CHAPTER 5: Probabilistic Features of the Distributions of Certain Sample Statistics

5.1 Introduction:

In this Chapter we will discuss the probability distributions of some statistics.

As we mention earlier, a statistic is measure computed form the random sample. As the sample values vary from sample to sample, the value of the statistic varies accordingly.

A statistic is a random variable; it has a probability distribution, a mean and a variance.

5.2 Sampling Distribution:

The probability distribution of a statistic is called the sampling distribution of that statistic.

The sampling distribution of the statistic is used to make statistical inference about the unknown parameter.

5.3 Distribution of the Sample Mean: (Sampling Distribution of the Sample Mean $\overline{X}$):

Suppose that we have a population with mean μ and variance σ^2. Suppose that $X_1, X_2, \ldots, X_n$ is a random sample of size (n) selected randomly from this population. We know that the sample mean is:

$$\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}.$$

Suppose that we select several random samples of size n=5.

	1st sample	2nd sample	3rd sample	…	Last sample
Sample values	28 30 34 34 17	31 20 31 40 28	14 31 25 27 32	. . . .	17 32 29 31 30
Sample mean $\overline{X}$	28.4	29.9	25.8	…	27.8

- The value of the sample mean $\overline{X}$ varies from random sample to another.
- The value of $\overline{X}$ is random and it depends on the random sample.
- The sample mean $\overline{X}$ is a random variable.
- The probability distribution of $\overline{X}$ is called the sampling distribution of the sample mean $\overline{X}$.
- Questions:
  - What is the sampling distribution of the sample mean $\overline{X}$?
  - What is the mean of the sample mean $\overline{X}$?
  - What is the variance of the sample mean $\overline{X}$?

Some Results about Sampling Distribution of $\overline{X}$:

Result (1): (mean & variance of $\overline{X}$)

If $X_1, X_2, \ldots, X_n$ is a random sample of size n from any distribution with mean μ and variance σ^2; then:

1. The mean of $\overline{X}$ is: $\mu_{\overline{X}} = \mu$.
2. The variance of $\overline{X}$ is: $\sigma^2_{\overline{X}} = \dfrac{\sigma^2}{n}$.
3. The Standard deviation of $\overline{X}$ is call the standard error and is defined by: $\sigma_{\overline{X}} = \sqrt{\sigma^2_{\overline{X}}} = \dfrac{\sigma}{\sqrt{n}}$.

Result (2): (Sampling from normal population)

If $X_1, X_2, \ldots, X_n$ is a random sample of size n from a normal population with mean μ and variance σ^2; that is Normal(μ, σ^2), then the sample mean has a normal distribution with mean μ and variance σ^2 / n, that is:

1. $\overline{X} \sim$ Normal $\left(\mu, \dfrac{\sigma^2}{n}\right)$.
2. $Z = \dfrac{\overline{X} - \mu}{\sigma / \sqrt{n}} \sim$ Normal (0,1).

We use this result when sampling from normal distribution with known variance σ^2.

Result (3): (Central Limit Theorem: Sampling from Non-normal population)

Suppose that $X_1, X_2, \ldots, X_n$ is a random sample of size n from non-normal population with mean μ and variance σ^2. If the sample size n is large $(n \geq 30)$, then the sample mean has approximately a normal distribution with mean μ and variance σ^2 / n, that is

1. $\overline{X} \approx$ Normal $\left(\mu, \frac{\sigma^2}{n}\right)$ (approximately)
2. $Z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}} \approx$ Normal $(0,1)$ (approximately)

Note: "$\approx$" means "approximately distributed".
We use this result when sampling from non-normal distribution with known variance σ^2 and with large sample size.

Result (4): (used when σ^2 is unknown + normal distribution)

If $X_1, X_2, \ldots, X_n$ is a random sample of size n from a normal distribution with mean μ and unknown variance σ^2; that is Normal(μ, σ^2), then the statistic:

$$T = \frac{\overline{X} - \mu}{S / \sqrt{n}}$$

has a t- distribution with $(n-1)$ degrees of freedom, where S is the sample standard deviation given by:

$$S = \sqrt{S^2} = \sqrt{\frac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{n-1}}$$

We write:

$$T = \frac{\overline{X} - \mu}{S / \sqrt{n}} \sim t(n-1)$$

Notation: degrees of freedom = df = ν

The t-Distribution: (Section 6.3. pp 172-174)

- Student's t distribution.
- t-distribution is a distribution of a continuous random variable.
- Recall that, if $X_1, X_2, \ldots, X_n$ is a random sample of size n from a normal distribution with mean μ and variance σ^2, i.e. $N(\mu,\sigma^2)$, then

$$Z = \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

We can apply this result only when σ^2 is known!

- If σ^2 is unknown, we replace the population variance σ^2 with the sample variance $S^2 = \frac{\sum_{i=1}^{n}(X_i-\overline{X})^2}{n-1}$ to have the following statistic

$$T = \frac{\overline{X}-\mu}{S/\sqrt{n}}$$

Recall:

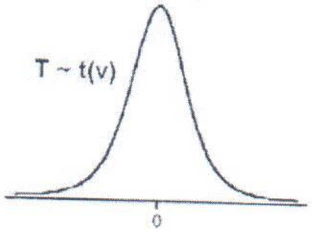
If $X_1, X_2, \ldots, X_n$ is a random sample of size n from a normal distribution with mean μ and variance σ^2, i.e. $N(\mu,\sigma^2)$, then the statistic:

$$T = \frac{\overline{X}-\mu}{S/\sqrt{n}}$$

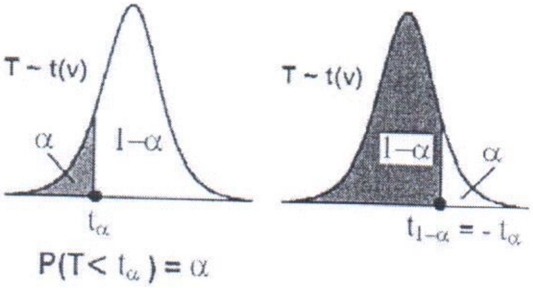
has a t-distribution with $(n-1)$ degrees of freedom ($df = \nu = n-1$), and we write $T \sim t(\nu)$ or $T \sim t(n-1)$.

Note:

- t-distribution is a continuous distribution.
- The value of t random variable range from $-\infty$ to $+\infty$ (that is, $-\infty<t<\infty$).
- The mean of t distribution is 0.
- It is symmetric about the mean 0.
- The shape of t-distribution is similar to the shape of the standard normal distribution.
- t-distribution → Standard normal distribution as $n \to \infty$.



Notation: (t_α)



- t_α = The t-value under which we find an area equal to α
  = The t-value that leaves an area of α to the left.
- The value t_α satisfies: $P(T< t_\alpha) = \alpha$.
- Since the curve of the pdf of T~ t(ν) is symmetric about 0, we have

$$t_{1-\alpha} = -t_\alpha$$

For example: $t_{0.35} = -t_{1-0.35} = -t_{0.65}$

$t_{0.82} = -t_{1-0.86} = -t_{0.14}$

- Values of t_α are tabulated in a special table for several values of α and several values of degrees of freedom. (Table E, appendix p. A-40 in the textbook).

Example:

Find the t-value with ν=14 (df) that leaves an area of:

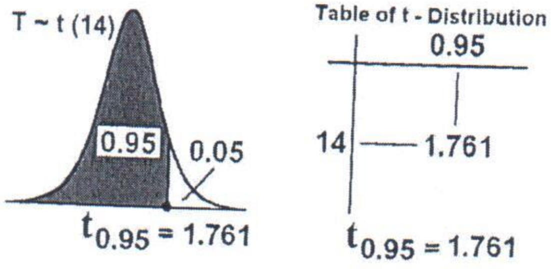
(a) 0.95 to the left.

(b) 0.95 to the right.

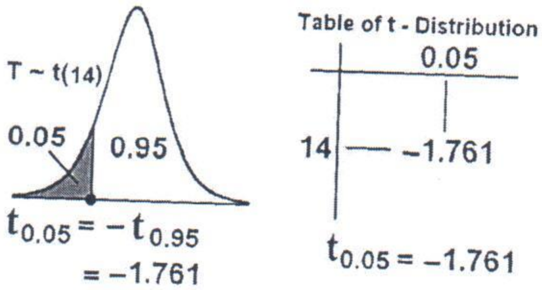
Solution:

ν = 14 (df); T~ t(14)

(a) The t-value that leaves an area of 0.95 to the left is $t_{0.95} = 1.761$.



(b) The t-value that leaves an area of 0.95 to the right is
$t_{0.05} = -\,t_{1-0.05} = -\,t_{0.95} = -\,1.761$



Note: Some t-tables contain values of α that are greater than or equal to 0.90. When we search for small values of α in these tables, we may use the fact that:

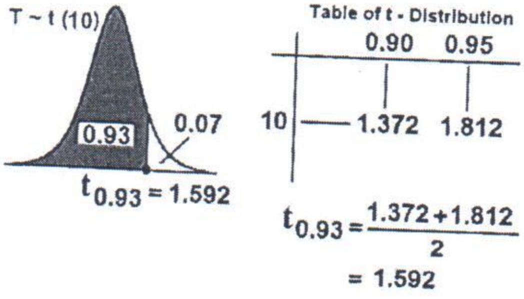
$$t_{1-\alpha} = -\,t_{\alpha}$$

Example:

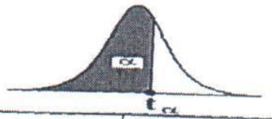
For $\nu = 10$ degrees of freedom (df), find $t_{0.93}$ and $t_{0.07}$.

Solution:

$t_{0.93} = (1.372+1.812)/2 = 1.592$ (from the table)
$t_{0.07} = -\,t_{1-0.07} = -\,t_{0.93} = -\,1.592$ (using the rule: $t_{1-\alpha} = -\,t_{\alpha}$)



Critical Values of the t-distribution (t_α)



ν=df	$t_{0.90}$	$t_{0.95}$	$t_{0.975}$	$t_{0.99}$	$t_{0.995}$
1	3.078	6.314	12.706	31.821	63.657
2	1.886	2.920	4.303	6.965	9.925
3	1.638	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.747	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	3.499
8	1.397	1.860	2.306	2.896	3.355
9	1.383	1.833	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169
11	1.363	1.796	2.201	2.718	3.106
12	1.356	1.782	2.179	2.681	3.055
13	1.350	1.771	2.160	2.650	3.012
14	1.345	1.761	2.145	2.624	2.977
15	1.341	1.753	2.131	2.602	2.947
16	1.337	1.746	2.120	2.583	2.921
17	1.333	1.740	2.110	2.567	2.898
18	1.330	1.734	2.101	2.552	2.878
19	1.328	1.729	2.093	2.539	2.861
20	1.325	1.725	2.086	2.528	2.845
21	1.323	1.721	2.080	2.518	2.831
22	1.321	1.717	2.074	2.508	2.819
23	1.319	1.714	2.069	2.500	2.807
24	1.318	1.711	2.064	2.492	2.797
25	1.316	1.708	2.060	2.485	2.787
26	1.315	1.706	2.056	2.479	2.779
27	1.314	1.703	2.052	2.473	2.771
28	1.313	1.701	2.048	2.467	2.763
29	1.311	1.699	2.045	2.462	2.756
30	1.310	1.697	2.042	2.457	2.750
35	1.3062	1.6896	2.0301	2.4377	2.7238
40	1.3030	1.6840	2.0210	2.4230	2.7040
45	1.3006	1.6794	2.0141	2.4121	2.6896
50	1.2987	1.6759	2.0086	2.4033	2.6778
60	1.2958	1.6706	2.0003	2.3901	2.6603
70	1.2938	1.6669	1.9944	2.3808	2.6479
80	1.2922	1.6641	1.9901	2.3739	2.6387
90	1.2910	1.6620	1.9867	2.3685	2.6316
100	1.2901	1.6602	1.9840	2.3642	2.6259
120	1.2886	1.6577	1.9799	2.3578	2.6174
140	1.2876	1.6558	1.9771	2.3533	2.6114
160	1.2869	1.6544	1.9749	2.3499	2.6069
180	1.2863	1.6534	1.9732	2.3472	2.6034
200	1.2858	1.6525	1.9719	2.3451	2.6006
∞	1.282	1.645	1.960	2.326	2.576

Application:

Example: (Sampling distribution of the sample mean)

Suppose that the time duration of a minor surgery is approximately normally distributed with mean equal to 800 seconds and a standard deviation of 40 seconds. Find the probability that a random sample of 16 surgeries will have average time duration of less than 775 seconds.

Solution:

X= the duration of the surgery

μ=800 , σ=40 , $\sigma^2 = 1600$

X~N(800, 1600)

Sample size: n=16

Calculating mean, variance, and standard error (standard deviation) of the sample mean $\overline{X}$:

Mean of $\overline{X}$: $\mu_{\overline{X}} = \mu$=800

Variance of $\overline{X}$: $\sigma^2_{\overline{X}} = \frac{\sigma^2}{n} = \frac{1600}{16} = 100$

Standard error (standard deviation) of $\overline{X}$: $\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}} = \frac{40}{\sqrt{16}} = 10$

Using the central limit theorem, $\overline{X}$ has a normal distribution with mean $\mu_{\overline{X}} = 800$ and variance $\sigma^2_{\overline{X}} = 100$, that is:

$$\overline{X} \sim \mathrm{N}(\mu, \frac{\sigma^2}{n}) = \mathrm{N}(800,100)$$

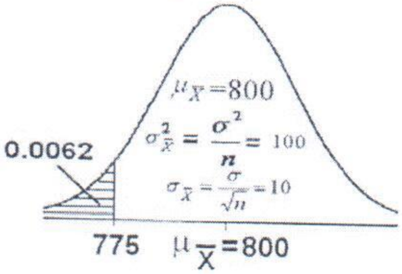
$$\Leftrightarrow \quad Z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}} = \frac{\overline{X} - 800}{10} \sim \mathrm{N}(0,1)$$

The probability that a random sample of 16 surgeries will have an average time duration that is less than 775 seconds equals to:

$$P(\overline{X} < 775) = P\left(\frac{\overline{X} - \mu}{\sigma / \sqrt{n}} < \frac{775 - \mu}{\sigma / \sqrt{n}}\right) = P\left(\frac{\overline{X} - 800}{10} < \frac{775 - 800}{10}\right)$$

$$= P\left(Z < \frac{775 - 800}{10}\right) = P(Z < -2.50) = 0.0062$$

$$\bar{X} \sim N(\mu, \frac{\sigma^2}{n}) = N(800,100)$$



Example:

If the mean and standard deviation of serum iron values for healthy mean are 120 and 15 microgram/100ml, respectively, what is the probability that a random sample of size 50 normal men will yield a mean between 115 and 125 microgram/100ml?

Solution:

X= the serum iron value

$\mu=120$, $\sigma=15$, $\sigma^2 = 225$

X~N(120, 225)

Sample size: n=50

Calculating mean, variance, and standard error (standard deviation) of the sample mean $\bar{X}$:

Mean of $\bar{X}$: $\mu_{\bar{X}} = \mu = 120$

Variance of $\bar{X}$: $\sigma^2_{\bar{X}} = \frac{\sigma^2}{n} = \frac{225}{50} = 4.5$

Standard error (standard deviation) of $\bar{X}$: $\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{15}{\sqrt{50}} = 2.12$

Using the central limit theorem, $\bar{X}$ has a normal distribution with mean $\mu_{\bar{X}} = 120$ and variance $\sigma^2_{\bar{X}} = 4.5$, that is:

$$\bar{X} \sim N(\mu, \frac{\sigma^2}{n}) = N(120, 4.5)$$

$$\Leftrightarrow \; Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} = \frac{\bar{X} - 120}{2.12} \sim N(0,1)$$

The probability that a random sample of 50 men will yield a mean between 115 and 125 microgram/100ml equals to:

$$P(115 < \bar{X} < 125) = P\left(\frac{115 - \mu}{\sigma / \sqrt{n}} < \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} < \frac{125 - \mu}{\sigma / \sqrt{n}}\right)$$

$$= P\left(\frac{115-120}{2.12} < \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} < \frac{125-120}{2.12}\right) = P(-2.36 < Z < 2.36)$$

$$= \ P(Z < 2.36) \ - P(Z < -2.36)$$

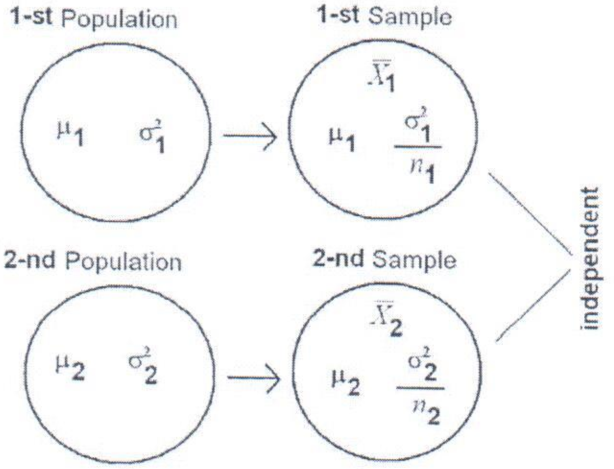
$$= 0.9909 - 0.0091$$

$$= 0.9818$$

5.4 Distribution of the Difference Between Two Sample Means ($\overline{X}_1 - \overline{X}_2$):

Suppose that we have two populations:

- 1-st population with mean μ_1 and variance $\sigma_1{}^2$
- 2-nd population with mean μ_2 and variance $\sigma_2{}^2$
- We are interested in comparing μ_1 and μ_2, or equivalently, making inferences about the difference between the means $(\mu_1-\mu_2)$.
- We independently select a random sample of size n_1 from the 1-st population and another random sample of size n_2 from the 2-nd population:
- Let $\overline{X}_1$ and S_1^2 be the sample mean and the sample variance of the 1-st sample.
- Let $\overline{X}_2$ and S_2^2 be the sample mean and the sample variance of the 2-nd sample.
- The sampling distribution of $\overline{X}_1 - \overline{X}_2$ is used to make inferences about $\mu_1-\mu_2$.



The sampling distribution of $\overline{X}_1 - \overline{X}_2$:

Result:

The mean, the variance and the standard deviation of $\overline{X}_1 - \overline{X}_2$ are:

Mean of $\overline{X}_1 - \overline{X}_2$ is: $\mu_{\overline{X}_1 - \overline{X}_2} = \mu_1 - \mu_2$

Variance of $\overline{X}_1 - \overline{X}_2$is: $\sigma^2_{\overline{X}_1 - \overline{X}_2} = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$

Standard error (standard) deviation of $\overline{X}_1 - \overline{X}_2$ is:

$$\sigma_{\overline{X}_1 - \overline{X}_2} = \sqrt{\sigma^2_{\overline{X}_1 - \overline{X}_2}} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Result:

If the two random samples were selected from normal distributions (or non-normal distributions with large sample sizes) with known variances σ_1^2 and σ_2^2, then the difference between the sample means $(\overline{X}_1 - \overline{X}_2)$ has a normal distribution with mean $(\mu_1 - \mu_2)$ and variance $((\sigma_1^2 / n_1) + (\sigma_2^2 / n_2))$, that is:

- $\overline{X}_1 - \overline{X}_2 \sim N\left(\mu_1 - \mu_2\, , \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)$
- $Z = \dfrac{(\overline{X}_1 - \overline{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim \mathrm{N}(0,1)$

Application:

Example:

Suppose it has been established that for a certain type of client (type A) the average length of a home visit by a public health nurse is 45 minutes with standard deviation of 15 minutes, and that for second type (type B) of client the average home visit is 30 minutes long with standard deviation of 20 minutes. If a nurse randomly visits 35 clients from the first type and 40 clients from the second type, what is the probability that the average length of home visit of first type will be greater than the average length of home visit of second type by 20 or more minutes?

Solution:

For the first type:

$\mu_1 = 45$

$\sigma_1 = 15$

$\sigma_1^2 = 225$

$n_1 = 35$

For the second type:

$\mu_2 = 30$

$\sigma_2 = 20$

$\sigma_2^2 = 400$

$n_2 = 40$

The mean, the variance and the standard deviation of $\overline{X}_1 - \overline{X}_2$ are:

Mean of $\overline{X}_1 - \overline{X}_2$ is:

$$\mu_{\overline{X}_1 - \overline{X}_2} = \mu_1 - \mu_2 = 45 - 30 = 15$$

Variance of $\overline{X}_1 - \overline{X}_2$is:

$$\sigma^2_{\overline{X}_1 - \overline{X}_2} = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} = \frac{225}{35} + \frac{400}{40} = 16.4286$$

Standard error (standard) deviation of $\overline{X}_1 - \overline{X}_2$ is:

$$\sigma_{\overline{X}_1 - \overline{X}_2} = \sqrt{\sigma^2_{\overline{X}_1 - \overline{X}_2}} = \sqrt{16.4286} = 4.0532$$

The sampling distribution of $\overline{X}_1 - \overline{X}_2$ is:

$$\overline{X}_1 - \overline{X}_2 \sim N(15\,,16.4286)$$

$$Z = \frac{(\overline{X}_1 - \overline{X}_2) - 15}{\sqrt{16.4286}} \sim \mathrm{N}(0,1)$$

The probability that the average length of home visit of first type will be greater than the average length of home visit of second type by 20 or more minutes is:

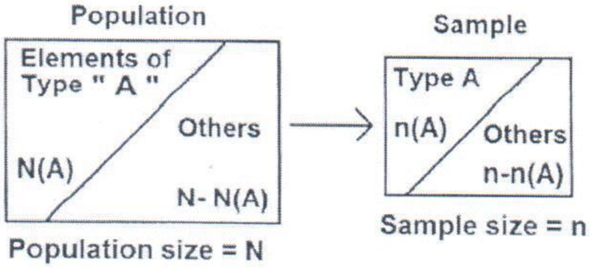
$$P(\overline{X}_1-\overline{X}_2>20)=P\left(\frac{(\overline{X}_1-\overline{X}_2)-(\mu_1-\mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}>\frac{20-(\mu_1-\mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}\right)$$

$$=P\left(Z>\frac{20-15}{4.0532}\right)=\text{P}(\text{Z}>1.23)=1-\text{P}(\text{Z}<1.23)$$

$$=1-0.8907$$

$$=0.1093$$

5.5 Distribution of the Sample Proportion ($\hat{p}$):



- For the population:

  $N(A)$ = number of elements in the population with a specified characteristic "A"

  N = total number of elements in the population (population size)

The population proportion is

$$p=\frac{N(A)}{N} \qquad \text{(p is a parameter)}$$

- For the sample:

  $n(A)$ = number of elements in the sample with the same characteristic "A"

  n = sample size

The sample proportion is

$$\hat{p}=\frac{n(A)}{n} \qquad (\hat{p}\ \text{is a statistic})$$

- The sampling distribution of $\hat{p}$ is used to make inferences about p.

Result:

The mean of the sample proportion ($\hat{p}$) is the population proportion (p); that is:

$$\mu_{\hat{p}} = p$$

The variance of the sample proportion ($\hat{p}$) is:

$$\sigma^2_{\hat{p}} = \frac{p(1-p)}{n} = \frac{pq}{n}. \qquad \text{(where q=1 −p)}$$

The standard error (standard deviation) of the sample proportion ($\hat{p}$) is:

$$\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{pq}{n}}$$

Result:

For large sample size ($n \geq 30, np > 5, nq > 5$), the sample proportion ($\hat{p}$) has approximately a normal distribution with mean $\mu_{\hat{p}} = p$ and a variance $\sigma^2_{\hat{p}} = pq/n$, that is:

$$\hat{p} \sim N\left(p, \frac{pq}{n}\right) \qquad \text{(approximately)}$$

$$Z = \frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}} \sim N(0,1) \qquad \text{(approximately)}$$

Example:

Suppose that 45% of the patients visiting a certain clinic are females. If a sample of 35 patients was selected at random, find the probability that:

1. the proportion of females in the sample will be greater than 0.4.
2. the proportion of females in the sample will be between 0.4 and 0.5.

Solution:

- .n = 35 (large)
- p = The population proportion of females = $\frac{45}{100} = 0.45$

- $\hat{p}$ = The sample proportion
  (proportion of females in the sample)
- The mean of the sample proportion ($\hat{p}$) is p = 0.45
- The variance of the sample proportion ($\hat{p}$) is:

$$\frac{p(1-p)}{n} = \frac{pq}{n} = \frac{0.45(1-0.45)}{35} = 0.0071.$$

- The standard error (standard deviation) of the sample proportion ($\hat{p}$) is:

$$\sqrt{\frac{p(1-p)}{n}} = \sqrt{0.0071} = 0.084$$

- $n \geq 30,\ np = 35 \times 0.45 = 15.75 > 5, nq = 35 \times 0.55 = 19.25 > 5$

1. The probability that the sample proportion of females ($\hat{p}$) will be greater than 0.4 is:

$$\text{P}(\hat{p} > 0.4) = 1 - \text{P}(\hat{p} < 0.4) = 1 - \text{P}\left(\frac{\hat{p}-p}{\sqrt{\frac{p(1-p)}{n}}} < \frac{0.4-p}{\sqrt{\frac{p(1-p)}{n}}} \right)$$

$$= 1-\text{P}\left(Z < \frac{0.4-0.45}{\sqrt{\frac{0.45(1-0.45)}{35}}} \right) = 1 - \text{P}(Z < -0.59)$$

$$= 1 - 0.2776 = 0.7224$$

2. The probability that the sample proportion of females ($\hat{p}$) will be between 0.4 and 0.5 is:

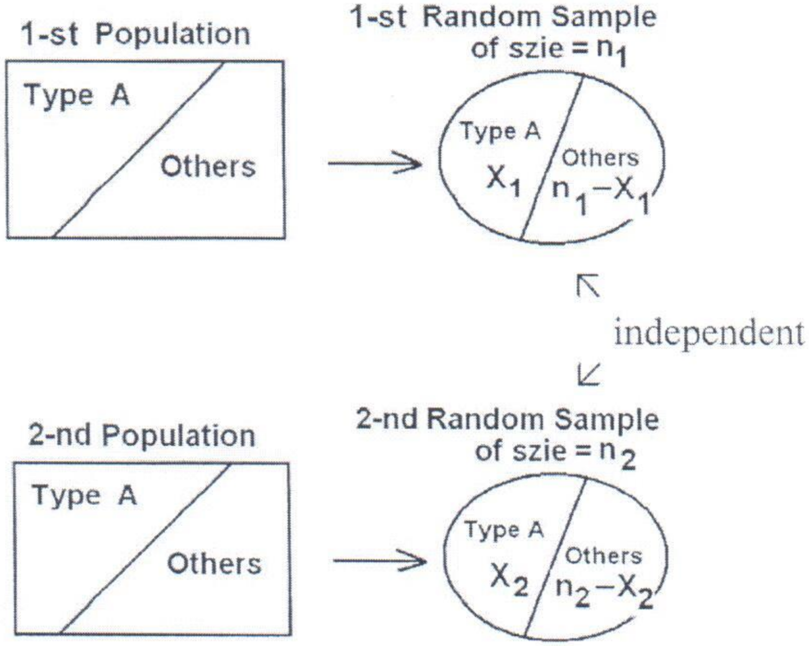
$$\text{P}(0.4 < \hat{p} < 0.5) = \text{P}(\hat{p} < 0.5) - \text{P}(\hat{p} < 0.4)$$

$$= \text{P}\left(\frac{\hat{p}-p}{\sqrt{\frac{p(1-p)}{n}}} < \frac{0.5-p}{\sqrt{\frac{p(1-p)}{n}}} \right) - 0.2776$$

$$= \text{P}\left(Z < \frac{0.5-0.45}{\sqrt{\frac{0.45(1-0.45)}{35}}} \right) - 0.2776$$

$$= P(Z < 0.59) - 0.2776$$
$$= 0.7224 - 0.2776$$
$$= 0.4448$$

5.6 Distribution of the Difference Between Two Sample Proportions ($\hat{p}_1 - \hat{p}_2$):



Suppose that we have two populations:

- p_1 = proportion of elements of type (A) in the 1-st population.
- p_2 = proportion of elements of type (A) in the 2-nd population.
- We are interested in comparing p_1 and p_2, or equivalently, making inferences about $p_1 - p_2$.
- We independently select a random sample of size n_1 from the 1-st population and another random sample of size n_2 from the 2-nd population:
- Let X_1 = no. of elements of type (A) in the 1-st sample.
- Let X_2 = no. of elements of type (A) in the 2-nd sample.
- $\hat{p}_1 = \dfrac{X_1}{n_1}$ = sample proportion of the 1-st sample

- $\hat{p}_2 = \dfrac{X_2}{n_2}$ = sample proportion of the 2-nd sample
- The sampling distribution of $\hat{p}_1 - \hat{p}_2$ is used to make inferences about $p_1 - p_2$.

The sampling distribution of $\hat{p}_1 - \hat{p}_2$:

Result:

The mean, the variance and the standard error (standard deviation) of $\hat{p}_1 - \hat{p}_2$ are:

- Mean of $\hat{p}_1 - \hat{p}_2$ is:

$$\mu_{\hat{p}_1 - \hat{p}_2} = p_1 - p_2$$

- Variance of $\hat{p}_1 - \hat{p}_2$ is:

$$\sigma^2_{\hat{p}_1 - \hat{p}_2} = \frac{p_1\, q_1}{n_1} + \frac{p_2\, q_2}{n_2}$$

- Standard error (standard deviation) of $\hat{p}_1 - \hat{p}_2$ is:

$$\sigma_{\hat{p}_1 - \hat{p}_2} = \sqrt{\frac{p_1\, q_1}{n_1} + \frac{p_2\, q_2}{n_2}}$$

- $q_1 = 1 - p_1$ and $q_2 = 1 - p_2$

Result:

For large samples sizes $(n_1 \geq 30, n_2 \geq 30, n_1 p_1 > 5, n_1 q_1 > 5, n_2 p_2 > 5, n_2 q_2 > 5)$, we have that $\hat{p}_1 - \hat{p}_2$ has approximately normal distribution with mean $\mu_{\hat{p}_1 - \hat{p}_2} = p_1 - p_2$ and variance $\sigma^2_{\hat{p}_1 - \hat{p}_2} = \frac{p_1\, q_1}{n_1} + \frac{p_2\, q_2}{n_2}$, that is:

$$\hat{p}_1 - \hat{p}_2 \sim N\left(p_1 - p_2, \frac{p_1\, q_1}{n_1} + \frac{p_2\, q_2}{n_2}\right) \quad \text{(Approximately)}$$

$$Z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\frac{p_1\, q_1}{n_1} + \frac{p_2\, q_2}{n_2}}} \sim \mathrm{N}(0,1) \quad \text{(Approximately)}$$

Example:

Suppose that 40% of Non-Saudi residents have medical insurance and 30% of Saudi residents have medical insurance in a certain city. We have randomly and independently selected a sample of 130 Non-Saudi residents and another sample of 120 Saudi residents. What is the probability that the difference between the sample proportions, $\hat{p}_1 - \hat{p}_2$, will be between 0.05 and 0.2?

Solution:

p_1 = population proportion of non-Saudi with medical insurance.

p_2 = population proportion of Saudi with medical insurance.

$\hat{p}_1$ = sample proportion of non-Saudis with medical insurance.

$\hat{p}_2$ = sample proportion of Saudis with medical insurance.

$p_1 = 0.4 \qquad n_1=130$

$p_2 = 0.3 \qquad n_2=120$

$$\mu_{\hat{p}_1-\hat{p}_2} = p_1 - p_2 = 0.4 - 0.3 = 0.1$$

$$\sigma^2_{\hat{p}_1-\hat{p}_2} = \frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2} = \frac{(0.4)(0.6)}{130} + \frac{(0.3)(0.7)}{120} = 0.0036$$

$$\sigma_{\hat{p}_1-\hat{p}_2} = \sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}} = \sqrt{0.0036} = 0.06$$

The probability that the difference between the sample proportions, $\hat{p}_1 - \hat{p}_2$, will be between 0.05 and 0.2 is:

$$P(0.05 < \hat{p}_1 - \hat{p}_2 < 0.2) = P(\hat{p}_1 - \hat{p}_2 < 0.2) - P(\hat{p}_1 - \hat{p}_2 < 0.05)$$

$$= P\left(\frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}} < \frac{0.2 - (p_1 - p_2)}{\sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}} \right)$$

$$- P\left(\frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\frac{p_1\,q_1}{n_1} + \frac{p_2\,q_2}{n_2}}} < \frac{0.05 - (p_1 - p_2)}{\sqrt{\frac{p_1\,q_1}{n_1} + \frac{p_2\,q_2}{n_2}}} \right)$$

$$= P\left(Z < \frac{0.2 - 0.1}{0.06} \right) - P\left(Z < \frac{0.05 - 0.1}{0.06} \right)$$

$$= P(Z < 1.67) - P(Z < -0.83)$$

$$= 0.9515 - 0.2033$$

$$= 0.7482$$