begin{aligned}2x - 3y - z &= 5\\ x + 2y &= 4\\ 10y + 3z &= -2\\ 3x + 3y + 2z &= 1\end{aligned}$$

Ans: $x = 2, y = 1, z = -4$

17.

$$\begin{aligned}3x - y + 7z &= 0\\ 2x - y + 4z &= \frac{1}{2}\\ x - y + z &= 1\\ 6x - 4y + 10z &= 3\end{aligned}$$

Ans: $x = -\frac{1}{2} - 3t, y = -\frac{3}{2} - 2t, z = t,$ where t is arbitrary.

18.

$$\begin{aligned}x_1 + x_2 - x_3 + 2x_4 &= 10\\ -5x_1 + 3x_2 - 15x_3 - 6x_4 &= 9\\ 3x_1 - x_2 + 7x_3 + 4x_4 &= -1\end{aligned}$$

Ans: System is inconsisitent.

1.12 Conditions on Solutions

Example:7. For which values of 'a' will be following system

$$\begin{aligned} 2x+3y+z &= -1 \\ x+2y+z &= 0 \\ 3x+y+(a^2-6)z &= a-3 \end{aligned}$$

(i) infinitely many solutions?
(ii) No solution?
(iii) Exactly one solution?

Solution:

Using Gaussian Elimination method: Reducing the Augmented matrix to row Echelon form

The augmented matrix:

[A:b]≡

$$\begin{bmatrix} 2 & 3 & 1 & -1 \\ 1 & 2 & 1 & 0 \\ 3 & 1 & a^2-6 & a-3 \end{bmatrix} R_1 \leftrightarrow R_2 \equiv \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 3 & 1 & -1 \\ 3 & 1 & a^2-6 & a-3 \end{bmatrix} \begin{matrix} R_2+(-2R_1) \\ R_3+(-3R_1) \end{matrix}$$

$$\equiv \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & -1 \\ 0 & -5 & a^2-9 & a-3 \end{bmatrix} \begin{matrix} -R_2 \\ R_3+(5R_2) \end{matrix} \equiv \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & a^2-4 & a-2 \end{bmatrix}$$

I. *infinitely many solutions:* $a = 2$, $a^2 - 4 = a - 2 \Leftrightarrow 0 = 0$, *as number of equations are reduced to two and number of variables are three.*

II. *no solution:* $a = -2$, $a^2 - 4 = a - 2 \Leftrightarrow 0 \neq -4$, It is never true statement

III. *one solution:* $a = R - \{2, -2\}$, *for every value of a in the given interval there will have only one solutions.*

Note: *System is inconsistent is case* $a = -2$, *otherwise the system is consistent.*

Conditions on Solutions

Example:7.1. For what values of λ does the system of equations

$$\begin{aligned} 3x \quad\quad\quad + \lambda z &= 2 \\ 3x + 3y + 4z &= 4 \\ y + 2z &= 3 \end{aligned}$$

have (i) unique solution, (ii) infinitely many solutions and (iii) no solution.

Solution: (a) Augmented matrix is Form

$$[A|b] \equiv \begin{bmatrix} 3 & 0 & \lambda & 2 \\ 3 & 3 & 4 & 4 \\ 0 & 1 & 2 & \lambda \end{bmatrix}$$

$$\xrightarrow{-R_1+R_2} \begin{bmatrix} 3 & 0 & \lambda & 2 \\ 0 & 3 & 4-\lambda & 2 \\ 0 & 1 & 2 & \lambda \end{bmatrix}$$

$$\xrightarrow{3R_3-R_2} \begin{bmatrix} 3 & 0 & \lambda & 2 \\ 0 & 3 & 4-\lambda & 2 \\ 0 & 0 & \lambda+2 & 3\lambda-2 \end{bmatrix}$$

considering last row the Augmented matrix

$$0x + 0y + (\lambda+2)z = 3\lambda - 2$$

(i) If $\lambda = -2$, then $0 = -8$, but $0 \neq -8$ which is not possible, so there is no solution.

(ii) If $\lambda \neq -2$, then , we have three equations and three unkowns, so we have unique solution.

(iii) As both side of last row of the matrix will not have all zero value for any value of λ, so system will not have infinitly many solutions.

Example:8. What conditions must a, and b satisfy in order for the system of equations

$$\begin{aligned} x-2y+3z &= 4 \\ 2x-3y+az &= 5 \\ 3x-4y+5z &= b \end{aligned}$$

to have (i)infinitely many solutions? (ii) No solution? (iii) Exactly one solution?

Solution: The augmented matrix is

$$[A:B]=\begin{bmatrix} 1 & -2 & 3 & 4 \\ 2 & -3 & a & 5 \\ 3 & -4 & 5 & b \end{bmatrix}$$

reducing it to reduced row echelon form

$$\approx\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & a-6 & -3 \\ 0 & 2 & -4 & b-12 \end{bmatrix} \quad R_2\text{-}2R_1,\ R_3\text{-}3R_1$$

$$\approx\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & a-6 & -3 \\ 0 & 0 & -2a+8 & b-6 \end{bmatrix} \quad R_3\text{-}2R_2$$

(i) Infinitely many solutions?

If $a=4$ and $b=6$ then

$$\approx\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(ii) No solution?

If $a=4$ and $b\neq 6$ then

$$\approx\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & b-6 \end{bmatrix}$$

(iii) Exactly one solution?

If $a\neq 4$ and $b\in R$ then

$$\approx\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & \dfrac{b-6}{-2a+8} \end{bmatrix}$$

Example:9. What conditions must a, b, and c satisfy in order for the system of equations

$$\begin{aligned} x+y+2z&=a \\ x \quad\;\; +z &= b \\ 2x+y+3z&=c \end{aligned}$$

to be consistent.

Solution: The augmented matrix is

$$\approx\begin{bmatrix} 1 & 1 & 2 & a \\ 1 & 0 & 1 & b \\ 2 & 1 & 3 & c \end{bmatrix}$$ reducing it to reduced row echelon form

$$\approx\begin{bmatrix} 1 & 1 & 2 & a \\ 0 & -1 & -1 & b-a \\ 0 & -1 & -1 & c-2a \end{bmatrix}$$ R_2-R_1, R_3-$2R_1$

$$\approx\begin{bmatrix} 1 & 1 & 2 & a \\ 0 & -1 & -1 & b-a \\ 0 & 0 & 0 & c-a-b \end{bmatrix}$$ R_3-R_1

The system will be consistent if only if $c-a-b=0$

Or $c=a+b$

Thus the required condition for system to be consistent is

$$c=a+b.$$

EXERCISE 1.2

Conditions on solutions

Find the relationship between a, b and c for which the system of equations will be consistent:

1.
$$\begin{aligned} 2x - y - z &= a \\ x + 2y + z &= b \\ 5x + 4y - z &= c \end{aligned}$$

2.
$$\begin{aligned} x_1 - x_2 + x_3 &= a \\ 2x_1 - x_2 + 3x_3 &= b \\ x_1 \quad + 2x_3 &= c \end{aligned}$$

3.
$$\begin{aligned} x_1 - 2x_2 + x_3 &= a \\ 2x_1 + x_2 + x_3 &= b \\ 5x_2 - x_3 &= c \end{aligned}$$

4.
$$\begin{aligned} x_1 - x_2 + 3x_3 &= a \\ 3x_1 - 3x_2 + 9x_3 &= b \\ -2x_1 + 2x_2 - 6x_3 &= c \end{aligned}$$
Ans: $2b = c + 4a$

5.
$$\begin{aligned} x + 2y - 2z - 2t &= a \\ -y + 3z + 2t &= b \\ -x + y + 4z + 3t &= c \\ 4y - z - t &= 2 \end{aligned}$$
Solution : $a - b + c = 2$

ans. $x = -\frac{3}{2}t, y = \frac{1}{2}t, z = t$

For what values of λ does the following system of linear equations have (a) Unique solution, (b) infinite many solutions, and (c) no solution.

6.
$$\begin{aligned} 3x \quad + \lambda z &= 2 \\ 3x + 3y + 4z &= 4 \\ y + 2z &= \lambda \end{aligned}$$
Ans. , (a) $\lambda \neq -2$,(b)$\lambda = -2$ and $\lambda = 2/3$, (c) $\lambda = -2$.

7.
$$\begin{aligned} x + y + z &= 3 \\ 2x - y - z &= 1 \\ 3x + \lambda y + \lambda^2 z &= 7 \end{aligned}$$

Ans. , (a)$R - \{0,1\}$,(b)$\lambda = 9/5$, (c) $\lambda = 0$ *or* $\lambda = 1$

8. $$\begin{aligned} 2x + 3y + z &= -1 \\ x + 2y + z &= 0 \\ 3x + y + (\lambda^2-13)z &= \lambda-3 \end{aligned}$$

Ans. ,(a) $\lambda \neq \pm\sqrt{11}$,(b)$\lambda = \pm\sqrt{11}$ and $\lambda = -2$,(c)$\lambda = \pm\sqrt{11}$

9. $$\begin{aligned} x + y + 2z &= \lambda \\ x - 3z &= \lambda^2 \\ 2x + y - z &= 0 \end{aligned}$$

Ans. (a) No unique solution,(b) $\lambda = 0$ *or* $\lambda = -1$, (c) $\lambda \neq -1, \lambda \neq 0$.

10. $$\begin{aligned} 2x + y - z &= 1 \\ x + y + z &= 2 \\ 3x + 2y &= \lambda \end{aligned}$$

11. $$\begin{aligned} x + 2y - 4z &= 3 \\ 3x - y + 13z &= 3 \\ 4x + y + \lambda^2 z &= \lambda+3 \end{aligned}$$

12. $$\begin{aligned} x + y - z &= 2 \\ x + y + z &= 3 \\ x + y + (\lambda^2-5)z &= \lambda \end{aligned}$$

Ans. , (a) $\lambda = 2$,(b)$\lambda \neq \pm 2$, (c) $\lambda = -2$.

For what values of λ and μ does the following system of linear equations have (a)Unique solution, (b) infinite many solutions, and (c) no solution.

13.

$$\begin{aligned} x + y + z &= 6 \\ x + 2y + 3z &= 10 \\ x + 2y + \lambda z &= \mu \end{aligned}$$

$Ans:\ (a)\ \lambda \neq 3\ \mu \in R,\ (b)\lambda = 3, \mu = 10\ (c)\ \lambda = 3, \mu \neq 10$

14.

$$\begin{aligned} x - 2y + 3z &= 4 \\ 2x - 3y + \lambda z &= 5 \\ 3x - 4y + 5z &= \mu \end{aligned}$$

15. For what values of λ does the following system of linear equations have(a) Unique solution, (b) infinite many solutions, and (c) no solution.

$$\begin{aligned} x+y+z&=2\\ 2x+y-z&=4\\ 3x+2y\quad&=\lambda \end{aligned}$$

Ans: $\lambda=6$, *system will have many solutions*

16. For what values of λ does the following system has non-trivial solution

$$\begin{aligned} 2x+y-z&=1\\ x+y+z&=2\\ 3x+2y\quad&=\lambda \end{aligned}$$

17.Find a relation between a and c so that the system

$$\begin{aligned} ax+3z&=2\\ 3y-2z&=1\\ x+cz&=2 \end{aligned}$$ has a unique solution

1.14 HOMOGENEOUS SYSTEM

Let AX= b be asystem of linear equations. This system is called **homogeneous system of linear equations** if and only if b = 0.
A system of equations of the form

AX = 0.,

That is with all constants b's taken as zero.

Example:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ... + a_{2n}x_n = 0$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + ... + a_{3n}x_n = 0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + ... + a_{mn}x_n = 0$$

is homogeneous system of linear equations.

1. The homogeneous system AX=0 always has at least one solution, $x_1 = x_2 = x_3 = ... = x_n = 0$ called a **trivial solution.**
2. The homogeneous system has infinitely many **non- trivial solutions** in addition to the trivial solutions.
3. The homogeneous system will have a non- trivial solution if and only if A is a singular matrix $\Rightarrow$ determinant of A is zero.

Example:10. Solve the homogeneous system of linear equations

$$\begin{aligned} 2x+2y+4z&=0 \\ w \quad\quad - y-3z&=0 \\ 2w+3x+\ \ y+\ z&=0 \\ -2w+\ x+3y-2z&=0 \end{aligned}$$

Solution: The augmented matrix is

$$\begin{bmatrix} 0 & 2 & 2 & 4 & 0 \\ 1 & 0 & -1 & -3 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{bmatrix} \Leftrightarrow \overset{R_1 \leftrightarrow R_2}{\begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 2 & 2 & 4 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{bmatrix}} \Leftrightarrow$$

$R_2/2,\ R_3-2R_1,\ R_4+2R_1$ $\quad$ $R_3\text{-}3R_2,\ -R_2+R_4$

$$\begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 3 & 3 & 7 & 0 \\ 0 & 1 & 1 & -8 & 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -10 & 0 \end{bmatrix} \Leftrightarrow$$

$R_1+3R_3,\ R_2\text{-}2R_3,\ R_4+10R_3$

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

System form is;

$$\begin{aligned} w-y \quad &=0 \\ x+y \quad &=0 \\ z &=0 \end{aligned}$$

leading entries are w, x, and z , free entry is y

let $y=t$

$w=y=t$

$x=-y=-t$

$z=0$

solution is $w=t, x=-t, y=t, z=0$, where $t\in R$, $t\neq 0$.

so there are inifitly many solutions.

Example:11. Solve the homogeneous system of linear equations

$$
\begin{aligned}
x_1 + 3x_2 \qquad + x_4 &= 0 \\
x_1 + 4x_2 + 2x_3 \qquad &= 0 \\
-2x_2 - 2x_3 - x_4 &= 0 \\
2x_1 - 4x_2 + x_3 + x_4 &= 0 \\
x_1 - 2x_2 - x_3 + x_4 &= 0
\end{aligned}
$$

Solution: The augmented matrix is

$$
\begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 1 & 4 & 2 & 0 & 0 \\ 0 & -2 & -2 & -1 & 0 \\ 2 & -4 & 1 & 1 & 0 \\ 1 & -2 & -1 & 1 & 0 \end{bmatrix}
\underset{R_2-R_1,\ -R_3,\ R_4-2R_1,\ R_5-R_1}{\approx}
\begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 2 & 2 & 1 & 0 \\ 0 & -10 & 1 & -1 & 0 \\ 0 & -5 & -1 & 0 & 0 \end{bmatrix}
$$

$$
\underset{R_3-R_2,\ R_4-10R_2,\ R5+5R_2}{\approx}
\begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & -2 & 3 & 0 \\ 0 & 0 & 21 & -11 & 0 \\ 0 & 0 & 9 & -5 & 0 \end{bmatrix}
\underset{2R_4+21R_3,\ 2R_5+9R_3}{\approx}
\begin{bmatrix} 1 & & 30 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & -2 & 3 & 0 \\ 0 & 0 & 0 & 41 & 0 \\ 0 & 0 & 0 & 17 & 0 \end{bmatrix}
$$

$$
\underset{R_4/41,\ -R_5/17-R_4/41}{\approx}
\begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & -2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\underset{R_1-R_4,\ R_2+R_4,\ (R_3-3R_4)/-2}{\approx}
\begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
\underset{R_1-2R_3}{\approx}
\begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\underset{R_1-3R_2}{\approx}
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

Equation form is $x_1 = 0, x_2 = 0, x_3 = 0$ and $x_4 = 0$.

There is only trivial solution and there are no additional non-trivial solutions.

Example:12. For which value (s) of λ, the system of equations have non – trivial solutions,

$$(\lambda-3)x+\quad y=0$$
$$x+(\lambda-3)y=0$$

Solution:

The augmented matrix is $\begin{bmatrix}\lambda-3 & 1 & 0\\ 1 & \lambda-3 & 0\end{bmatrix}$

$$\approx\begin{bmatrix}1 & \lambda-3 & 0\\ \lambda-3 & 1 & 0\end{bmatrix}\approx\begin{bmatrix}1 & \lambda-3 & 0\\ 0 & 1-(\lambda-3)^2 & 0\end{bmatrix}$$

The equation form is

$$x+(\lambda-3)y=0 \qquad .1$$
$$[1-(\lambda-3)^2]y=0 \qquad .2$$

if y = 0 ⇒ x = 0, which are trivial solution,

hence $y\neq 0 \Rightarrow [1-(\lambda-3)^2]=0$

$$1-(\lambda^2-6\lambda+9)=0$$
$$1-\lambda^2+6\lambda-9=0$$
$$-\lambda^2+6\lambda-8=0$$
$$\lambda^2-6\lambda+8=0$$
$$(\lambda-2)(\lambda-4)=0$$

1. Both equations are identical for $\lambda=2$ or $\lambda=4 \Rightarrow$ there are infinitely many solutions.

2. If $\lambda=4$, then $x+y=0$, when $x=t$
   $y=-t$, where $t\in\mathbb{R}$, $t\neq 0$.

3. If $\lambda=2$, then $x-y=0$, when $x=t$
   $y=t$, where $t\in\mathbb{R}$, $t\neq 0$.

4. If $\lambda\neq 2$, and $\lambda\neq 4$, then the system will have trivial solution.

Example:13. For what values of λ the following system of equations

$$x+(\lambda-1)y=0$$
$$(\lambda-1)x+y=0$$

has (a) unique solution, and (b) infinitly many solutions.

Solution:

Augmented form of the homogeneous system

$$\begin{bmatrix} 1 & \lambda-1 & 0 \\ \lambda-1 & 1 & 0 \end{bmatrix}$$

Reducing it to reduced row echlon form,

multiply R_1 by -$(\lambda-1)$and add to R_2

$$\approx\begin{bmatrix} 1 & \lambda-1 & 0 \\ 0 & 1-(\lambda-1)^2 & 0 \end{bmatrix}$$

From R_2, the system will have solution, if

$$\left[1-(\lambda-1)^2\right]y=0 \Rightarrow \lambda(2-\lambda)y=0.$$

If $y=0$, then $x=0$ and system will have unique trivial solution $x=0$, $y=0$.

If $y\neq 0$, then $\lambda(2-\lambda)=0 \Rightarrow \lambda=0$ or $\lambda=2$,

(a) if$\lambda=0$ the $x+y=0$ or $x=-y$, then the system will have infinite many solutions,

(b) if$\lambda=2$ the $x-y=0$ or $x=y$, then the system will have infinite many solutions, and

(c) if $\lambda\neq 0$ or $\lambda\neq 2$ then the system will have unique solution and that is trivail solution $x=0$, $y=0$.

EXERCISE 1.3

Homogenous system of equations

1. Solve the homogenous system of equations by reducing it to reduced row echelon form:

$$x_1 - x_2 + x_3 = 0$$
$$2x_1 - x_2 + 4x_3 = 0$$
$$3x_1 + 2x_2 + 11x_3 = 0$$

2. Solve the homogenous system of equations by reducing it to reduced row echelon form:

$$2x - 9y + 3z + 2w = 0$$
$$x - 4y - w = 0$$
$$2x - 6y - 2z + 5w = 0.$$

3. Solve the homogenous system of equations by reducing it to reduced row echelon form:

$$x + y + z = 0$$
$$x - y + 2z = 0$$
$$3x - y + 5z = 0$$

4. For what values of λ does the following system has non-trivial solution

$$x + 2y + 4z = 0$$
$$3x - 2y + \lambda z = 0$$
$$5x + 3y + z = 0$$

5. For what values of λ does the following system has non-trivial solution

$$3x + y - \lambda z = 0$$
$$4x - 2y - 3z = 0$$
$$2\lambda x + 4y + \lambda z = 0$$

$Ans:\ \lambda = 1, -9$

2.4 Inverse of a 2x2 matrix

Consider a 2x2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

If $ad - bc \neq 0$, then $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Example:3. Find inverse of matrix $A = \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$

$3 \times 5 - 2 \times 4 = 15 - 8 = 7$

$A^{-1} = \frac{1}{7}\begin{bmatrix} 5 & -2 \\ -4 & 3 \end{bmatrix}$.

Properties of Inverse

1. $A^{-1}A = A\,A^{-1} = I$
2. If A and B are invertible matrices of the same size , then AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$

2.5 Power of a matrix

1. $A^0 = I$
2. $A^n = A.A.A\ldots A$ (n-factors), where n>0.
3. $A^{-n} = (A^{-1})^n = A^{-1}.A^{-1}.A^{-1}....A^{-1}$ (n- factors) , where n>0.
4. $A^rA^s = A^{r+s}$
5. $(A^r)^s = A^{rs}$
6. $(A^{-1})^{-1} = A$
7. $(A^n)^{-1} = (A^{-1})^n$, $n = 0,1,2,...$
8. $(kA)^{-1} = \frac{1}{k}A^{-1}$, where k is a scalar.

Example:4. Let A be an invertible matrix and suppose that inverse of 7A is $\begin{bmatrix} -2 & 7 \\ 1 & -3 \end{bmatrix}$, find matrix A

Solution: $(7A)^{-1} = \frac{1}{7}A^{-1} = \begin{bmatrix} -2 & 7 \\ 1 & -3 \end{bmatrix}$

$$A^{-1} = 7\begin{bmatrix} -2 & 7 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} -14 & 49 \\ 7 & -21 \end{bmatrix}$$

$$A = (A^{-1})^{-1} = -\frac{1}{49}\begin{bmatrix} -21 & -49 \\ -7 & -14 \end{bmatrix} = \frac{7}{49}\begin{bmatrix} 3 & 7 \\ 1 & 2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3 & 7 \\ 1 & 2 \end{bmatrix}.$$

Example:5. Let A be a matrix $\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$ compute A^3, A^{-3}, $A^2 - 2A + I$.

Solution:

$$A^2 = AA = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 28 & 1 \end{bmatrix}$$

$$A^{-3} = (A^3)^{-1} = \frac{1}{8}\begin{bmatrix} 1 & 0 \\ -28 & 8 \end{bmatrix}$$

$$A^2 - 2A + I = \begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 8 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix}$$

Example:6. Find inverse of the matrix

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Solution:

$$ad - bc = \cos^2\theta + \sin^2\theta = 1,$$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

2.6 Elementary Matrix

An nxn matrix is called *elementary matrix,* if it can be obtained from nxn identity matrix by performing a single row operation.

Examples: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a 3x3 identity matrix.

Elementary matrices E_1,E_2 and E_3 can be obtained by single row operation.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \; -3R_3$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -2 \end{bmatrix} \; -2R_3+R_2$$

$$E_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \; R_1 \leftrightarrow R_3$$

NOTE:

When a matrix A is multiplied from the left by an elementary matrices E, the effect is same as to perform an elementary row operation on A.

Example: 1.

Let A be a 3x4 matrix, $A=\begin{bmatrix}1 & 0 & 2 & 3\\ 2 & -1 & 3 & 6\\ 1 & 4 & 4 & 0\end{bmatrix}$ and

E be 3x3 elementary matrix obtained by row operation $3R_1+R_3$ from an Identity matrix

$$E=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 3 & 0 & 1\end{bmatrix}$$

$$EA=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 3 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 2 & 3\\ 2 & -1 & 3 & 6\\ 1 & 4 & 4 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 2 & 3\\ 2 & -1 & 3 & 6\\ 4 & 4 & 10 & 9\end{bmatrix}, 3R_1+R_3 .$$

2.7 Method for finding Inverse of a matrix

To find an inverse of matrix A, we perform a sequence of elementary row operations that reduce

$$[A \mid I] \text{ to } [I \mid A^{-1}]$$

Example:2. Find inverse of a matrix $A=\begin{bmatrix}1 & 4\\ 2 & 7\end{bmatrix}$ by using Elementary matrix method.

Solution:

$$[A|I]=\left[\begin{array}{cc|cc}1 & 4 & 1 & 0\\ 2 & 7 & 0 & 1\end{array}\right]$$

$$\approx\left[\begin{array}{cc|cc}1 & 4 & 1 & 0\\ 0 & -1 & -2 & 1\end{array}\right] \quad -2R_1+R_2$$

$$\approx\left[\begin{array}{cc|cc}1 & 4 & 1 & 0\\ 0 & 1 & 2 & -1\end{array}\right] \quad -R_2$$

$$\approx\left[\begin{array}{cc|cc}1 & 0 & -7 & 4\\ 0 & 1 & 2 & -1\end{array}\right] \quad -4R_2+R_1$$

$$=[I|A^{-1}]$$

$$A^{-1}=\begin{bmatrix}-7 & 4\\ 2 & -1\end{bmatrix}$$

Example:4. Use Elementary matrix method to find inverses of

$$A=\begin{bmatrix}3 & 4 & -1\\ 1 & 0 & 3\\ 2 & 5 & -4\end{bmatrix} \qquad \text{if A is invertible.}$$

Solution:

$$[A|I] = \left[\begin{array}{ccc|ccc}3 & 4 & -1 & 1 & 0 & 0\\ 1 & 0 & 3 & 0 & 1 & 0\\ 2 & 5 & -4 & 0 & 0 & 1\end{array}\right]$$

$$\approx \left[\begin{array}{ccc|ccc}1 & 0 & 3 & 0 & 1 & 0\\ 3 & 4 & -1 & 1 & 0 & 0\\ 2 & 5 & -4 & 0 & 0 & 1\end{array}\right] R_1 \leftrightarrow R_2$$

$$\approx \left[\begin{array}{ccc|ccc}1 & 0 & 3 & 0 & 1 & 0\\ 0 & 4 & -10 & 1 & -3 & 0\\ 0 & 5 & -10 & 0 & -2 & 1\end{array}\right] -3R_1 + R_2, -2R_1 + R_3$$

$$\approx \left[\begin{array}{ccc|ccc}1 & 0 & 3 & 0 & 1 & 0\\ 0 & 4 & -10 & 1 & -3 & 0\\ 0 & 1 & 0 & -1 & -2 & 1\end{array}\right] -R_2 + R_3$$

$$\approx \left[\begin{array}{ccc|ccc}1 & 0 & 3 & 0 & 1 & 0\\ 0 & 1 & 0 & -1 & 1 & 1\\ 0 & 0 & -1 & \frac{1}{2} & \frac{-7}{10} & \frac{-2}{5}\end{array}\right] R_2 \leftrightarrow R_3, \frac{(-4R_3 + R_2)}{10}$$

$$\approx \left[\begin{array}{ccc|ccc}1 & 0 & 0 & \frac{3}{2} & \frac{-11}{10} & \frac{-6}{5}\\ 0 & 1 & 0 & -1 & 1 & 1\\ 0 & 0 & 1 & \frac{-1}{2} & \frac{7}{10} & \frac{2}{5}\end{array}\right] -3R_3 + R_1, -R_3$$

$$\approx [I|A^{-1}]$$

$$A^{-1} = \begin{bmatrix}\frac{3}{2} & \frac{-11}{10} & \frac{-6}{5}\\ -1 & 1 & 1\\ \frac{-1}{2} & \frac{7}{10} & \frac{2}{5}\end{bmatrix}.$$

2.8 Solving Linear system by Inverse Matrix

Let a given linear system of equations is

$$AX = B$$

Find A^{-1}

Multiply with A^{-1} from left

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B \text{ is a solution.}$$

Note: To find A^{-1} we use *Elementary Matrix method.*

Example:5.

Write the system of equations in a matrix form, find A^{-1}, use A^{-1} to solve the system

$$x_1 + 3x_2 + x_3 = 4$$
$$2x_1 + 2x_2 + x_3 = -1$$
$$2x_1 + 3x_2 + x_3 = 3$$

Solution: 1. Matrix Form is:

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} \quad \text{is in form of } AX = B$$

2. Find A^{-1} by using Elementary Matrix method

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -4 & -1 & -2 & 1 & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array}\right] \quad R_2 - 2R_1, R_3 - 2R_1$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -4 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -2 & -3 & 4 \end{array}\right] \quad 4R_3 - 3R_2$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & -1 & -3 & 4 \\ 0 & -4 & 0 & 0 & 4 & -4 \\ 0 & 0 & -1 & -2 & -3 & 4 \end{array}\right] \quad R_1 + R_3,\ R_2 - R_3$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & -1 & -3 & 4 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -2 & -3 & 4 \end{array}\right] \quad -\tfrac{1}{4}R_2$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & -1 & -3 & 4 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 3 & -4 \end{array}\right] \quad -R_3$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 3 & -4 \end{array}\right] \quad -3R_2 + R_1$$

$$\equiv \left[I \,|\, A^{-1} \right]$$

$$A^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \end{bmatrix}$$

$$X = A^{-1}B = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -7 \end{bmatrix}$$

Solution set is $x_1 = -1, x_2 = 4, x_3 = -7$.

Example:6.

Write the system of equations in a matrix form, find A^{-1}, use A^{-1} to solve the system

$$x + z = -y + 4$$
$$x + 2z = y + 1$$
$$y + z = 3$$

Solution: 1. Matrix Form is:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} \quad \text{is in form of } AX = B$$

2. Find A^{-1} by using Elementary Matrix method

$$[A|I] \approx \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_2 - R_1$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & -2 & 1 & -1 & 1 & 0 \end{array}\right] \quad R_3 \leftrightarrow R_2$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 3 & -1 & 1 & 2 \end{array}\right] \quad R_1 - R_2,\ 2R_2 + R_3$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{-1}{3} & \frac{1}{3} & \frac{2}{3} \end{array}\right] \quad \frac{1}{3}R_3$$

$$\approx\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{-1}{3} & \frac{1}{3} & \frac{2}{3} \end{array}\right] \quad -R_3 + R_2$$

$$\equiv \left[\, I \,|A^{-1}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$X = A^{-1}B = \begin{bmatrix} 1 & 0 & -1 \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Solution set is $x = 1, y = 2, y = 1$.

EXERCISE 3

SOLUTION OF SYSTEM OF EQUATION BY FINDING INVERSE OF MATRIX BY ELEMENTARY ROW OPERATIONS

1. Use elementary matrix method (using row operations) to find A^{-1} of the matrix

$$A = \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$$

2. Use elementary matrix method to find A^{-1} of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

3. Use elementary matrix method to find A^{-1} of the matrix

$$A = \begin{bmatrix} 2 & 4 & 3 & 2 \\ 3 & 6 & 5 & 2 \\ 2 & 5 & 2 & -3 \\ 4 & 5 & 14 & 14 \end{bmatrix}$$

4. Solve the matrix equation by finding A^{-1}, by using the Elementary Row Operation (Row Reduction)

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix}$$

In problems 3 to 5, write the following system of equations in matrix equation and solve by using row operations (elementary matrix method) for find the inverse of the coefficients matrix

5.

$$\begin{aligned} 2x + 4y + 3z &= 6 \\ y - z &= -4 \\ 3x + 3y + 7z &= 7 \end{aligned}$$

4.

$$\begin{aligned} -1x + z &= 2 \\ -5x + y + 3z &= -2 \\ 7x - y - 4z &= 1 \end{aligned}$$

3.1 Determinant of a Matrix

A number is associated with each matrix that is referred as determinant of matrix A is denoted by $|A|$ or det(A)

3.2 Evaluating determinant by direct multiplication

The determinant of a 2x2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ is}$$

$$\text{Det (A)} = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix} \qquad a_{11}a_{22} - a_{21}a_{12}$$

The determinant of 3x3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ is}$$

$$\det \text{(A)} = \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{matrix} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{23} + a_{13}a_{21}a_{32} \\ - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12} \end{matrix}$$

Note: This method does not work for 4 x 4 or higher order.

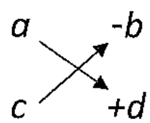
3.1 Determinant of a matrix

A number us associated with each matrix A that is referred as determinant of matrix A and is denoted by | A | or det (A)

3.2 Evaluating determinant of Matrix

For 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$



$$|A| = \det A = ad - cb$$

For 3x3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det A = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - (a_{12}a_{21}a_{33} - a_{12}a_{31}a_{23}) + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}$$

Example .1: Find the determinant of matrics

$$A=\begin{bmatrix}2 & 3\\5 & 9\end{bmatrix}, \quad B=\begin{bmatrix}2 & 4 & 5\\3 & 6 & 8\\4 & 5 & 9\end{bmatrix}$$

Solution: Determinant of 2x2 matrix

$$A=\begin{bmatrix}2 & 3\\5 & 9\end{bmatrix} \qquad \det A = 2x9-3x5=18-15=3.$$

Determinant of 3x3 matrix

$$B=\begin{bmatrix}2 & 4 & 5\\3 & 6 & 8\\4 & 5 & 9\end{bmatrix}$$

$$\begin{aligned}\det(B) &= 2\begin{vmatrix}6 & 8\\5 & 9\end{vmatrix}-4\begin{vmatrix}3 & 8\\4 & 9\end{vmatrix}+5\begin{vmatrix}3 & 6\\4 & 5\end{vmatrix}\\ &=2(54-40)-4(27-32)+5(15-24)\\ &=2(14)-4(-5)+5(-9)\\ &=28+20-45\\ &=48-45\\ &=3\end{aligned}$$

Note: We can write it as det(A) or $|A|$ or $\begin{vmatrix}2 & 3\\5 & 9\end{vmatrix}$ or $\det\begin{bmatrix}2 & 3\\5 & 9\end{bmatrix}$

3.3 Finding determinant by method of co-factors

Minor The minor of an element of a matrix a_{ij} of a matrix A, denoted by M_{ij}, is the determinant of the matrix obtained by deleting the row and column containing a_{ij}.

$$\text{Example: } A=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{bmatrix}$$

The minor M_{23} of the element a_{23} of matrix

$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is the determinant of 2x2 matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$. Thus

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}a_{32} - a_{31}a_{12}.$$

Cofactor of an element a_{ij} of a matrix A , denoted by C_{ij}, is defined as

$$C_{ij} = (-1)^{i+j} M_{ij}, \text{ where } M_{ij} \text{ is minor of the element } a_{ij}.$$

NOTE: $C_{ij} = M_{ij}$ when $i + j$ is even, and $C_{ij} = - M_{ij}$ when $i + j$ is odd.

Determinant of matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ by method of cofactor is

$$Det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

NOTE: Determinant of a matrix can be obtained by expanding about any column or row.

Finding determinant by expanding it through second row:

$$Det(A) = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

Example:2. Find the determinant of matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 4 & 3 & 1 \end{bmatrix}$$

Solution:

$$\begin{aligned} Det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= (1)\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} - (2)\begin{vmatrix} 2 & 2 \\ 4 & 1 \end{vmatrix} + (3)\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} \\ &= (1)(-5) - (2)(-6) + (3)(2) \\ &= -5 + 12 + 6 \\ &= 13 \end{aligned}$$

Example:3. Find determinant of matrix if $A = \begin{bmatrix} 0 & 1 & 2 & 5 \\ 2 & -1 & 2 & 3 \\ 3 & 2 & 1 & 5 \\ 1 & 0 & 4 & 0 \end{bmatrix}$

Solution: Expanding from 4^{th} row

$$\text{Det}(A) = -(1)\begin{vmatrix} 1 & 2 & 5 \\ -1 & 2 & 3 \\ 2 & 1 & 5 \end{vmatrix} + (0)\begin{vmatrix} 0 & 2 & 5 \\ 2 & 2 & 3 \\ 3 & 1 & 5 \end{vmatrix} - (4)\begin{vmatrix} 0 & 1 & 5 \\ 2 & -1 & 3 \\ 3 & 2 & 5 \end{vmatrix} + (0)\begin{vmatrix} 0 & 1 & 2 \\ 2 & -1 & 2 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= -(1)(4) + (0)(?) - (4)(34) + (0)(?)$$

$$= -4 - 136 = -140.$$

Example : 4. Find all values of λ for which $\det(A) = 0$ for matrix

$$A = \begin{bmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda - 1 \end{bmatrix}$$

Solution: $\det(A) = (\lambda-4)\begin{vmatrix} \lambda & 2 \\ 3 & \lambda-1 \end{vmatrix} - (0)\begin{vmatrix} 0 & 2 \\ 0 & \lambda-1 \end{vmatrix} + (0)\begin{vmatrix} 0 & \lambda \\ 0 & 3 \end{vmatrix}$

$= (\lambda-4)\,[\lambda(\lambda-1) - 6]$

$= (\lambda-4)\,[\lambda^2 - \lambda - 6]$

$= (\lambda-4)(\lambda-3)(\lambda+2)$

$\det(A) = 0.$

$(\lambda-4)(\lambda-3)(\lambda+2) = 0.$

$\Rightarrow \lambda = 4,\ \lambda = 3,\ \lambda = -2.$

3.4 Evaluating Determinant by row operations

1. If matrix A_1 is obtained from matrix A by the interchange of two rows , then **$\det(A_1) = -\det(A)$.**

2. If matrix A_2 is obtained from matrix A by the multiplication of a row of A by a constant k , then **$\det(A_2) = k\det(A)$.**

3. If matrix A_3 is obtained from the matrix A by addition of a multiple of one row to another row , then **$\det(A_3) = \det(A)$.**

Example:5. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 8 \end{bmatrix}$, and $\det(A) = 2$. Find determinant of

(i) $A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 8 \\ 0 & 1 & 2 \end{bmatrix}$, (ii) $A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$, (iii) $A_3 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$.

Solution: (i) A_1 is obtained from A by interchanging R_2 and R_3 of A,

$\det(A_1) = -\det(A) = -2.$

(ii) A_2 is obtained from A by multiplying R_3 of A by $\frac{1}{2}$,

$\det(A_2) = \frac{1}{2}\det(A) = \frac{1}{2}(2) = 1.$

(iii) A_3 is obtained by row operation $-2R_2+R_1$,

$\det(A_3) = \det(A) = 2.$

NOTE:

1. If A is any square matrix that contains a row of zeros, then $\det(A) = 0$.
2. If a square matrix has two proportional rows, then $\det(A) = 0$.
3. In case of upper or lower triangular matrix, determinant is the product of the diagonal elements.

Lower **triangular matrix**

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \det(A) = a_{11}a_{22}a_{33}$$

Upper **triangular matrix**

$$B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}, \quad \det(B) = a_{11}a_{22}a_{33}$$

Diagonal matrix

$$D = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

Example:6.

Given that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 6$, find (a) $\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix}$, (b) $\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix}$

(c) $\begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix}$, (d) $\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix}$.

Solution:

(a) $\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix} = -\underset{R_1 \leftrightarrow R_3}{\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}} = (-1)(-1)\underset{R_2 \leftrightarrow R_3}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} = (-1)(-1)(6) = 6$

(b) $\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix} = (3)(-1)(4)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (-12)(6) = -72$

(c) $\begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix} = \underset{R_1 - R_3}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} = 6$

(d) $\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix} = (-3)\underset{4R_2 + R_3}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} = (-3)(6) = -18$

Example:7. Evaluate the determinant by row reduction

$$\text{Det A} = \begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ -2 & -7 & 0 & -4 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}$$

Solution:

$$\det A = \begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & -1 & 2 & 6 & 8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} \qquad 2R_1 + R_2, -2R_3 + R_4$$

$$= \begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & -1 & 2 & 6 & 8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix} \qquad -R_4 + R_5$$

$$= (1)(-1)(1)(1)(2) = -2$$

Example:8. Find the value(s) of x if det A = -12, where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & x-3 & -3 \\ 1 & x-4 & 0 \end{bmatrix}$$

Solution: Performing row operations $-2R_1+R_2$, $-R_1+R_3$

$$\det A = \begin{vmatrix} 1 & 0 & 0 \\ 0 & x-3 & -3 \\ 0 & x-4 & 0 \end{vmatrix} = (1)\begin{vmatrix} x-3 & -3 \\ x-4 & 0 \end{vmatrix} - (0) + (0)$$

$$= 3(x-4)$$

$$\det A = -12 \quad \Rightarrow -3x - 12 = -12$$

$$-3x = 0$$

$$x = 0.//$$

NOTE: Operations on columns are same as on rows.

Theorem:

For an nxn matrix A, following are equivalent:

1. $\det(A) \neq 0$,
2. A^{-1} exists, and
3. $AX = B$ has a unique solution for any B.
4. A is invertible

3.5 Properties of Determinantial Function

1. **If A is a nxn matric $\det(kA) = k^n \det(A)$,**
2. **$\det(A+B) \neq \det(A) + \det(B)$,**
3. **$\det(AB) = \det(A).\det(B)$,**
4. $\det(A^{-1}) = \dfrac{1}{\det A}$,
5. **A square matrix is invertible if and only if $\det(A) \neq 0$,and**
6. **$\det(A^t) = \det(A)$**

7. If $\det A = 0$, then matrix A is singular matrix.
8. $AX = 0$, will have non-trivial solution iff $\det A = 0$

Example :9. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and det(A) = -7 find

(a) det (3A) , (b) $\det(2A)^{-1}$, (c) $\det(2A^{-1})$ and (d) $\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix}$

Solution: a. $\det(3A) = 3^3 \det A = 27(-7) = -189$

b. $\det(2A)^{-1} = \dfrac{1}{\det(2A)} = \dfrac{1}{2^3\det(A)} = \dfrac{1}{8(-7)} = \dfrac{-1}{56}$

c. $\det(2A^{-1}) = 2^3\det(A) = \dfrac{2^3}{\det(A)} = \dfrac{8}{-7} = \dfrac{-8}{7}$

d. $\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix}^t = \begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -(-7) = 7$

Example:10. Use row reduction to show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$$

Solution.

det(A) = det (A^t)

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$$

$R_2 - R_1, R_3 - R_1$

$$= (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix}$$

$R_3 - R_2$

$$= (b-c)(c-a)(c-b)$$

Example:11. Without directly evaluating by using properties of determinant show that

$$\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Solution:

$$\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} R_1 + R_2$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 0.$$

EXERCISE 4

DETERMINANTS AND PROPERTIES OF DETERMINANT

In problems 1 to 8 evaluate the determinant by the definition of a determinant or by direct multiplication.

1. $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$

2. $\begin{vmatrix} 5 & -6 \\ 7 & 2 \end{vmatrix}$

3. $\begin{vmatrix} 1 & 6 \\ 5 & 4 \end{vmatrix}$

4. $\begin{vmatrix} 3 & 6 \\ 4 & 8 \end{vmatrix}$

5. $\begin{vmatrix} 2 & -5 & -1 \\ -1 & 6 & 1 \\ -1 & -3 & -3 \end{vmatrix}$

6. $\begin{vmatrix} 3 & 1 & 2 \\ 1 & 2 & -4 \\ 4 & 3 & 6 \end{vmatrix}$

7. $\begin{vmatrix} 1 & 3 & 0 \\ 2 & 4 & 1 \\ -6 & 7 & 2 \end{vmatrix}$

8. $\begin{vmatrix} 1 & 5 & -1 \\ 4 & 1 & 0 \\ 2 & -3 & 0 \end{vmatrix}$

In problems 9 to 16 evaluate the determinant by using row operations to introduce zeros,

9. $\begin{vmatrix} 8 & -1 & -1 \\ -1 & 7 & -2 \\ 1 & 1 & -6 \end{vmatrix}$

10. $\begin{vmatrix} 2 & 1 & 3 & 4 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 2 \\ 4 & 2 & 1 & 1 \end{vmatrix}$

11. $\begin{vmatrix} 2 & 3 & -3 & 8 \\ 4 & 0 & 1 & 5 \\ 9 & 0 & -5 & 9 \\ 2 & 0 & 12 & 8 \end{vmatrix}$

12. $\begin{vmatrix} 3 & 1 & 2 & 6 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 5 & 10 \\ 1 & 2 & 1 & 3 \end{vmatrix}$

13. $\begin{vmatrix} 2 & 4 & 6 & 8 \\ 1 & 3 & 4 & 5 \\ 2 & 5 & 6 & 7 \\ 3 & 2 & 4 & 5 \end{vmatrix}$

14. $\begin{vmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix}$

15. $\begin{vmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 1 \\ 0 & 3 & 8 & 1 \\ 0 & 2 & 3 & 4 \end{vmatrix}$

16. $\begin{vmatrix} 1 & 4 & 1 & 2 & 3 \\ -3 & -4 & -3 & -5 & -8 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & -3 & -4 & 1 \\ 0 & 0 & 0 & 3 & 6 \end{vmatrix}$

17. If $A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 4 \\ 5 & -2 & -3 \end{bmatrix}$, find (i) det A (ii) det A^T

18. Find values of λ the determinant of the matrix

$$\begin{bmatrix} \lambda^2 & 4 & 1 \\ -4 & -\lambda & 2 \\ 6 & 3 & \lambda^2 \end{bmatrix}, \text{ if the inverse of matrix } \begin{bmatrix} \lambda^2 & 1 \\ 1 & \lambda \end{bmatrix}$$

does not exist.

19. For what values of λ, A is not invertible.

(i) $A = \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda-1 & 0 \\ -1 & 0 & \lambda-4 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 0 & \lambda \end{bmatrix}$

20. Find the values of λ for which the following matrix does not have an inverse

(i) $\begin{bmatrix} 1 & \lambda-2 & 0 \\ 1 & 2\lambda & 0 \\ 0 & 2 & \lambda \end{bmatrix}$

(ii) $\begin{bmatrix} -\lambda & \lambda-1 & \lambda+1 \\ 1 & 2 & 3 \\ 2-\lambda & \lambda+3 & \lambda+7 \end{bmatrix}$

20. Find the values of λ for which an inverse of the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2\lambda & 0 \\ 0 & 2 & \lambda \end{bmatrix} \text{ exists.}$$

21. Find the value of x if det A = -12, where

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & x-3 & 1 \\ 1 & x-4 & 2 \end{bmatrix}$$

22. Solve the determinantal equation

$$\begin{vmatrix} x & -3 & 2 \\ 2 & x & -3 \\ -3 & 2 & x \end{vmatrix} = -19$$

23. Use properties of the determinant to evaluate

$$\det(2A) + \det(2A^{-1}) + 2\det A \cdot \det A^{-1},$$

$$\text{if } A = \begin{bmatrix} 1 & -2 & 3 \\ -1 & -3 & 2 \\ 0 & 0 & 4 \end{bmatrix}.$$

24. Use properties of the determinant to prove

$$\begin{bmatrix} 2x_1+2y & 2x_1-2y_1 & z_1 \\ 2x_2+2y_2 & 2x_2-2y_2 & z_2 \\ 2x_3+2y_3 & 2x_3-2y_3 & z_2 \end{bmatrix} = -8\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}$$

25. Show that inverse of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ c+b & c+a & a+b \end{bmatrix}$ does not exist.

26. If $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0,$

then prove that $xyz = -1$, where *x, y, z* are unequal.

27. Prove that $\det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = (a-b)(b-c)(c-a)$

28. Use properties of the determinant to evaluate

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} & \frac{1}{d} \\ a & b & c & d \\ bcd & acd & abd & abc \end{vmatrix}$$

29. Use properties of the determinant to show that

$$\begin{vmatrix} a-3b & r-3s & x-3y \\ b-2c & s-2t & y-2z \\ 5c & 5t & 5z \end{vmatrix} = 5\begin{vmatrix} a & r & x \\ b & s & y \\ c & t & z \end{vmatrix}$$

30. Find the value of $\det[(2A)^2 (A^{-1})^2]$, where

$$A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} \text{ and } \quad \det A = 3.$$

31. Without expanding the determinant show that

$$\det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{bmatrix} = (b-c)(c-a)(c-b)(a+b+c)$$

32. Find the values of x for which the matrix

$$\begin{bmatrix} 2x-1 & 2-x & x \\ x & 1 & x \\ 2x & 1+x & 1+x \end{bmatrix} \text{ has no inverse.}$$

33. For what values of x matrix A is singular

$$A = \begin{bmatrix} x & -1 & 0 \\ 0 & x-3 & -2 \\ 2 & 1 & x+2 \end{bmatrix}$$

34. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and detA = 5, find

(a) $\det[(2A)^{-1}]$ (b) $\det(2A)$

(c) $\det(3A^{-1}) + \det[(3A)^{-1}] + \det(2A)$

(d) $\begin{bmatrix} -a & -b & -c \\ 3a+d & 3b+e & 3c+f \\ 4g & 4h & 4i \end{bmatrix}$

35. If $\begin{vmatrix} a & b & c \\ x & y & z \\ l & m & n \end{vmatrix} = 3$ evaluate $\begin{vmatrix} a+x & b+y & c+z \\ l-x & m-y & n-z \\ a+x+l & b+y+m & c+z+n \end{vmatrix}$

36. Find the matrix A if $(5A)^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

37. If A is 3x3 matrix and if $\det(A) = 10$, find

(a) $\det(7A)$ (b) $\det(5A^{-1})$

38. Find det (A) if (i) $\det(A^{-1}) = 5$
(iii) $\det(2A) = 25$ where A is 3x3 matrix.

39. Find non – trivial solutions, if any, of

$(-4I_3 - A)\,x = 0$ where $A = \begin{bmatrix} 1 & 0 & 5 \\ 1 & 1 & 1 \\ 0 & 1 & -4 \end{bmatrix}$ and $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

40. Find the values of " a" for which an inverse of the matrix A exists

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & a^2 \end{bmatrix}$$

41. Find the values of "a" for which the matrix A does not have an inverse

(i) $$A = \begin{bmatrix} 1 & a-2 & 0 \\ 1 & 2-a & 0 \\ 0 & 2 & a \end{bmatrix}$$

(ii) $$A = \begin{bmatrix} -a & a-1 & a+1 \\ 1 & 2 & 3 \\ 2-a & a+3 & a+7 \end{bmatrix}$$

42. For what values of "λ" , matrix A is not invertible

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 0 & \lambda \end{bmatrix}$$

43. **If** $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ **and show that** $\det(A^{-1}) = \dfrac{1}{\det(A)}$

44. Solve by inspection

(i)

$$\begin{vmatrix} x & 5 & 7 \\ 0 & x+1 & 6 \\ 0 & 0 & 2x+1 \end{vmatrix} = 0$$

$$Ans: x = 0, x = -1, x = -\frac{1}{2}$$

(ii)

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & 1 & 1 \\ 1 & -3 & 9 \end{vmatrix} = 0$$

$$Ans: x = 1, x = -3$$

45. Solve $$\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix} = 0$$

Ans: $x = -1$, $x = -1$, $x = -2$

3.5 Minors and cofactors of a Matrix

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Minor of $a_{ij} \equiv M_{ij}$**,**

is determinant obtained by deleting ith row and jth column.

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \text{ is determinant obtained by deleting 1st row and 1st column}$$

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}, M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Cofactor of $a_{ij} = C_{ij} = (-1)^{i+j} M_{ij}$

Signs of Cofactors

$$\text{For 2x2 – matrix } \begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

$$\text{For 3x3 – matrix } \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\text{For 4x4 – matrix } \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

Example:1. Find all minors and cofactors of the matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$$

Solution:

$$M_{11} = \begin{vmatrix} 0 & 3 \\ 5 & -4 \end{vmatrix} = -15, M_{12} = \begin{vmatrix} 1 & 3 \\ 2 & -4 \end{vmatrix} = -10, M_{13} = \begin{vmatrix} 1 & 0 \\ 2 & -5 \end{vmatrix} = 5$$

$$M_{21} = \begin{vmatrix} 4 & -1 \\ 5 & -4 \end{vmatrix} = -11, M_{22} = \begin{vmatrix} 3 & -1 \\ 2 & -4 \end{vmatrix} = -10, M_{23} = \begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix} = 7$$

$$M_{31} = \begin{vmatrix} 4 & -1 \\ 0 & 3 \end{vmatrix} = 12, \quad M_{32} = \begin{vmatrix} 3 & -1 \\ 1 & 0 \end{vmatrix} = 10, \quad M_{33} = \begin{vmatrix} 3 & 4 \\ 1 & 0 \end{vmatrix} = -4$$

Cofactor of $a_{ij} = C_{ij} = (-1)^{i+j} M_{ij}$

$$\begin{aligned} &C_{11} = -15, \quad C_{12} = 10, \quad C_{13} = 5 \\ &C_{21} = 11, \quad C_{22} = -10, \quad C_{23} = -7 \\ &C_{31} = 12, \quad C_{32} = -10, \quad C_{33} = -4 \end{aligned}$$

NOTE: Matrix of cofactors , $C = \begin{bmatrix} -15 & 10 & 5 \\ 11 & -10 & -7 \\ 12 & -10 & -4 \end{bmatrix}$

<u>NOTE: Determinant of matrix of Cofactors by the method of Cofactors</u>

$$\begin{aligned} \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ \det(A) &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \\ \det(A) &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \end{aligned}$$

The above equations can be used to check that the cofactors are found correctly as the values of determinants found must be equal, we open matrix from any row or column.

Example: 2 . Find the determinant of the matrix A by method of cofactors,

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$$

Solution: **: Using the cofactors found in the last example**

$$\begin{aligned}\det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= 3(-15)+4(10)+(-1)(5) \\ &= -45 + 40 - 5 = -10\end{aligned}$$

NOTE: 3. We can find determinant by opening matrix from second or third row or first column, the value of the determinant will be same

$$\begin{aligned}\det(A) &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \\ &= (1)(11)+0(-10)+3(-7)=11-21=-10\end{aligned}$$

$$\begin{aligned}\det(A) &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \\ &= 2(12)+5(-10)+(-4)(-4)=24-50+16=-10\end{aligned}$$

NOTE : 4. Determinant of A can be obtained by multiplying any row or any column of matrix A with the corresponding row or column of the matrix of cofactors.

NOTE: 5. Determinant of matrix A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\textbf{det A} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

3.7 . Inverse by method of Cofactors:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \det A \neq 0.$$

Step:1. **Find Matrix of cofactors**

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Step : 2. **Find Adjoint of matrix A , adj(A)**

$$Adj(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

Step: 3.

If A is an invertible matrix, det(A) ≠ 0, then

$$A^{-1} = \frac{1}{\det A}[adj(A)]$$

Example: 3 . Find A^{-1} of matrix A

$$A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}$$ by the method of cofactors.

Solution: Cofactors of the matrix A are

$$C_{11} = \begin{vmatrix} 3 & 2 \\ 0 & -4 \end{vmatrix} = -12, C_{12} = -\begin{vmatrix} 0 & 2 \\ -2 & -4 \end{vmatrix} = -4, C_{13} = \begin{vmatrix} 0 & 3 \\ -2 & 0 \end{vmatrix} = 6$$

$$C_{21} = -\begin{vmatrix} 0 & 3 \\ 0 & 4 \end{vmatrix} = 0, \quad C_{22} = \begin{vmatrix} 2 & 3 \\ -2 & -4 \end{vmatrix} = -2, \; C_{23} = -\begin{vmatrix} 2 & 0 \\ -2 & 0 \end{vmatrix} = 0,$$

$$C_{31} = \begin{vmatrix} 0 & 3 \\ 3 & 2 \end{vmatrix} = -9, \quad C_{32} = -\begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} = -4, \quad C_{33} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$$

Matrix of cofactors, $C = \begin{bmatrix} -12 & -4 & 6 \\ 0 & -2 & 0 \\ -9 & -4 & 6 \end{bmatrix}$

Adjoint of matrix A, $\operatorname{adj}(A) = \begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$

$$\begin{aligned} \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= 2(-12)+0(-4)+3(6) \\ &= -24 + 18 = -6 \neq 0 \end{aligned}$$

Inverse of matrix A is

$$A^{-1} = \frac{1}{\det A}[adj(A)] = \frac{1}{-6}\begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$$

NOTE :

If we can find A^{-1}, then solution of linear system $AX = B$ is $X = A^{-1}B$

EXERCISE 5

FINDING INVERSE OF A MATRIX and SOLUTION OF SYSYTEM OF EQUATIONS

1. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$ be a matrix. Find adjA and hence find A^{-1} if exists.

2. Let $A = \begin{bmatrix} 2 & 2 & 0 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ be a matrix. Find adjA and hence find A^{-1} if exists.

3. Consider the following system of equations

$$\begin{aligned} x - y + 3z &= 8 \\ 4x + y + 6z &= 24 \\ x - z &= -2 \end{aligned}$$

(a) write the system in the form AX = B,
(b) Find A^{-1},if exists,
(c) Use (a) and (b) to solve the given system.

4. Consider the following system of equations

$$\begin{aligned} x - z &= 2 \\ y + z &= 3 \\ 3x - 2z &= 5 \end{aligned}$$

(a) write the system in the form AX = B,
(b) Find A^{-1}, if exists,
(c) Use (a) and (b) to solve the given system.

5. Let A be the coefficient matrix of the following system of equations

$$\begin{aligned} 2x - 3y + 4z &= -19 \\ 3x + 2y - z &= 4 \\ x + 5y + 4z &= -23 \end{aligned}$$

(a) find adj(A)
(b) Use adj (A) to find A^{-1} , if exists,
(c) Use A^{-1} to solve the given system.

6. Solve the system of equations by finding A^{-1} by method of cofactors:

(ii)
$$\begin{aligned} x + 2y + 8z &= 5 \\ -x - y &= 1 \\ x + 2y + 7z &= 4 \end{aligned}$$

3.8 Cramer's Rule

If A is nxn marix with $\det(A) \neq 0$, then the linear system $AX = B$ has a unique solution $X = (x_j)$ given by

$$x_j = \frac{\det(A_j)}{\det(A)} \quad , j = 1, 2, \ldots, n$$

Where A_j is the matrix obtained by replacing the jth column of A by B.

NOTE: If A is 3x3 matrix , then the solution of the system $AX = B$ is

$$x = \frac{\det(A_1)}{\det(A)}, \quad y = \frac{\det(A_2)}{\det(A)}, \quad z = \frac{\det(A_{31})}{\det(A)}$$

Example: Use Cramer's Rule to solve

$$\begin{aligned} 4x + 5y \quad &= 2 \\ 11x + y + 2z &= 3 \\ x + 5y + 2z &= 1 \end{aligned}$$

Solution: $A = \begin{bmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{bmatrix}$, $A_1 = \begin{bmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{bmatrix}$; $A_2 = \begin{bmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$, $A_3 = \begin{bmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{bmatrix}$

$\det(A) = -132$, $\det(A_1) = -36$, $\det(A_2) = -24$, $\det(A_3) = 12$

$$x = \frac{\det(A_1)}{\det(A)} = \frac{-36}{-132} = \frac{3}{11},$$

$$y = \frac{\det(A_2)}{\det(A)} = \frac{-24}{-132} = \frac{2}{11},$$

$$z = \frac{\det(A_{31})}{\det(A)} = \frac{12}{-132} = \frac{-1}{11}$$

NOTE: when $\det(A) = 0$, then there does not exist any solution of the system.

EXERCISE 6

Cramer's Rule

In problem 1 to 9 , use Cramer's Rule to solve the system of linear equations:

1. $$\begin{aligned} 2x + 2y &= 2 \\ 3x - y &= -6 \end{aligned}$$

2. $$\begin{aligned} 3x + 5y &= 21 \\ 2x + 3y &= 12 \end{aligned}$$

3. $$\begin{aligned} x + 2y + 3z &= 1 \\ 2x + 5y + 3z &= -2 \\ x + 8z &= 8 \end{aligned}$$ Ans: $x = 0, y = -1, z = 1$

4. $$\begin{aligned} 2x + 2y &= 1 \\ -2x + y + z &= 0 \\ 3x + z &= 1 \end{aligned}$$ Ans: $x = \frac{1}{4}, y = \frac{1}{4}, z = \frac{1}{4}$

5. $$\begin{aligned} 3x + 2y + z &= 7 \\ 2x - y + 3z &= 3 \\ 5x + 4y - 2z &= 1 \end{aligned}$$ Ans: $x = \frac{13}{7}, y = \frac{31}{7}, z = \frac{26}{7}$

6. $$\begin{aligned} 2x - 3y + 4z &= -19 \\ 3x + 2y - z &= 4 \\ x + 5y + 4z &= 23 \end{aligned}$$

7. $$\begin{aligned} 4y + 3z &= -2 \\ -2x + 5y - z &= 1 \\ 3x + 4y + 5z &= 6 \end{aligned}$$ Ans: $x = \frac{180}{41}, y = \frac{59}{41}, z = -\frac{106}{41}$

8. $$\begin{aligned} x + y + 2z &= 0 \\ -2x + 5y + 2z &= 1 \\ 3x + y + 4z &= -1 \end{aligned}$$ Ans: $x = -3, y = -2, z = \frac{5}{2}$

9. $$\begin{aligned} x - y + z + w &= -2 \\ -x + 3y + 2w &= 0 \\ y + z - w &= 1 \\ x + z - w &= 1 \end{aligned}$$