Theory of statistics 2

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Sequential likelihood ratio test

All previous tests γ for the hypothesis $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$ use Likelihood Ratio (LR) λ under a fixed sample size n and rejects as $\lambda \leq k$ where k is properly found. Whereas Sequential Likelihood Ratio Test (SLRT) γ_{SLRT} uses also LR λ_N but under a random sample size N and decides to reject or accept using two bands $k_0 < k_1$. Precisely, γ_{SLRT} computes from the one observation x_1 the well-known LR:

$$\lambda_1 = \frac{\ell(x_1; \theta_0)}{\ell(x_1; \theta_1)}.$$

Then γ_{SLRT} rejects if $\lambda_1 \leq k_0$ or accepts if $\lambda_1 \geq k_1$ and stop.

If $k_0 < \lambda_1 < k_1$, then another observation is drawn

$$\lambda_2 = \frac{\ell(x_1, x_2; \theta_0)}{\ell(x_1, x_2; \theta_1)}.$$

is computed to rejects if $\lambda_2 \leq k_0$ or accepts if $\lambda_2 \geq k_1$ or to draw another more observation.

This sequential strategy makes the sample size N for which the decision of rejection or acceptance is random. This strategy may be summarize as following:

1. To reject H_0 at N means that:

$$\lambda_N \leq k_0; \quad k_0 < \lambda_j < k_1, j \in \{1, \ldots, N-1\}.$$

2. To accept H_0 at N means that:

$$\lambda_N \geq k_1; \quad k_0 < \lambda_j < k_1, j \in \{1, \ldots, N-1\}.$$

Where

$$\lambda_N = \frac{\ell(x_1, x_2, \ldots, x_N; \theta_0)}{\ell(x_1, x_2, \ldots, x_N; \theta_1)}.$$

The rejection area here is:

$$C = \bigcup_{N \geq 1} C_N,$$

where

$$C_N = \{x_1, x_2, \ldots, x_n;\ k_0 < \lambda_i < k_1 \text{ for } i = 1, \ldots, N-1;\ \lambda_N \leq k_0\}.$$

The acceptance area here is:

$$A = \bigcup_{N \geq 1} A_N,$$

where

$$A_N = \{x_1, x_2, \ldots, x_n;\ k_0 < \lambda_i < k_1 \text{ for } i = 1, \ldots, N-1;\ \lambda_N \geq k_1\}.$$

If the sizes of errors of γ_{SLRT} are α^* and β^*, then

$$\alpha^* = \mathbb{P}(\text{Reject } H_0 | \theta_0) = \sum_{N \geq 1} \int_{C_N} \ell(x_1, x_2, \ldots, x_N; \theta_0)$$

and

$$\beta^* = \mathbb{P}(\text{Accept } H_0 | \theta_1) = \sum_{N \geq 1} \int_{A_N} \ell(x_1, x_2, \ldots, x_N; \theta_1).$$

Theorem

- The test γ_{SLRT} of sizes of errors α^* and β^* minimizes $\mathbb{E}(N)$ among all other tests γ of sizes of errors $\alpha \leq \alpha^*$ and $\beta \leq \beta^*$.
- The approximations

$$k_0 \approx k_0' = \frac{\alpha^*}{1-\beta^*} \text{ and } k_1 \approx k_1' = \frac{1-\alpha^*}{\beta^*}.$$

Satisfy

$$k_0' \leq k_0 < k_1 \leq k_1'.$$

Computation of $\mathbb{E}(N)$

$$\mathbb{E}(N|H_0) = \frac{\alpha^* \log(k_0') + (1-\alpha^*)\log(k_1')}{\mathbb{E}(Z_i|H_0)}$$

$$\mathbb{E}(N|H_1) = \frac{(1-\beta^*)\log(k_0') + \beta^* \log(k_1')}{\mathbb{E}(Z_i|H_1)},$$

$$\text{where } Z_i = \log\left(\frac{f(x_i;\theta_0)}{f(x_i;\theta_1)}\right)$$

Example

It was found γ_{MP} in testing $H_0 : \theta = 3$ vs $H_1 : \theta = 5$ for the normal distribution $N(\theta, 16)$ using $n = 25$, $\alpha_{MP} = 0.05$ and $\beta_{MP} = 0.1949$

1) Approximate the bands k_0 and k_1 of γ_{SLRT} with the errors of γ_{MP}:

$$k_0 \approx k_0' = \frac{\alpha_{MP}}{1-\beta_{MP}} = 0.0621 \text{ and } k_1 \approx k_1' = \frac{1-\alpha_{MP}}{\beta_{MP}} = 4.874.$$

2) Determine $\mathbb{E}(Z_i|H_0)$ and $\mathbb{E}(Z_i|H_1)$

$$Z_i = \log\left(\frac{f(x_i;\theta_0)}{f(x_i;\theta_1)}\right) = -\frac{1}{2\times 16}[(x_i-3)^2-(x_i-5)^2] = \frac{1}{8}(4-x_i).$$

$$\mathbb{E}(Z_i|H_0) = \frac{1}{8}(4-\mathbb{E}(x_i|H_0)) = \frac{1}{8}(4-3) = \frac{1}{8}.$$

$$\mathbb{E}(Z_i|H_1) = \frac{1}{8}(4-\mathbb{E}(x_i|H_1)) = \frac{1}{8}(4-5) = -\frac{1}{8}.$$

3) Compute $\mathbb{E}(N|H_0)$ and $\mathbb{E}(N|H_1)$

$$\begin{aligned}\mathbb{E}(N|H_0) &= \frac{\alpha^* \log(k_0') + (1-\alpha^*)\log(k_1')}{\mathbb{E}(Z_i|H_0)} \\ &= \frac{-2.779 \times 0.05 + 1.584 \times 0.95]}{1/8} = 10.93\end{aligned}$$

$$\begin{aligned}\mathbb{E}(N|H_1) &= \frac{(1-\beta^*)\log(k_0') + \beta^* \log(k_1')}{\mathbb{E}(Z_i|H_1)} \\ &= \frac{2.779 \times 0.8051 + 1.584 \times 0.1949}{(-1/8)} = 15.43\end{aligned}$$

Homework

It was found γ_{MP} in testing $H_0 : \theta = 2$ vs $H_1 : \theta = 3$ for the exponential distribution $f(x;\theta) = \theta e^{-\theta x}, \quad x > 0$ using $n = 10$, $\alpha_{MP} = 0.05$ and $\beta_{MP} = 0.7$

1) Approximate the bands k_0 and k_1 of γ_{SLRT} with the errors of γ_{MP}:
2) Determine $\mathbb{E}(Z_i|H_0)$ and $\mathbb{E}(Z_i|H_1)$
3) Compute $\mathbb{E}(N|H_0)$ and $\mathbb{E}(N|H_1)$

Homework

It was found γ_{MP} in testing $H_0 : \sigma = 2$ vs $H_1 : \sigma = 4$ for the normal distribution $N(\theta, \sigma^2)$, with θ known, using $n = 10$, $\alpha_{MP} = 0.05$ and $\beta_{MP} = 0.0824$

1) Approximate the bands k_0 and k_1 of γ_{SLRT} with the errors of γ_{MP}:
2) Determine $\mathbb{E}(Z_i|H_0)$ and $\mathbb{E}(Z_i|H_1)$
3) Compute $\mathbb{E}(N|H_0)$ and $\mathbb{E}(N|H_1)$

Homework

It was found γ_{MP} in testing $H_0 : \theta = 2$ vs $H_1 : \theta = 3$ for the distribution $f(x;\theta) = \theta x^{\theta-1}, \quad 0 < x < 1$ using $n = 10$, $\alpha_{MP} = 0.05$ and $\beta_{MP} = 0.23$

1) Approximate the bands k_0 and k_1 of γ_{SLRT} with the errors of γ_{MP}:
2) Determine $\mathbb{E}(Z_i|H_0)$ and $\mathbb{E}(Z_i|H_1)$
3) Compute $\mathbb{E}(N|H_0)$ and $\mathbb{E}(N|H_1)$

Thank you