Selected Solutions to *Complex Analysis* by Lars Ahlfors

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Chapter 4 - Complex Integration

Cauchy's Integral Formula

4.2.2 Exercise 1

Applying the Cauchy integral formula to $f(z) = e^z$,

$$1 = f(0) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{f(z)}{z} dz \iff 2\pi i = \oint_{|z|=1} \frac{e^z}{z} dz$$

Section 4.2.2 Exercise 2

Using partial fractions, we may express the integrand as

$$\frac{1}{z^2+1} = \frac{i}{2(z+i)} - \frac{i}{2(z-i)}$$

Applying the Cauchy integral formula to the constant function $f(z) = 1$,

$$\frac{1}{2\pi i} \oint_{|z|=2} \frac{1}{z^2+1} dz = \frac{i}{2}\left(\frac{1}{2\pi i}\right) \oint_{|z|=2} \frac{1}{z+i} dz - \frac{i}{2}\left(\frac{1}{2\pi i}\right) \oint_{|z|=2} \frac{1}{z-i} dz = 0$$

4.2.3 Exercise 1

1. Applying Cauchy's differentiation formula to $f(z) = e^z$,

$$1 = f^{(n-1)}(0) = \frac{(n-1)!}{2\pi i} \oint_{|z|=1} \frac{e^z}{z^n} dz \iff \frac{2\pi i}{(n-1)!} = \oint_{|z|=1} \frac{e^z}{z^n} dz$$

2. We consider the following cases:

   (a) If $n \geq 0, m \geq 0$, then it is obvious from the analyticity of $z^n(1-z)^m$ and Cauchy's theorem that the integral is 0.

   (b) If $n \geq 0, m < 0$, then by the Cauchy differentiation formula,

$$\oint_{|z|=2} z^n(1-z)^m dz = (-1)^m \oint_{|z|=2} \frac{z^n}{(z-1)^{|m|}} dz = \begin{cases} 0 & n < |m| - 1 \\ \frac{(-1)^m 2\pi i}{(|m|-1)!} \frac{n!}{(n-|m|+1)!} = (-1)^{|m|} 2\pi i \binom{n}{|m|-1} & n \geq |m| \end{cases}$$

   (c) If $n < 0, m \geq 0$, then by a completely analogous argument,

$$\oint_{|z|=2} z^n(1-z)^m dz = \oint_{|z|=2} \frac{(1-z)^m}{z^{|n|}} dz = \begin{cases} 0 & m < |n| - 1 \\ \frac{(-1)^{|n|-1} 2\pi i}{(|n|-1)!} \frac{m!}{(m-|n|+1)!} = (-1)^{|n|-1} 2\pi i \binom{m}{|n|-1} & m \geq n \end{cases}$$

   (d) If $n < 0, m < 0$, then since$n(|z| = 2, 0) = n(|z| = 2, 1) = 1$, we have by the residue formula that

$$\oint_{|z|=2} (1-z)^m z^n = 2\pi i \text{res}(f; 0) + 2\pi i \text{res}(f; 1) = \oint_{|z|=\frac{1}{2}} (1-z)^m z^n dz + \oint_{|z-1|=\frac{1}{2}} (1-z)^m z^n dz$$

Using Cauchy's differentiation formula, we obtain

$$\oint_{|z|=2}(1-z)^m z^n dz = \left[\oint_{|z|=\frac{1}{2}}\frac{(1-z)^{-|m|}}{z^{|n|}}dz + \oint_{|z-1|=\frac{1}{2}}\frac{z^{-|n|}}{(1-z)^{|m|}}dz\right]$$

$$= \frac{2\pi i}{(|n|-1)!}\cdot\frac{(|m|+|n|-2)!}{(|m|-1)!} + \frac{(-1)^{|m|}2\pi i}{(|m|-1)!}\cdot\frac{(-1)^{|m|-1}(|n|+|m|-2)!}{(|n|-1)!}$$

$$= 2\pi i\left[\binom{|m|+|n|-2}{|n|-1} - \binom{|m|+|n|-2}{|n|-1}\right] = 0$$

3. If $\rho = 0$, then it is trivial that $\oint_{|z|=\rho}|z-a|^{-4}|dz| = 0$, so assume otherwise. If $a = 0$, then

$$\oint_{|z|=\rho}|z|^{-4}|dz| = \int_0^1 \rho^{-4}2\pi i\rho dt = \frac{2\pi i}{\rho^3}$$

Now, assume that $a \neq 0$. Observe that

$$\frac{1}{|z-a|^4} = \frac{1}{(z-a)^2\overline{(z-a)}^2}$$

$$\oint_{|z|=\rho}|z-a|^{-4}|dz| = \oint_{|z|=\rho}\frac{1}{(z-a)^2(\overline{z}-\overline{a})^2}|dz| = \int_0^1\frac{1}{(\rho e^{2\pi it}-a)^2(\rho e^{-2\pi it}-\overline{a})^2}\rho\frac{2\pi ie^{4\pi it}}{ie^{4\pi it}}dt$$

$$= -i\int_0^1\frac{\rho 2\pi ie^{4\pi it}}{(\rho e^{2\pi it}-a)^2(\rho-\overline{a}e^{2\pi it})^2}dt = \frac{-i}{\rho}\oint_{|z|=\rho}\frac{z}{(\rho-\frac{\overline{a}}{\rho}z)^2(z-a)^2}dz = \frac{-i\rho}{\overline{a}^2}\oint_{|z|=\rho}\frac{z}{(z-\frac{\rho^2}{\overline{a}})^2(z-a)^2}dz$$

We consider two cases. First, suppose $|a| > \rho$. Then $z(z-a)^{-2}$ is holomorphic on and inside $\{|z|=\rho\}$ and $\frac{\rho^2}{\overline{a}}$ lies inside $\{|z|=\rho\}$. By Cauchy's differentiation formula,

$$\oint_{|z|=\rho}|z-a|^{-4}|dz| = 2\pi i\frac{-i\rho}{\overline{a}^2}\left[(z-a)^{-2}-2z(z-a)^{-3}\right]_{z=\frac{\rho^2}{\overline{a}}} = \frac{2\pi\rho}{\overline{a}^2(\frac{\rho^2}{\overline{a}}-a)^2}\left[1-2\frac{\rho^2}{\overline{a}(\frac{\rho^2}{\overline{a}}-a)}\right]$$

$$= \frac{-2\pi\rho(\rho^2+|a|^2)}{(\rho^2-|a|^2)^3} = \frac{2\pi\rho(\rho^2+|a|^2)}{(|a|^2-\rho^2)^3}$$

Now, suppose $|a| < \rho$. Then $\frac{\rho^2}{\overline{a}}$ lies outside $|z| = \rho$, so the function $z(z-\frac{\rho^2}{\overline{a}})^{-2}$ is holomorphic on and inside $\{|z|=\rho\}$. By Cauchy's differentiation formula,

$$\oint_{|z|=\rho}|z-a|^{-4}|dz| = 2\pi i\frac{-i\rho}{\overline{a}^2}\left[(z-\frac{\rho^2}{\overline{a}})^{-2}-2z(z-\frac{\rho^2}{\overline{a}})^{-3}\right]_{z=a} = \frac{2\pi\rho}{\overline{a}^2(a-\frac{\rho^2}{\overline{a}})^2}\left[1-2\frac{a}{(a-\frac{\rho^2}{\overline{a}})}\right]$$

$$= \frac{-2\pi\rho}{(|a|^2-\rho^2)^2}\frac{(a+\frac{\rho^2}{\overline{a}})}{a-\frac{\rho^2}{\overline{a}}} = \frac{-2\pi\rho(|a|^2+\rho^2)}{(|a|^2-\rho^2)^3} = \frac{2\pi\rho(|a|^2+\rho^2)}{(\rho^2-|a|^2)^3}$$

4.2.3 Exercise 2

Let $f:\mathbb{C}\to\mathbb{C}$ be a holomorphic function satisfying the following condition: there exists $R > 0$ and $n \in \mathbb{N}$ such that $|f(z)| < |z|^n \ \forall |z| \geq R$. For every $r \geq R$, we have by the Cauchy differentiation formula that for all $m > n$,

$$\left|f^{(m)}(a)\right| \leq \frac{m!}{2\pi}\oint_{|z|=r}\frac{|z|^n}{|z|^{m+1}}|dz| \leq \frac{m!}{r^{m-n}}$$

Noting that $m - n \geq 1$ and letting $r \to \infty$, we have that $f^{(m)}(a) = 0$. Since f is entire, for every $a \in \mathbb{C}$, we may write

$$f(z) = f(a) + f'(a)(z-a) + \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^n + f_{n+1}(z)(z-a)^{n+1} \ \forall z \in \mathbb{C}$$

where f_{n+1} is entire. Since $f_{n+1}(a) = f^{(n+1)}(a) = 0$ and $a \in \mathbb{C}$ was arbitary, we have that $f_{n+1} \equiv 0$ on $\mathbb{C}$. Hence, f is a polynomial of degree at most n.

Local Properties of Analytical Functions

4.3.2 Exercise 2

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function with a nonessential singularity at ∞. Consider the function $g(z) = f\left(\frac{1}{z}\right)$ at $z = 0$. Let $n \in \mathbb{N}$ be minimal such that $\lim_{z\to 0} z^n g(z) = 0$. Then the function $z^{n-1}g(z)$ has an analytic continuation $h(z)$ defined on all of $\mathbb{C}$. By Taylor's theorem, we may express $h(z)$ as

$$z^{n-1}g(z) = h(z) = \underbrace{h(0)}_{c_{n-1}} + \underbrace{\frac{h'(0)}{1!}}_{c_{n-2}} z + \frac{h''(0)}{2!} z^2 + \cdots + \underbrace{\frac{h^{(n-1)}(0)}{(n-1)!}}_{c_0} z^{n-1} + h_n(z)z^n \ \forall z \neq 0$$

where $h_n : \mathbb{C} \to \mathbb{C}$ is holomorphic. Hence,

$$\lim_{z\to 0} g(z) - \left[\frac{c_{n-1}}{z^{n-1}} + \frac{c_{n-2}}{z^{n-2}} + \cdots + c_0\right] = \lim_{z\to 0} z h_n(z) = 0$$

And

$$\lim_{z\to \infty} g(z) - \left[\frac{c_{n-1}}{z^{n-1}} + \frac{c_{n-2}}{z^{n-2}} + \cdots + c_0\right] = \lim_{z\to 0} f(z) = f(0)$$

since f is entire. Note that we also obtain that $c_0 = f(0)$. Hence, $g(z) - \left[\frac{c_{n-1}}{z^{n-1}} + \frac{c_{n-2}}{z^{n-2}} + \cdots + c_0\right]$ (we are abusing notation to denote the continuation to all of $\mathbb{C}$) is a bounded entire function and is therefore identically zero by Liouville's theorem. Hence,

$$\forall z \neq 0, f(z) = c_{n-1}z^{n-1} + c_{n-2}z^{n-2} + \cdots + c_0$$

Since $f(0) = c_0$, we obtain that f is a polynomial.

4.3.2 Exercise 4

Let $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ be a meromorphic function in the extended complex plane. First, I claim that f has finitely many poles. Since the poles of f are isolated points, they form an at most countable subset $\{p_k\}_{k=1}^{\infty}$ of $\mathbb{C}$. By definition, the function $\tilde{f}(z) = f\left(\frac{1}{z}\right)$ has either a removable singularity or a pole at $z = 0$. In either case, there exists $r > 0$ such that $\tilde{f}$ is holomorphic on $D'(0;r)$. Hence, $\{p_k\}_{k=1}^{\infty} \subset \overline{D}(0;r)$. Since this set is bounded, $\{p_k\}_{k=1}^{\infty}$ has a limit point p. By continuity, $f(p) = \infty$ and therefore p is a pole. Since p is an isolated point, there must exist $N \in \mathbb{N}$ such that $\forall k \geq N, p_k = p$.
Our reasoning in the preceding Exercise 2 shows that for any pole $p_k \neq \infty$ of order m_k, we can write in a neighborhood of p_k

$$f(z) = \underbrace{\left[\frac{c_{m_k}}{(z-p_k)^{m_k}} + \frac{c_{m_k-1}}{(z-p_k)^{m_k-1}} + \cdots + \frac{c_1}{z-p_k} + c_0\right]}_{f_k(z)} + g_k(z)$$

where g_k is holomorphic in a neighborhood of p_k. If $p = \infty$ is a pole, then analogously,

$$\tilde{f}(z) = \underbrace{\left[\frac{c_{m_\infty}}{z^{m_\infty}} + \frac{c_{m_\infty-1}}{z^{m_\infty-1}} + \cdots + \frac{c_1}{z} + c_0\right]}_{\tilde{f}_\infty(z)} + \tilde{g}_\infty(z)$$

where $\tilde{g}_\infty$ is holomorphic in a neighborhood of 0. For clarification, the coefficients c_n depend on the pole, but we omit the dependence for convenience. Set $f_\infty(z) = \tilde{f}_\infty\left(\frac{1}{z}\right)$ and

$$h(z) = f(z) - f_\infty(z) - \sum_{k=1}^{n} f_k(z)$$

I claim that h is (or rather, extends to) an entire, bounded function. Indeed, in a neighborhood of each z_k, h can be written as $h(z) = g_k(z) - \sum_{i\neq k} f_k(z)$ and in a neighborhood of z_∞ as $h(z) = g_\infty(z) - \sum_{k=1}^{n} f_k(z)$, which are sums of holomorphic functions. $\tilde{h}(z) = h\left(\frac{1}{z}\right)$ is evidently bounded in a neighborhood of 0 since the $f_k\left(\frac{1}{z}\right)$ are polynomials and $f\left(\frac{1}{z}\right) - f_\infty\left(\frac{1}{z}\right) = \tilde{g}_\infty(z)$, which is holomorphic in a neighborhood of 0. By Liouville's theorem, h is a constant. It is immediate from the definition of h that f is a rational function.

Calculus of Residues

4.5.2 Exercise 1

Set $f(z) = 6z^3$ and $g(z) = z^7 - 2z^5 - z + 1$. Clearly, f, g are entire, $|f(z)| > |g(z)| \ \forall |z| = 1$, and $f(z) + g(z) = z^7 - 2z^5 + 6z^3 - z + 1$. By Rouché's theorem, f and $f + g$ have the same number of zeros, which is 3 (counted with order), in the disk $\{|z| < 1\}$.

Section 4.5.2 Exercise 2

Set $f(z) = z^4$ and $g(z) = -6z + 3$. Clearly, f, g are entire, $|f(z)| > |g(z)| \ \forall |z| = 2$. By Rouché's theorem, $z^4 - 6z + 3$ has 4 roots (counted with order) in the open disk $\{|z| < 2\}$. Now set $f(z) = -6z$ and $g(z) = z^4 + 3$. Clearly, $|f(z)| > |g(z)| \ \forall |z| = 1$. By Rouché's theorem, $z^4 - 6z + 3 = 0$ has 1 root in the in the open disk $\{|z| < 1\}$. Observe that if $z \in \{1 \leq |z| < 2\}$ is root, then by the reverse triangle inequality,

$$3 = |z| \left|z^3 - 6\right| \geq |z| \left||z|^3 - 6\right|$$

So $|z| \in (1, 2)$. Hence, the equation $z^4 - 6z + 3 = 0$ has 3 roots (counted with order) with modulus strictly between 1 and 2.

4.5.3 Exercise 1

1. Set $f(z) = \frac{1}{z^2+5z+6} = \frac{1}{(z+3)(z+2)}$. Then f has poles $z_1 = -2, z_2 = -3$ and by Cauchy integral formula,

$$\text{res}(f; z_1) = \frac{1}{2\pi i} \oint_{|z+2|=\frac{1}{2}} \frac{(z+3)^{-1}}{(z+2)} dz = \frac{1}{z+3}|_{z=-2} = 1$$

$$\text{res}(f; z_2) = \frac{1}{2\pi i} \oint_{|z+3|=\frac{1}{2}} \frac{(z+2)^{-1}}{(z+3)} dz = \frac{1}{z+2}|_{z=-3} - 1$$

2. Set $f(z) = \frac{1}{(z^2-1)^2} = \frac{1}{(z-1)^2(z+1)^2}$. Then f has poles $z_1 = -1, z_2 = -1$. Applying Cauchy's differentiation formula, we obtain

$$\text{res}(f; z_1) = \frac{1}{2\pi i} \oint_{|z+1|=1} \frac{(z-1)^{-2}}{(z+1)^2} dz = -2(z-1)^{-3}|_{z=-1} = \frac{1}{4}$$

$$\text{res}(f; z_2) = \frac{1}{2\pi i} \oint_{|z-1|=1} \frac{(z+1)^{-2}}{(z-1)^2} dz = -2(z+1)^{-3}|_{z=1} = -\frac{1}{4}$$

3. $\sin(z)$ has zeros at $k\pi, k \in \mathbb{Z}$, hence $\sin(z)^{-1}$ has poles at $z_k = k\pi$. We can write $\sin(z) = (z - z_k)\left[\cos(z_k) + g_k(z)\right]$, where g_k is holomorphic and $g_k(z_k) = 0$. By the Cauchy integral formula,

$$\text{res}(f; z_k) = \frac{1}{2\pi i} \oint_{|z-z_k|=1} \frac{\left[f'(z_k) + g_k(z)\right]^{-1}}{(z - z_k)} dz = \frac{1}{f'(z_k) + g(z_k)} = (-1)^k$$

4. Set $f(z) = \cot(z)$. Since $\sin(z)$ has zeros at $z_k = k\pi, k \in \mathbb{Z}$ and $\cos(z_k) \neq 0$, $\cot(z)$ has poles at $z_k, k \in \mathbb{Z}$. We can write $\sin(z) = (z - z_k)\left[\cos(z_k) + g_k(z)\right]$, where g_k is holomorphic and $g_k(z_k) = 0$. By Cauchy's integral formula,

$$\text{res}(f; z_k) = \frac{1}{2\pi i} \oint_{|z-z_k|=1} \frac{\cos(z)\left[\cos(z_k) + g_k(z)\right]^{-1}}{(z - z_k)} dz = \frac{\cos(z_k)}{\cos(z_k) + g_k(z_k)} = 1$$

5. It follows from (3) that $f(z) = \sin(z)^{-2}$ has poles at $z_k = k\pi, k \in \mathbb{Z}$. We remark further that $g_k(z) = -\cos(z_k)(z - z_k)^2 + h_k(z)$, where $h_k(z)$ is holomorphic. By the Cauchy differentiation formula,

$$\text{res}(f; z_k) = \frac{1}{2\pi i} \oint_{|z-z_k|=1} \frac{\left[\cos(z_k) + g_k(z)\right]^{-2}}{(z - z_k)^2} dz = -2\frac{g_k'(z_k)}{(\cos(z_k) + g_k(z_k))^3} = 0$$

6. Evidently, the poles of $f(z) = \frac{1}{z^m(1-z)^n}$ are $z_1 = 0, z_2 = 1$. By Cauchy's differentiation formula,

$$\text{res}(f; z_1) = \frac{1}{2\pi i}\oint_{|z|=\frac{1}{2}} \frac{(1-z)^{-n}}{z^m}dz = \frac{(n+m-2)!}{(n-1)!(m-1)!} = \binom{n+m-2}{m-1}$$

$$\text{res}(f; z_2) = \frac{(-1)^n}{2\pi i}\oint_{|z-1|=\frac{1}{2}} \frac{z^{-m}}{(z-1)^n}dz = \frac{(-1)^n(-1)^{n-1}(m+n-2)!}{(m-1)!} = -\binom{n+m-2}{n-1}$$

4.5.3 Exercise 3

(a) Since $a + \sin^2(\theta) = a + \frac{1-\cos(2\theta)}{2} = 2\left[(2a+1) - \cos(2\theta)\right]$, we have

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{a+\sin^2(\theta)} = 2\int_0^{\frac{\pi}{2}} \frac{d\theta}{(2a+1)-\cos(2\theta)} = \int_0^{\pi} \frac{dt}{(2a+1)-\cos(t)} = \int_{-\pi}^{0} \frac{d\tau}{(2a+1)+\cos(\tau)}$$

$$= \int_0^{\pi} \frac{d\tau}{(2a+1)+\cos(\tau)}$$

where we make the change of variable $\tau = \theta - \pi$, and the last equality follows from the symmetry of the integrand. Ahlfors p. 155 computes $\int_0^{\pi} \frac{dx}{\alpha+\cos(x)} = \frac{\pi}{\sqrt{\alpha^2-1}}$ for $\alpha > 1$. Hence,

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{a+\sin^2(\theta)} = \frac{\pi}{\sqrt{(2a+1)^2-1}}$$

(b) Set

$$f(z) = \frac{z^2}{z^4+5z^2+6} = \frac{z^2}{(z^2+3)(z^2+2)} = \frac{z^2}{(z-\sqrt{3}i)(z+\sqrt{3}i)(z-\sqrt{2}i)(z+\sqrt{2}i)}$$

For $R >> 0$,

$$\gamma_1 : [-R, R] \to \mathbb{C}, \gamma_1(t) = t; \gamma_2 : [0,\pi] \to \mathbb{C}, \gamma_2(t) = Re^{it}$$

and let γ be the positively oriented closed curve formed by γ_1, γ_2. By the residue formula and applying the Cauchy integral formula to $\frac{e^{iz}}{z+ai}$ to compute $\text{res}(f; ai)$,

$$\int_\gamma f(z)dz = 2\pi i\text{res}(f; \sqrt{3}i) + 2\pi i\text{res}(f; \sqrt{2}i)$$

It is immediate from Cauchy's integral formula that

$$2\pi i\text{res}(f; \sqrt{3}i) = \int_{|z-i\sqrt{3}|=\epsilon} \frac{z^2(z+i\sqrt{3})^{-1}(z^2+2)^{-1}}{(z-i\sqrt{3})}dz = 2\pi i \cdot \frac{(i\sqrt{3})^2}{((i\sqrt{3})^2+2)(2i\sqrt{3})} = \sqrt{3}\pi$$

$$2\pi i\text{res}(f; \sqrt{2}i) = \int_{|z-i\sqrt{2}|=\epsilon} \frac{z^2(z+i\sqrt{2})^{-1}(z^2+3)^{-1}}{(z-i\sqrt{2})}dz = 2\pi i \cdot \frac{(i\sqrt{2})^2}{((i\sqrt{2})^2+3)(2i\sqrt{2})} = -\sqrt{2}\pi$$

Using the reverse triangle inequality, we obtain the estimate

$$\left|\int_{\gamma_2} f(z)dz\right| \leq \frac{\pi R^3}{|R^2-3|\,|R^2-2|} \to 0, R \to \infty$$

Hence,

$$2\int_0^{\infty} \frac{x^2}{x^4+5x^2+6} = \int_{-\infty}^{\infty} \frac{x^2}{x^4+5x^2+6}dx = (\sqrt{3}-\sqrt{2})\pi \Rightarrow \int_0^{\infty} \frac{x^2dx}{x^4+5x^2+6} = \frac{(\sqrt{3}-\sqrt{2})\pi}{2}$$

(e) We may write

$$\frac{\cos(x)}{x^2+a^2} = \text{Re } \frac{e^{ix}}{(x^2+a^2)}$$

So set $f(z) = \frac{e^{iz}}{z^2+a^2}$, which has simple poles at $\pm ai$. First, suppose that $a \neq 0$. For $R >> 0$, define

$$\gamma_1 : [-R, R] \to \mathbb{C}, \gamma_1(t) = t; \gamma_2 : [0, \pi] \to \mathbb{C}, \gamma_2(t) = Re^{it}$$

and let γ be the positively oriented closed curve formed by γ_1, γ_2. By the residue formula,

$$\int_\gamma f(z)dz = 2\pi i \text{res}(f; ai) = 2\pi i \cdot \frac{e^{i(ai)}}{(2ai)} = \frac{\pi e^{-a}}{a}$$

$$\left|\int_{\gamma_2} f(z)dz\right| = \left|\int_0^\pi \frac{e^{iR[\cos(t)+i\sin(t)]}}{R^2e^{2it}+a^2} Re^{it}dt\right| = \left|\int_0^\pi \frac{e^{iR\cos(t)}e^{-R\sin(t)}}{R^2e^{2it}+a^2} Re^{it}dt\right|$$

$$\leq \int_0^\pi \frac{Re^{-R\sin(t)}}{R^2-a^2}dt \leq \frac{\pi R}{R^2-a^2} \to 0, R \to \infty$$

since $e^{-R\sin(t)} \leq 1$ on $[0, \pi]$. Hence,

$$\int_0^\infty \frac{\cos(x)}{x^2+a^2} = \text{Re } \frac{1}{2}\int_{-\infty}^\infty \frac{e^{ix}}{x^2+a^2}dx = \frac{\pi e^{-a}}{2a}$$

If $a = 0$, then the integral does not converge.

(h) Define $f(z) = \frac{\log(z)}{(1+z^2)}$, where we take the branch of the logarithm with $\arg(z) \in [-\frac{\pi}{2}, \frac{3\pi}{2})$. For $R >> 0$, define

$$\gamma_1 : [-R, \frac{-1}{R}] \to \mathbb{C}, \gamma_1(t) = t; \gamma_2 : [, \pi] \to \mathbb{C}, \gamma_2(t) = \frac{-1}{R}e^{-it}; \gamma_3 : [\frac{1}{R}, R] \to \mathbb{C}, \gamma_3(t) = t; \gamma_4 : [0, \pi] \to \mathbb{C}, \gamma_4(t) = Re^{it}$$

and let γ be the positively oriented closed curve formed by the γ_i.

$$\left|\int_{\gamma_2} f(z)dz\right| \leq \int_0^\pi \frac{\left|\log|R|^{-1}\right| + \frac{3\pi}{2}}{\left|\frac{1}{R^2}-1\right|}\frac{1}{R}dt \leq \pi\frac{R(\log|R| + \frac{3\pi}{2})}{|R^2-1|} \to 0, R \to \infty$$

$$\left|\int_{\gamma_4} f(z)dz\right| \leq \int_0^\pi \frac{|\log|R| + it|}{R^2-1}Rdt \leq \pi\frac{R(\log|R| + \pi)}{R^2-1} \to 0, R \to \infty$$

By the residue formula and applying the Cauchy integral formula to $f(z)/(z+i)$ to compute $\text{res}(f; i)$,

$$\int_\gamma f(z)dz = 2\pi i \text{res}(f; i) = 2\pi i \cdot \frac{\log(z)}{(z+i)}|_{z=i} = 2\pi i \cdot \frac{\frac{\pi}{2}}{2i} = \frac{\pi^2}{2}$$

Hence,

$$\frac{\pi^2}{2} = \int_{\gamma_1} f(z)dz + \int_{\gamma_3} f(z)dz = \int_{-R}^{-\frac{1}{R}} \frac{\log(te^{i\pi})}{1+t^2}dt + \int_{\frac{1}{R}}^R \frac{\log(t)}{1+t^2}dt = \int_{-R}^{-\frac{1}{R}} \frac{\log(|t|)}{1+t^2}dt + \int_{\frac{1}{R}}^R \frac{\log(t)}{1+t^2}dt + \pi\int_{-R}^{-\frac{1}{R}} \frac{1}{1+t^2}dt$$

$$= 2\int_{\frac{1}{R}}^R \frac{\log(t)}{1+t^2} + \pi\int_{\frac{1}{R}}^R \frac{1}{1+t^2}dt = 2\int_{\frac{1}{R}}^R \frac{\log(t)}{1+t^2} + \frac{\pi^2}{2}$$

where we've used $\int_0^\infty \frac{1}{1+t^2}dt = \lim_{R\to\infty} \arctan(R) - \arctan(0) = \frac{\pi}{2}$. Hence,

$$\int_{\frac{1}{R}}^R \frac{\log(t)}{1+t^2}dt = 0 \Rightarrow \int_0^\infty \frac{\log(t)}{1+t^2}dt = 0$$

Lemma 1. *Let $U, V \subset \mathbb{C}$ be open sets, $F : U \to V$ a holomorphic function, and $u : V \to \mathbb{C}$ a harmonic function. Then $u \circ F : U \to \mathbb{C}$ is harmonic.*

Proof. Since $u \circ F$ is harmonic on U if and only if it is harmonic on any open disk contained in U about every point, we may assume without loss of generality that V is an open disk. Then there exists a holomorphic function $G : V \to \mathbb{C}$ such that $u = \text{Re}(G)$. Hence, $G \circ F : U \to \mathbb{C}$ is holomorphic and $\text{Re}(G \circ F) = u \circ F$, which shows that $u \circ F$ is harmonic. □

In what follows, a conformal map $f : \Omega \to \mathbb{C}$ is a bijective holomorphic map.

Harmonic Functions

4.6.2 Exercise 1

Let $u : D'(0;\rho) \to \mathbb{R}$ be harmonic and bounded. I am going to cheat a bit and assume Schwarz's theorem for the Poisson integral formula, even though Ahlfors discusses it in a subsequent section. Let

$$P_u(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\frac{re^{i\theta}-z}{re^{i\theta}+z}u(re^{i\theta})d\theta$$

denote the Poisson integral for u on some circle of fixed radius $r < \rho$. Since u is continuous, $P_u(z)$ is a harmonic function in the open disk $D(0;r)$ and is continuous on the boundary $\{|z| = r\}$. We want to show that u and P_u agree on the annulus, so that we can define a harmonic extension of u by setting $u(0) = P_u(0)$. Define

$$g(z) = u(z) - P_u(z)$$

and for $\epsilon > 0$ define

$$g_\epsilon(z) = g(z) + \epsilon \log\left(\frac{|z|}{r}\right) \ \forall 0 < |z| \leq r$$

Then g is harmonic in $D'(0;r)$ and continuous on the boundary. Furthermore, since u is bounded by hypothesis and P_u is bounded by construction on $\overline{D}(0;r)$, we have that g is bounded on $\overline{D}(0;r)$. $g_\epsilon(z)$ is harmonic in $D'(0;r)$ and continuous on the boundary since both its terms are. Since $\log\left(r^{-1}|z|\right) \to -\infty, z \to 0$, we have that

$$\limsup_{z\to 0} g_\epsilon(z) < 0$$

Hence, there exists $\delta > 0$ such that $0 < |z| \leq \delta \Rightarrow g_\epsilon(z) \leq 0$. Since g_ϵ is harmonic on the closed annulus $\{\delta \leq |z| \leq r\}$, we can apply the maximum principle. Hence, g_ϵ assumes its maximum in $\{|z| = \delta\} \cup \{|z| = r\}$. But, $g_\epsilon(z) \leq 0 \, \forall \, |z| = \delta$, by our choice of δ, and since u, P_u agree on $\{|z| = r\}$, we have that $g_\epsilon(z) = 0 \, \forall \, |z| = r$. Hence,

$$g_\epsilon(z) \leq 0 \ \forall 0 < |z| \leq r$$

Letting $\epsilon \to 0$, we conclude that $g(z) \leq 0 \ \forall 0 < |z| \leq r$, which shows that $u \leq P_u$ on the annulus. Applying the same argument to $h = P_u - u$, we conclude that $u = P_u$ on $0 < |z| \leq r$. Setting $u(0) = P_u(0)$ defines a harmonic extension of u on the closed disk.

4.6.2 Exercise 2

If $f : \Omega = \{r_1 < |z| < r_2\} \to \mathbb{C}$ is identically zero, then there is nothing to prove. Assume otherwise. Since the annulus is bounded, f has finitely many zeroes in the region. Hence, for $\lambda \in \mathbb{R}$, the function

$$g(z) = \lambda \log|z| + \log|f(z)|$$

is harmonic in $\Omega \setminus \{a_1, \cdots, a_n\}$, where $a_1, \cdots, a_n$ are the zeroes of f. Applying the maximum principle to $g(z)$, we see that $|g(z)|$ takes its maximum in $\partial\Omega$. Hence,

$$\lambda \log|z| + \log|f(z)| = g(z) \leq \max\{\lambda\log(r_1) + \log(M(r_1)), \lambda\log(r_2) + \log(M(r_2))\} \ \forall z \in \Omega \setminus \{a_1, \cdots, a_n\}$$

Thus, if $|z| = r$, then we have the inequality

$$\lambda\log(r) + \log(M(r)) \leq \max\{\lambda\log(r_1) + \log(M(r_1)), \lambda\log(r_2) + \log(M(r_2))\}$$

We now find $\lambda \in \mathbb{R}$ such that the two inputs in the maximum function are equal.

$$\lambda\log(r_1) + \log(M(r_1)) = \lambda\log(r_2) + \log(M(r_2)) \Rightarrow \lambda\log\left(\frac{r_1}{r_2}\right) = \log\left(\frac{M(r_2)}{M(r_1)}\right)$$

Hence, $\lambda = \log\left(\frac{M(r_2)}{M(r_1)}\right)\left(\log\left(\frac{r_1}{r_2}\right)\right)^{-1}$. Exponentiating both sides of the obtained inequality,

$$M(r) \leq \exp\left[\log(M(r_2)) + \log\left(\frac{M(r_2)}{M(r_1)}\right)\frac{\log\left(\frac{r_2}{r}\right)}{\log\left(\frac{r_1}{r_2}\right)}\right] = \exp\left[\log(M(r_2) + \log\left(\frac{M(r_1)}{M(r_2)}\right)\alpha\right]$$

$$= M(r_1)^{\alpha} M(r_2)^{1-\alpha}$$

where $\alpha = \log\left(\frac{r_2}{r}\right)\left(\log\left(\frac{r_2}{r_1}\right)\right)^{-1}$. I claim that equality holds if and only if $f(z) = az^{\lambda}$, where $a \in \mathbb{C}, \lambda \in \mathbb{R}$. It is obvious that equality holds if $f(z)$ is of this form. Suppose quality holds. Then by Weierstrass's extreme value theorem, for some $|z_0| = r$, we have

$$|f(z_0)| = M(r) = \left(\frac{r_1}{r}\right)^{\lambda} M(r_1) \Rightarrow \left|z_0^{\lambda} f(z_0)\right| = r_1^{\lambda} M(r_1)$$

But since the bound on the RHS holds for all $r_1 < |z| < r_2$, the Maximum Modulus Principle tells us that $z^{\lambda} f(z) = a \in \mathbb{C}\ \forall r_1 < |z| < r_2$. Hence, $f(z) = az^{-\lambda}$. But λ is an arbitrary real parameter, from which the claim follows.

4.6.4 Exercise 1

We seek a conformal mapping of the upper-half plane $\mathbb{H}^+$ onto the unit disk $\mathbb{D}$. lemma The map ϕ given by

$$\phi(z) = i\frac{1+z}{1-z}$$

is a conformal map of $\mathbb{D}$ onto $\mathbb{H}^+$ and is a bijective continuous map of $\partial\mathbb{D}$ onto $\mathbb{R} \cup \{\infty\}$, where $1 \mapsto \infty$. Its inverse is given by

$$\phi^{-1}(w) = \frac{w-i}{w+i}$$

Proof. The statements about conformality and continuity follow from a general theorem about the group of linear fractional transformations of the Riemann sphere (Ahlfors p. 76), so we just need to verify the images. For $z \in \mathbb{D}$,

$$\text{Im}(\phi(z)) = \text{Im}\left(i\frac{1+z}{1-z} \cdot \frac{1-\overline{z}}{1-\overline{z}}\right) = \frac{1-|z|^2}{|1-z|^2} > 0$$

since $|z| < 1$. Furthermore, observe that $\text{Im}(\phi(z)) = 0 \iff z \in \partial\mathbb{D}$. In particular, $\phi(1) = \infty$. For $w \in \mathbb{H}^+$,

$$\left|\phi^{-1}(w)\right|^2 = \frac{w-i}{w+i} \cdot \frac{\overline{w}+i}{\overline{w}-i} = \frac{|w|^2 - 2\text{Im}(w) + 1}{|w|^2 + 2\text{Im}(w) + 1} < 1$$

by hypothesis that $\text{Im}(z) > 0$. Furthermore, observe that $\left|\phi^{-1}(w)\right| = 1 \iff \text{Im}(w) = 0$. □

$\tilde{U} = U \circ \phi : \partial\mathbb{D} \to \mathbb{C}$ is a piecewise continuous function since U is bounded and we therefore can ignore the fact that $\phi(1) = \infty$. By Poisson's formula, the function

$$P_{\tilde{U}}(z) = \frac{1}{2\pi}\int_0^{2\pi} \text{Re}\ \frac{e^{i\theta}+z}{e^{i\theta}-z}\tilde{U}(e^{i\theta})d\theta$$

is a harmonic function in the open disk $\mathbb{D}$. By Lemma 1, the function

$$P_U(z) = P_{\tilde{U}} \circ \phi^{-1}(z) = \frac{1}{2\pi}\int_0^{2\pi} \text{Re}\ \frac{e^{i\theta}+\phi^{-1}(z)}{e^{i\theta}-\phi^{-1}(z)}\tilde{U}(e^{i\theta})d\theta$$

is harmonic in $\mathbb{H}^+$. Fix $w_0 \in \mathbb{D}$ and let $x_0 + iy_0 = z_0 = \phi^{-1}(w_0)$. Let $P_{w_0}(\theta)$ denote the Poisson kernel. We apply the change of variable $t = \varphi^{-1}(e^{i\theta})$ to obtain

$$\frac{1}{2\pi}P_{w_0}(\theta)\frac{d\theta}{dt} = \frac{1}{2\pi}\frac{1-\left|\frac{z_0-i}{z_0+i}\right|^2}{\left|\frac{z_0-i}{z_0+i} - \frac{t-i}{t+i}\right|^2} \cdot \frac{\left(1-\frac{t-i}{t+i}\right)^2}{-2\frac{t-i}{t+i}} = \frac{1}{2\pi}\frac{|z_0+i|^2 - |z_0-i|^2}{\left|(z_0-i) - \frac{t-i}{t+i}(z_0+i)\right|^2} \cdot \frac{((t+i)-(t-i))^2}{-2\left|t+i\right|^2}$$

$$= \frac{2}{\pi}\frac{y_0}{|(z_0-i)(t+i)-(t-i)(z_0+i)|^2} = \frac{2}{\pi} \cdot \frac{y_0}{2\left|z_0-t\right|^2} = \frac{1}{\pi} \cdot \frac{y_0}{(x_0-t)^2+y_0^2}$$

Hence,

$$P_U(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2} U(t)dt$$

is a harmonic function in $\mathbb{H}^+$. Furthermore, since the value of $P_{\tilde{U}}(z)$ for $|z|=1$ is given by $\tilde{U}(z)$ at the points of continuity and since $\phi^{-1}(\partial\mathbb{D}) = \mathbb{R}\cup\{\infty\}$, we conclude that

$$P_U(x,0) = P_{\tilde{U}} \circ \phi^{-1}(x,0) = \tilde{U}\circ\phi^{-1}(x,0) = U(x,0)$$

at the points of continuity $x \in \mathbb{R}$.

4.6.4 Exercise 5

I couldn't figure out how to show that $\log|f(z)|$ satisfies the mean-value property for $z_0 = 0, r = 1$ without first computing the value of $\int_0^\pi \log\sin(\theta)d\theta$.

Since $\sin(\theta) \leq \theta \; \forall \theta \in [0, \frac{\pi}{2}]$, $1 \geq \frac{\theta}{\sin(\theta)}$ is continuous on $[0, \frac{\pi}{2}]$, where we've removed the singularity at the origin. Hence, for $\delta > 0$,

$$\int_0^{\frac{\pi}{2}} \log\left|\frac{\theta}{\sin(\theta)}\right| d\theta = \lim_{\delta\to 0}\int_\delta^{\frac{\pi}{2}} \log\left|\frac{\theta}{\sin(\theta)}\right| d\theta = \int_0^{\frac{\pi}{2}} \log|\theta|\, d\theta - \lim_{\delta\to 0}\int_\delta^{\frac{\pi}{2}} \log|\sin(\theta)|\, d\theta$$

By symmetry, it follows that the improper integral $\int_{\frac{\pi}{2}}^{\pi} \log|\sin(\theta)|\, d\theta$ exists and therefore $\int_0^\pi \log|\sin(\theta)|\, d\theta$ exists. Again by symmetry, $\int_0^{\frac{\pi}{2}} \log(\sin(\theta))d\theta = \int_0^{\frac{\pi}{2}} \log\cos(\theta)d\theta$, hence

$$\int_0^{\frac{\pi}{2}} \log\sin(\theta)d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \log\sin(\theta)\cos(\theta)d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \log\left(\frac{1}{2}\sin(2\theta)\right)d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(2\theta)d\theta - \frac{\pi}{4}\log(2)$$

$$= \frac{1}{4}\int_0^{\pi} \sin(\vartheta)d\vartheta - \frac{\pi}{4}\log(2)$$

where we make the change of variable $\vartheta = 2\theta$ to obtain the last equality. Since $\int_0^\pi \log\sin(\theta)d\theta = 2\int_0^{\frac{\pi}{2}} \log\sin(\theta)d\theta$, we conclude that

$$\int_0^\pi \log\sin(\theta)d\theta = -\pi\log(2)$$

We now show that for $f(z) = 1+z$, $\log|f(z)|$ satisfies the mean-value property for $z_0 = 0, r = 1$. Observe that

$$\log\left|1+e^{i\theta}\right| = \frac{1}{2}\log\left|(1+\cos(\theta))^2+\sin^2(\theta)\right| = \frac{1}{2}\log\left|1+2\cos(\theta)+\cos^2(\theta)+\sin^2(\theta)\right| = \frac{1}{2}\log|2+2\cos(\theta)|$$

$$= \log|2| + \frac{1}{2}\log\left|\frac{1+\cos(\theta)}{2}\right|$$

Substituting and making the change of variable $2\vartheta = \theta$,

$$\int_0^{2\pi} \log\left|1+e^{i\theta}\right| d\theta = \int_0^{2\pi}\left[\log 2 + \frac{1}{2}\log\left|\cos^2(\theta)\right|\right] d\theta = 2\pi\log 2 + \int_0^{\pi} \log\left|\frac{1+\cos(2\vartheta)}{2}\right| d\vartheta = 2\pi\log 2 + \int_0^\pi \log\cos^2(\vartheta)d\vartheta$$

By symmetry, integrating $\log\cos^2(\theta)$ over $[0,\pi]$ is the same as integrating $\log\left|\sin^2(\theta)\right|$ over $[0,\pi]$. Hence,

$$\int_0^{2\pi} \log\left|1+e^{i\theta}\right| d\theta = 2\pi\log 2 + \int_0^\pi \log\left|\sin^2(\vartheta)\right| d\vartheta = 2\pi\log 2 + 2\int_0^\pi \log|\sin(\vartheta)|\, d\vartheta = 0$$

4.6.4 Exercise 6

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire holomorphic function, and suppose that $z^{-1}\text{Re}(f(z)) \to 0, z \to \infty$. By Schwarz's formula (Ahlfors (66) p. 168), we may write

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta|=R} \frac{\zeta+z}{\zeta-z}\text{Re}(f(\zeta))\frac{d\zeta}{\zeta} \ \forall |z| < R$$

Let $\epsilon > 0$ be given and $R_0 > 0$ such that $\forall R \geq R_0$, $\left|\frac{\text{Re}(f(z))}{z}\right| < \epsilon$. Let R be sufficiently large that $R > \frac{R}{2} > R_0$. By Schwarz's formula, $\forall \frac{R}{2} \leq |z| < R$,

$$|f(z)| \leq \frac{R\epsilon}{2\pi}\int_0^{2\pi}\left|\frac{Re^{i\theta}+z}{Re^{i\theta}-z}\right|d\theta \leq \frac{R\epsilon}{2\pi}\int_0^{2\pi}\frac{R+|z|}{R-|z|}d\theta = R\epsilon\cdot\frac{R+R}{R-\frac{R}{2}} = 4R\epsilon$$

Fix $z \in \mathbb{C}$ and let $\frac{R}{2} > \max\{R_0, |z|\}$. By Cauchy's differentiation formula,

$$|f'(z)| = \frac{1}{2\pi}\left|\int_{|w|=\frac{R}{2}}\frac{f(w)}{(w-z)^2}dw\right| \leq \frac{1}{2\pi}\int_0^{2\pi}\frac{\frac{R}{2}\left|f(\frac{R}{2}e^{i\theta})\right|}{\left|\frac{R}{2}e^{i\theta}-z\right|^2}d\theta$$

$$\leq \frac{1}{2\pi}\frac{R}{2}\cdot 4R\epsilon\int_0^{2\pi}\frac{1}{\left|\frac{R}{2}-|z|\right|^2}d\theta = 8\epsilon\frac{R^2}{(R-2|z|)^2}$$

Letting $R \to \infty$, we conclude that $|f'(z)| \leq 8\epsilon$. Since $z \in \mathbb{C}$ was arbitrary, we conclude that $|f'(z)|\, 8\epsilon \ \forall z \in \mathbb{C}$. Since $\epsilon > 0$ was arbitrary, we conclude that $f'(z) = 0$, which shows that f is constant.

4.6.5 Exercise 1

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire holomorphic function satisfying $f(\mathbb{R}) \subset \mathbb{R}$ and $f(i\cdot\mathbb{R}) \subset i\cdot\mathbb{R}$. Since $f(\mathbb{R}) \subset \mathbb{R}$, $f(z) - \overline{f(\overline{z})}$ vanishes on the real axis. By the limit-point uniqueness theorem that

$$f(z) = \overline{f(\overline{z})} \ \forall z \in \mathbb{C}$$

Since $f(i\mathbb{R}) \subset i\mathbb{R}$, $f(z) + \overline{f(-\overline{z})}$ vanishes on the imaginary axis. By the limit-point uniqueness theorem that

$$f(z) = -\overline{f(-\overline{z})} \ \forall z \in \mathbb{C}$$

Combining these two results, we have

$$f(z) = -\overline{f(-\overline{z})} = -\overline{f(\overline{-z})} = -f(-z) \ \forall z \in \mathbb{C}$$

4.6.5 Exercise 3

Let $f : \overline{\mathbb{D}} \to \mathbb{C}$ be holomorphic and satisfy $|f(z)| = 1 \ \forall |z| = 1$. Let $\phi : \mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}$ be the linear fractional transformation

$$\phi(z) = \frac{z-i}{z+i}$$

Consider the function $g = \phi^{-1}\circ f\circ\phi : \overline{\mathbb{H}}^+ \to \mathbb{C}$. By the maximum modulus principle, $|f(z)| \leq 1 \ \forall |z| \leq 1$. Hence, $g : \overline{\mathbb{H}}^+ \to \overline{\mathbb{H}}^+$. Since $|f(z)| = 1 \ \forall |z| = 1$, $\phi^{-1}(f(z)) \in \mathbb{R} \ \forall |z| = 1$. Hence, $\underline{\tilde{f}(\mathbb{R})} \subset \mathbb{R}$. By the Schwarz Reflection Principle, g extends to an entire function $g : \mathbb{C} \to \mathbb{C}$ satisfying $g(z) = \overline{g(\overline{z})}$. Define

$$\tilde{f} = \phi\circ g\circ\phi^{-1} : \mathbb{C} \to \mathbb{C}$$

Then $\tilde{f}$ is meromorphic in $\mathbb{C}$ since ϕ has a pole at $z = -i$ and ϕ^{-1} has a pole at $z = 1$. In particular, $\tilde{f}$ has finitely many poles. We proved in Problem Set 1 (Ahlfors Section 4.3.2 Exercise 4) that a function meromorphic in the extended complex plane is a rational function, so we need to verify that $\tilde{f}$ doesn't have an essential singularity at ∞. But in a neighborhood of 0,

$$\tilde{f}\left(\frac{1}{z}\right) = \phi\circ g\left(i\frac{1+\frac{1}{z}}{1-\frac{1}{z}}\right) = \phi\circ g\left(i\frac{z+1}{z-1}\right)$$

which is evidently a meromorphic function. Alternatively, we note that $\forall |z| \geq 1$, $\left|\tilde{f}(z)\right| \geq 1$ since g maps $\overline{\mathbb{H}}^-$ onto $\overline{\mathbb{H}}^-$. So the image of $\tilde{f}$ in a suitable neighborhood of ∞ is not dense in $\mathbb{C}$. The Casorati-Weierstrass theorem then tells us that $\tilde{f}$ cannot have an essential singularity at ∞.

Chapter 5 - Series and Product Developments

Power Series Expansions

5.1.1 Exercise 2

We know that in the region $\Omega = \{z : \text{Re}(z) > 1\}$, $\zeta(z)$ exists since

$$\left|\frac{1}{n^z}\right| = \frac{1}{n^{\text{Re}(z)}\left|n^{\text{Im}(z)i}\right|} = \frac{1}{n^{\text{Re}(z)}\left|e^{\log(n)\text{Im}(z)i}\right|} = \frac{1}{n^{\text{Re}(z)}}$$

and therefore $\sum_{n=1}^{\infty}\left|\frac{1}{n^z}\right|$ is a convergent harmonic series; absolute convergence implies convergence by completeness. Define $\zeta_N(z) = \sum_{n=1}^{N}\frac{1}{n^z}$. Clearly, ζ_N is the sum of holomorphic functions on the region Ω. I claim that $(\zeta_N)_{N\in\mathbb{N}}$ converge uniformly to ζ on any compact subset $K \subset \Omega$. Since K is compact and $z \mapsto \text{Re}(z)$ is continuous, by Weierstrass's Extreme Value Theorem $\exists z_0 \in K$ such that $\text{Re}(z_0) = \inf_{z\in K}\text{Re}(z)$. In particular, $\text{Re}(z_0) > 1$ since $z_0 \in \Omega$. Hence, $\left|\frac{1}{n^z}\right| \leq \frac{1}{n^{\text{Re}(z)}} \leq \frac{1}{n^{\text{Re}(z_0)}}$. So by the Triangle Inequality,

$$\forall z \in \Omega, \left|\sum_{n=1}^{N}\frac{1}{n^z}\right| \leq \sum_{n=1}^{N}\left|\frac{1}{n^z}\right| \leq \sum_{n=1}^{N}\frac{1}{n^{\text{Re}(z_0)}} < \sum_{n=1}^{\infty}\frac{1}{n^{\text{Re}(z_0)}} < \infty$$

By Weierstrass's M-test, we attain that $\zeta_n \to \zeta$ uniformly on K. Therefore by Weierstrass's theorem, ζ is holomorphic in Ω and

$$\zeta'(z) = \lim_{N\to\infty}\zeta_N'(z) = \lim_{N\to\infty}\sum_{n=1}^{N} -\log(n)e^{-\log(n)z} = \lim_{N\to\infty}\sum_{n=1}^{N}\frac{-\log(n)}{n^z} = \sum_{n=1}^{\infty}\frac{-\log(n)}{n^z}$$

Section 5.1.1 Exercise 3

Lemma 2. *Set $a_n = (-1)^{n+1}$. If $\sum_{n=1}^{\infty}\frac{a_n}{n^z}$ converges for some z_0. Then $\sum_{n=1}^{\infty}\frac{a_n}{n^z}$ converges uniformly on $\forall z \in \mathbb{C}$ with $Re(z) \geq Re(z_0)$.*

Proof. If $\sum_{n=1}^{\infty}\frac{a_n}{n^{z_0}}$ conveges, there exists an $M > 0$ which bounds the partial sums. Let $m \leq N \in \mathbb{N}$. Using summation by parts, we may write

$$\sum_{n=m}^{N}\frac{a_n}{n^z} = \sum_{n=m}^{N}\frac{a_n}{n^{z_0}}\frac{1}{n^{z-z_0}} = \frac{1}{N^{z-z_0}}\sum_{n=1}^{m-1}\frac{a_n}{n^{z_0}} - \sum_{n=m}^{N-1}\left(\sum_{k=1}^{n}\frac{a_k}{k^{z_0}}\right)\left(\frac{1}{(n+1)^{z-z_0}} - \frac{1}{n^{z-z_0}}\right)$$

Hence,

$$\left|\sum_{n=m}^{N}\frac{a_n}{n^z}\right| \leq M\frac{1}{|N^{z-z_0}|} + M\frac{1}{|n^{z-z_0}|} + M\sum_{n=m}^{N-1}\left|\frac{1}{(n+1)^{z-z_0}} - \frac{1}{n^{z-z_0}}\right|$$

Observe that

$$\left|\frac{1}{(n+1)^{z-z_0}} - \frac{1}{n^{z-z_0}}\right| = \left|e^{-\log(n+1)(z-z_0)} - e^{-\log(n)(z-z_0)}\right| = \left|\frac{-1}{z-z_0}\int_{\log(n)}^{\log(n+1)} e^{-t(z-z_0)}dt\right|$$

$$\leq \frac{1}{|z-z_0|}\int_{\log(n)}^{\log(n+1)} e^{-t(\mathrm{Re}(z)-\mathrm{Re}(z_0))}dt = \frac{|\mathrm{Re}(z)-\mathrm{Re}(z_0)|}{|z-z_0|}\left|e^{-\log(n+1)(\mathrm{Re}(z)-\mathrm{Re}(z_0))} - e^{-\log(n)(\mathrm{Re}(z)-\mathrm{Re}(z_0))}\right|$$

$$\leq e^{-\log(n)(\mathrm{Re}(z)-\mathrm{Re}(z_0))} - e^{-\log(n+1)(\mathrm{Re}(z)-\mathrm{Re}(z_0))} = \frac{1}{n^{\mathrm{Re}(z)-\mathrm{Re}(z_0)}} - \frac{1}{(n+1)^{\mathrm{Re}(z)-\mathrm{Re}(z_0)}}$$

Since this last expression is telescoping as the summation ranges over n, we have that

$$\left|\sum_{n=m}^{N}\frac{a_n}{n^z}\right| \leq \frac{M}{N^{\mathrm{Re}(z)-\mathrm{Re}(z_0)}} + \frac{M}{m^{\mathrm{Re}(z)-\mathrm{Re}(z_0)}} + M\left|\frac{1}{N^{\mathrm{Re}(z)-\mathrm{Re}(z_0)}} - \frac{1}{m^{\mathrm{Re}(z)-\mathrm{Re}(z_0)}}\right|$$

$$\leq \frac{2M}{m^{\mathrm{Re}(z)-\mathrm{Re}(z_0)}} + \frac{M}{m^{\mathrm{Re}(z)-\mathrm{Re}(z_0)}}\left|\left(\frac{m}{N}\right)^{\mathrm{Re}(z)-\mathrm{Re}(z_0)} - 1\right| \leq \frac{4M}{m^{\mathrm{Re}(z)-\mathrm{Re}(z_0)}} \to 0, m\to\infty$$

Hence, the partial sums of $\sum_{n=1}^{N}\frac{a_n}{n^{z-z_0}}$ are Cauchy and therefore converge by completeness. □

Corollary 3. *If $\sum_{n=1}^{\infty}\frac{a_n}{n^z}$ converges for some $z=z_0$, then $\sum_{n=1}^{\infty}\frac{a_n}{n^z}$ conveges uniformly on compact subsets of $\{Re(z)\geq Re(z_0)\}$.*

Proof. Let $K\subset\{\mathrm{Re}(z)\geq\mathrm{Re}(z_0)\}$ be compact. Since $z\mapsto\mathrm{Re}(z)$ is continuous, there exists $z_1\in K$ such that $\mathrm{Re}(z)\geq\mathrm{Re}(z_1)\ \forall z\in K$. Since $\mathrm{Re}(z_1)\geq\mathrm{Re}(z_0)$, $\sum_{n=1}^{\infty}\frac{a_n}{n^{z_1}}$ converges. The proof of the preceding lemma shows that we have a uniform bound

$$\left|\sum_{n=m}^{N}\frac{a_n}{n^z}\right| \leq \frac{4M}{m^{\mathrm{Re}(z)-\mathrm{Re}(z_0)}} \leq \frac{4M}{m^{\mathrm{Re}(z_1)-\mathrm{Re}(z_0)}}$$

where M depends only on z_0. The claim follows immediately from the M-test and completeness. □

Since the series $f(z)=\sum_{n=1}^{\infty}\frac{a_n}{n^z}$ converges if we take $z\in\mathbb{R}^{>0}$ (the well-known alternating series), we have by the lemma that $\sum_{n=1}^{\infty}\frac{a_n}{n^z}$ converges $\forall\mathrm{Re}(z)>0$. We now show that this series is holomorphic on the region $\{\mathrm{Re}(z)>0\}$.
Define a sequence of functions $(f_N)_{N\in\mathbb{N}}$ by

$$f_N = \sum_{n=1}^{N}\frac{(-1)^{n+1}}{n^z}$$

It is clear that f_N is holomorphic, being the finite sum of holomorphic functions. Set $\Omega=\{z\in\mathbb{C}:\mathrm{Re}(z)>0\}$ and let $K\subset\Omega$ be compact. Since the f_N are just the partial sums of the series, we have by the corollary to the lemma that $f_N\to f$ uniformly on K. By Weierstrass's theorem, f is holomorphic in Ω.
To see that $(1-2^{1-z})\zeta(z)=\sum_{n=1}^{\infty}\frac{a_n}{n^z}$ on $\{\mathrm{Re}(z)>1\}$, observe that

$$(1-2^{1-z})\sum_{n=1}^{N}\frac{1}{n^z} - \sum_{n=1}^{N}\frac{(-1)^{n+1}}{n^z} = \sum_{n=1}^{N}\frac{1}{n^z} - 2\sum_{n=1}^{N}\frac{1}{(2n)^z} - \sum_{n=1}^{N}\frac{(-1)^{n+1}}{n^z} = 2\sum_{\substack{N<n\leq 2N\\ n\text{ is even}}}\frac{1}{n^z}$$

Since $\sum_{n=1}^{\infty}\frac{1}{n^z}$ is absolutely convergent, we see that by taking N sufficiently large, the RHS can be made less than ϵ for $\epsilon>0$ given.

5.1.2 Exercise 2

Differentiating $\left(1-2\alpha z+z^2\right)^{-\frac{1}{2}}$ with respect to z, we obtain

$$p_1(\alpha) = \frac{2\alpha-2z}{2(1-2\alpha z+z^2)^{\frac{3}{2}}}|_{z=0} = \alpha$$

To compute higher order Legendre polynomials, we differentiate $\left(1-2\alpha z+z^2\right)^{-\frac{1}{2}}$ and its Taylor series to obtain the equality

$$\frac{\alpha-z}{(1-2\alpha z+z^2)^{\frac{3}{2}}} = \sum_{n=1}^{\infty}nP_n(\alpha)z^{n-1} \Rightarrow \frac{\alpha-z}{\sqrt{1-2\alpha z+z^2}} = (1-2\alpha z+z^2)\sum_{n=1}^{\infty}nP_n(\alpha)z^{n-1}$$

Hence,

$$\sum_{n=0}^{\infty} \alpha P_n(\alpha) z^n - \sum_{n=0}^{\infty} P_n(\alpha) z^{n+1} = \sum_{n=0}^{\infty} n P_n(\alpha) z^{n-1} - \sum_{n=0}^{\infty} 2\alpha n P_n(\alpha) z^n + \sum_{n=0}^{\infty} n P_n(\alpha) z^{n+1}$$

Invoking elementary limit properties and using the fact that a function is zero if and only if all its Taylor coefficients are zero, we may equate terms to obtain the recurrence

$$\alpha P_{n+1}(\alpha) - P_n(\alpha) = (n+2)P_{n+2}(\alpha) - 2\alpha(n+1)P_{n+1}(\alpha) + nP_n(\alpha)$$

$$\Rightarrow P_{n+2}(\alpha) = \frac{1}{n+2}\left[(2n+3)\alpha P_{n+1}(\alpha) - (n+1)P_n(\alpha)\right]$$

So,

$$P_2(\alpha) = \frac{1}{2}\left(3\alpha^2 - 1\right)$$

$$P_3(\alpha) = \frac{1}{3}\left(5\alpha\frac{1}{2}(3\alpha^2-1) - 2\alpha\right) = \frac{1}{2}\left(5\alpha^3 - 3\alpha\right)$$

$$P_4(\alpha) = \frac{1}{4}\left(7\alpha\frac{1}{2}(5\alpha^3-3\alpha) - 3\frac{1}{2}(3\alpha^2-1)\right) = \frac{1}{8}(35\alpha^4 - 30\alpha^2 + 3)$$

5.1.2 Exercise 3

Observe that

$$\frac{\sin(z)}{z} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n}$$

So, $\frac{\sin(z)}{z} \neq 0$ in some open disk about $z=0$. Hence, the function $z \mapsto \log\left(\frac{\sin(z)}{z}\right)$ is holomorphic in an open disk about $z=0$, where we take the principal branch of the logarithm. Substituting,

$$\log\left(\frac{\sin(z)}{z}\right) = \log\left(1 - \left(1 - \frac{\sin(z)}{z}\right)\right) = -\sum_{m=1}^{\infty}\frac{\left(1 - \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n}\right)^m}{m}$$

$$= -\sum_{m=1}^{\infty}\frac{\left(\frac{1}{3!}z^2 - \frac{1}{5!}z^4 + \frac{1}{7!}z^6 - [z^8]\right)^m}{m}$$

Set $P(z) = \frac{1}{3!}z^2 - \frac{1}{5!}z^4$. Then

$$\log\left(\frac{\sin(z)}{z}\right) = -\left[\frac{z^6}{7!} + \frac{P(z)+[z^8]}{1} + \frac{P(z)^2+[z^8]}{2} + \frac{P(z)^3+[z^8]}{3}\right]$$

$$= -\left[\frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} + \frac{1}{2}\left(\frac{z^4}{(3!)^2} - \frac{2z^6}{(3!)(5!)}\right) + \frac{z^6}{3(3!)^3} + [z^8]\right]$$

$$= -\frac{1}{6}z^2 - \frac{1}{180}z^4 - \frac{1}{2835}z^6 + [z^8]$$

Partial Fractions and Factorization

5.2.1 Exercise 1

From Ahlfors p. 189, we obtain for $|z| < 1$,

$$z\pi\cot(\pi z) = z\left(\frac{1}{z} + 2\sum_{n=1}^{\infty}\frac{z}{z^2-n^2}\right) = 1 - 2z^2\sum_{n=1}^{\infty}\frac{1}{n^2}\cdot\frac{1}{1-\frac{z^2}{n^2}} = 1 - 2z^2\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\sum_{k=0}^{\infty}\left(\frac{z^2}{n^2}\right)^k\right)$$

where we expand $\frac{z^2}{n^2}$ using the geometric series. Since both series are absolutely convergent, we may interchange the order of summation to obtain

$$z\pi\cot(\pi z) = 1 - 2z^2\sum_{k=0}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{n^{2(k+1)}}\right)z^{2k} = 1 - 2\sum_{k=0}^{\infty}\zeta(2k)z^{2k}$$

We now compute the Taylor series for $\pi z\cot(\pi z)$.

$$\pi z\cot(\pi z) = \pi z\frac{\cos(\pi z)}{\sin(\pi z)} = \pi iz\frac{e^{i\pi z}+e^{-i\pi z}}{e^{i\pi z}-e^{-i\pi z}} = \pi iz\frac{e^{i2\pi z}+1}{e^{i2\pi z}-1} = \frac{2\pi iz}{e^{2\pi iz}-1} + \frac{\pi iz(e^{2\pi iz}-1)}{e^{2\pi iz}-1} = \pi iz + \frac{2\pi iz}{e^{2\pi iz}-1}$$

Let $|z| < \frac{1}{2\pi}$. Then

$$z\pi\cot(\pi z) = \pi iz + \frac{2\pi iz}{\sum_{k=1}^{\infty}\frac{(2\pi iz)^k}{k!}} = \pi iz + \frac{1}{1-\left(-\sum_{k=1}^{\infty}\frac{(2\pi iz)^k}{(k+1)!}\right)} = \pi iz + \sum_{n=0}^{\infty}\left(-\sum_{k=1}^{\infty}\frac{(2\pi iz)^k}{(k+1)!}\right)^n$$

$$= \pi iz + \sum_{k=0}^{\infty}\frac{B_k}{k!}(2\pi iz)^k$$

where we may use the geometric expansion since $\left|\sum_{k=1}^{\infty}\frac{(2\pi iz)^k}{(k+1)!}\right| \leq \sum_{k=1}^{\infty}|2\pi z|^k < 1$ ($|z| < \frac{1}{2\pi}$), and the change in the order of summation is permitted since the series are absolutely convergent. According to Ahlfors, the numbers B_k are called Bernoulli numbers, the values of which one can look up. Since the two series representations for $\pi z\cot(\pi z)$ are equal, the coefficients must agree. Hence,

$$\zeta(2) = \frac{-1}{2}\frac{(2\pi i)^2B_2}{2!} = \frac{\pi^2}{6}$$

$$\zeta(4) = \frac{-1}{2}\frac{(2\pi i)^4B_4}{4!} = \frac{16\pi^4}{6}\cdot 60 = \frac{\pi^4}{90}$$

$$\zeta(6) = \frac{-1}{2}\frac{(2\pi i)^6B_6}{6!} = \frac{32\pi^6}{42\cdot 6!} = \frac{\pi^6}{21\cdot 45} = \frac{\pi^6}{945}$$

5.2.1 Exercise 2

We first observe that

$$\sum_{n=-\infty}^{\infty}\frac{1}{z^3-n^3}$$

converges absolutely, being comparable to $\sum_{n=1}^{\infty}\frac{1}{n^3}$. For $z \neq 0$, we may write (after some laborious computation, which can be found at the end of the solutions)

$$\frac{1}{z^3-n^3} = \frac{1}{(z-n)(z-ne^{i\frac{2\pi}{3}})(z-ne^{i\frac{4\pi}{3}})} = \frac{1}{(z-n)(e^{i\frac{2\pi}{3}}z-n)(e^{i\frac{4\pi}{3}}z-n)} = \frac{A}{z-n} + \frac{B}{ze^{i\frac{4\pi}{3}}-n} + \frac{C}{ze^{i\frac{2\pi}{3}}-n}$$

where

$$C = \frac{e^{\frac{2\pi}{3}i}}{3z^2}\ B = \frac{e^{\frac{4\pi}{3}i}}{3z^2}\ A = \frac{1}{3z^2}$$

Ahlfors p. 189 shows that $\lim_{m\to\infty}\sum_{-m}^{m}\frac{1}{z-n} = \pi\cot(\pi z), 0 < |z| < 1$. Hence, for $0 < |z| < 1$,

$$\lim_{m\to\infty}\sum_{-m}^{m}\frac{1}{z^3-n^3} = \frac{1}{3z^2}\lim_{m\to\infty}\sum_{-m}^{m}\frac{1}{z-n} + \frac{e^{\frac{2\pi}{3}i}}{3z^2}\lim_{m\to\infty}\sum_{-m}^{m}\frac{1}{ze^{i\frac{2\pi}{3}}-n} + \frac{e^{\frac{4\pi}{3}i}}{3z^2}\lim_{m\to\infty}\sum_{-m}^{m}\frac{1}{ze^{i\frac{4\pi}{3}}-n}$$

$$= \frac{\pi\cot(\pi z)}{3z^2} + \frac{\pi e^{\frac{2\pi}{3}i}\cot(\pi e^{\frac{2\pi}{3}i}z)}{3z^2} + \frac{\pi e^{\frac{4\pi}{3}i}\cot(\pi e^{\frac{4\pi}{3}i})}{3z^2}$$

5.2.2 Exercise 2

In what follows, we will restrict ourselves to $z \in D(0;1)$. For $n \in \mathbb{Z}^{\geq 0}$, define

$$P_n(z) = (1+z)\left(1+z^2\right)\cdots\left(1+z^{2^n}\right) = \prod_{i=1}^{n}\left(1+z^{2^i}\right)$$

First, I claim that $(1-z)P_n(z) = (1-z^{2^{n+1}})$. Suppose the claim is true for some n, then

$$(1-z)P_{n+1}(z) = [(1-z)P_n(z)]\,(1+z^{2^{n+1}}) = (1-z^{2^{n+1}})(1+z^{2^{n+1}}) = \left(1-z^{2^{n+2}}\right) = \left(1-z^{2^{(n+1)+1}}\right)$$

The base case is trivial, so the result follows by induction. Therefore,

$$\left|P_n(z) - \frac{1}{1-z}\right| \leq \frac{1}{1-|z|}\,|(1-z)P_n(z)-1| = |z|^{2^{n+1}} \to 0, n\to\infty$$

since $|z|<1$. Since $\frac{1}{1-|z|}, |z|$ are bounded on any compact subset of $D(0;1)$, we remark that the convergence is uniform on compact subsets of $D(0;1)$.

5.2.3 Exercise 3

First, note that even though the function $z \mapsto \sqrt{z}$ is not entire for any branch choice, the function $f(z) = \cos(\sqrt{z})$ is. Indeed, substituting into the definition of $\cos(z)$,

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(\sqrt{z})^{2n}$$

Since changing the choice of branch only results in a sign change, we see that $(\sqrt{z})^{2n} = z^n$, and therefore

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^n$$

which is evidently an entire function, being a power series with infinite radius of convergence. Observe that $f(z)$ has zero set $\left\{\left(\frac{(2n+1)\pi}{2}\right)^2 : n \in \mathbb{Z}\right\}$. Since $\sin(z+\frac{\pi}{2}) = \cos(z)$, $\cos(\pi z)$ can be written as

$$\cos(\pi z) = \pi\left(z+\frac{1}{2}\right)\prod_{n\neq 0}\left(1-\frac{z+\frac{1}{2}}{n}\right)e^{\frac{z+\frac{1}{2}}{n}} = \pi\left(z+\frac{1}{2}\right)\prod_{n\neq 0}\left(\frac{2n-1}{2n}-\frac{2z}{2n}\right)e^{\frac{1}{2n}+\frac{z}{n}}$$

$$= \frac{\pi}{2}\left(1-\frac{z}{\frac{-1}{2}}\right)\prod_{n\neq 0}\frac{2n-1}{2n}\left(1-\frac{z}{\frac{2n-1}{2}}\right)e^{\frac{1}{2n}+\frac{z}{n}} = \frac{\pi}{2}\prod_{n=1}^{\infty}\left(1-\frac{1}{4n^2}\right)\left(1-\frac{z^2}{\frac{(2n-1)^2}{4}}\right)$$

Using the infinite product representation of $\sin(z)$, we have

$$\frac{2}{\pi} = \frac{\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} = \prod_{n=1}^{\infty}\left(1-\frac{\left(\frac{\pi}{2}\right)^2}{n^2\pi^2}\right) = \prod_{n=1}^{\infty}\left(1-\frac{4}{n^2}\right)$$

Hence,

$$\cos(\pi z) = \prod_{n=1}^{\infty}\left(1-\frac{z^2}{\frac{(2n-1)^2}{4}}\right) \Rightarrow \cos(z) = \prod_{n=1}^{\infty}\left(1-\frac{z^2}{\left(\frac{(2n-1)\pi}{2}\right)^2}\right)$$

Hence, $f(z)$ has the canonical product representation

$$f(z) = \prod_{n=1}^{\infty}\left(1-\frac{z}{\left(\frac{(2n-1)\pi}{2}\right)^2}\right)$$

Since

$$\sum_{n=1}^{\infty}\left(\frac{(2n-1)\pi}{2}\right)^{-2} = \frac{\pi^2}{4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} < \infty$$

being comparable to $\sum \frac{1}{n^2}$, we see that $f(z)$ is an entire function of genus zero.

5.2.3 Exercise 4

Let $f(z)$ be an entire function of genus h. Let $\{a_n \neq 0\}_{n\in\mathbb{N}}$ denotes the (at most countable) set of nonzero zeroes of f and h_c denote the genus of the canonical product. We may write

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{h_c}\left(\frac{z}{a_n}\right)^{h_c}}$$

where $g(z)$ is a polynomial and $h = \max\left(\deg(g(z)), h_c\right)$. Hence,

$$f(z^2) = z^{2m} e^{g\left(z^2\right)} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{a_n}\right) e^{\frac{z^2}{a_n} + \frac{1}{2}\left(\frac{z^2}{a_n}\right)^2 + \cdots + \frac{1}{h_c}\left(\frac{z^2}{a_n}\right)^{h_c}}$$

$$= z^{2m} e^{g\left(z^2\right)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\sqrt{a_n}}\right)\left(1 + \frac{z}{\sqrt{a_n}}\right) e^{\frac{z}{\sqrt{a_n}} + \frac{1}{2}\left(\frac{z}{\sqrt{a_n}}\right)^2 + \cdots + \frac{1}{2h_c+1}\left(\frac{z}{\sqrt{a_n}}\right)^{2h_c+1}} e^{\frac{z}{-\sqrt{a_n}} + \frac{1}{2}\left(\frac{z}{-\sqrt{a_n}}\right)^2 + \cdots + \frac{1}{2h_c+1}\left(\frac{z}{-\sqrt{a_n}}\right)^{2h_c+1}}$$

where we've chosen some branch of the square root. If we define $b_1 = \sqrt{a_1}, b_2 = -\sqrt{a_1}, \cdots$. Then

$$\tilde{f}(z) = f(z^2) = z^{2m} e^{g\left(z^2\right)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{b_n}\right) e^{\frac{z}{b_n} + \frac{1}{2}\left(\frac{z}{b_n}\right)^2 + \cdots + \frac{1}{2h_c+1}\left(\frac{z}{b_n}\right)^{2h_c+1}}$$

the breaking up of the product being justified since the individual products converge absolutely by virtue of

$$\sum \frac{1}{|b_n|^{2h_c+1+1}} = \sum \frac{1}{|a_n|^{h_c+1}} < \infty$$

I claim that the genus of $\tilde{f}$ is bounded from below by h. If $h = 0$, then there is nothing to prove; assume otherwise. If $h = \deg(g(z)) > 0$, then $\tilde{h} \geq \deg(g(z^2)) > h$; so assume that $h = h_c$. We will show that the genus $\tilde{h}_c$ of the canonical product associated to (b_n) is bounded from below by $2h_c$. Suppose $\tilde{h}_c < 2h_c$. Since $a_n \to \infty$ and therefore $b_n \to \infty$ by continuity, we have that for all n sufficiently large $|b_n| > 1$. So it suffices to consider the case $\tilde{h}_c = 2h_c - 1$. Then

$$\infty > \sum_{n=1}^{\infty} \frac{1}{|b_n|^{\tilde{h}_c+1}} = \sum_{n=1}^{\infty} \frac{1}{|b_n|^{2h_c-1+1}} = \sum_{n=1}^{\infty} \frac{1}{|a_n|^{h_c}}$$

But this shows that the genus of the canonical product associated to (a_n) is at most $h_c - 1$, which is obviously a contradiction. Taking f to be a polynomial shows that this bound is sharp.
I claim that the genus of $\tilde{f}$ is bounded from above by $2h+1$. Indeed, $2h+1 \geq 2\deg(g(z)) = \deg(g(z^2))$, and we showed above that $\tilde{h}_c \leq 2h_c + 1 \leq 2h+1$. This bound is also sharp since we can take

$$f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n^2}\right) \Rightarrow f(z^2) = \prod_{n\neq 0} \left(1 - \frac{z}{n}\right) e^{\frac{z}{n}}$$

$f(z)$ is clearly an entire function of genus 0, and the genus of the canonical product associated to $(n)_{n\in\mathbb{Z}}$ is 1, from which we conclude the genus of $f(z^2)$ is 1 .

5.2.4 Exercise 2

Using Legendre's duplication formula for the gamma function (Ahlfors p. 200),

$$\Gamma\left(\frac{1}{6}\right) = \sqrt{\pi}\Gamma\left(2 \cdot \frac{1}{6}\right) 2^{1-2\cdot\frac{1}{6}} \Gamma\left(\frac{1}{6} + \frac{1}{2}\right)^{-1} = \sqrt{\pi} 2^{\frac{2}{3}} \Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{2}{3}\right)$$

Applying the formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ (Ahlfors p. 199), we obtain

$$\Gamma\left(\frac{1}{6}\right) = \sqrt{\pi} 2^{\frac{2}{3}} \Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{1}{3}\right) \frac{\sin\left(\pi \cdot \frac{1}{3}\right)}{\pi} = 2^{-\frac{1}{3}} \left(\frac{3}{\pi}\right)^{\frac{1}{2}} \left(\Gamma\left(\frac{1}{3}\right)\right)^2$$

5.2.4 Exercise 3

It is clear from the definition of the Gamma function that for each $k \in \mathbb{Z}^{\leq 0}$,

$$f(z) = \begin{cases} \left(1 + \frac{z}{k}\right)\Gamma(z) & k \neq 0 \\ z\Gamma(z) & k = 0 \end{cases}$$

extends to a holomorphic function in an open neighborhood of k. We abuse notation and denote the extension also by $\left(1+\frac{z}{k}\right)\Gamma(z)$ and $z\Gamma(z)$. lemma For any $k \in \mathbb{Z}^{>0}$,

$$\Gamma(z) = \frac{\Gamma(z+k)}{\prod_{j=1}^{k}(z+j-1)} \ \forall z \notin \mathbb{Z}$$

Proof. Recall that $\Gamma(z)$ has the property that the $\Gamma(z+1) = z\Gamma(z)$. We proceed by induction. The base case is trivial, so assume that $\Gamma(z) = \frac{\Gamma(z+k)}{\prod_{j=1}^{k}(z+j-1)}$ for some $k \in \mathbb{N}$. Then

$$\frac{\Gamma(z+(k+1))}{\prod_{j=1}^{k+1}(z+j-1)} = \frac{\Gamma((z+k)+1)}{\prod_{j=1}^{k+1}(z+j-1)} = \frac{(z+k)\Gamma(z+k)}{\prod_{j=1}^{k+1}(z+j-1)} = \frac{\Gamma(z+k)}{\prod_{j=1}^{k}(z+j-1)!} = \Gamma(z)$$

□

Corollary 4. *For any* $k \in \mathbb{Z}^{\leq 0}$,

$$\lim_{z\to k}(z-k)\Gamma(z) = \frac{(-1)^k}{|k|!}$$

Proof. Fix $k \in \mathbb{Z}^{\leq 0}$. Immediate from the preceding lemma is that

$$\lim_{z\to -|k|}(z+|k|)\Gamma(z) = \lim_{z\to -|k|}(z+|k|)\frac{\Gamma(z+|k|+1)}{\prod_{j=1}^{k+1}(z+|j|-1)} = \frac{\Gamma(1)}{(-1)(-2)\cdots(-|k|)} = \frac{(-1)^k}{|k|!}$$

□

Let $k \in \mathbb{Z}^{\leq 0}$. Then

$$\operatorname{res}(\Gamma; k) = \frac{1}{2\pi i}\int_{|z-k|=\frac{1}{2}}\Gamma(z)dz = \frac{1}{2\pi i}\int_{|z-k|=\frac{1}{2}}\frac{(1-\frac{z}{k})\Gamma(z)}{1-\frac{z}{k}}dz = \frac{1}{2\pi i}\int_{|z-k|=\frac{1}{2}}\frac{(z-k)\Gamma(z)}{z-k}dz$$

Since the function $\left(1+\frac{z}{k}\right)\Gamma(z)$ extends to a holomorphic function in a neighborhood of k, by Cauchy's integral formula,

$$\frac{1}{2\pi i}\int_{|z-k|=\frac{1}{2}}\frac{(z-k)\Gamma(z)}{z-k}dz = (z-k)\Gamma(z)|_{z=k} = \frac{(-1)^k}{|k|!}$$

where use the preceding lemma to obtain the last equality. Thus,

$$\operatorname{res}(\Gamma; k) = \frac{(-1)^k}{|k|!} \ \forall k \in \mathbb{Z}^{\leq 0}$$

5.2.5 Exercise 2

Lemma 5.

$$\int_0^\infty \log\left(\frac{1}{1-e^{-2\pi x}}\right)dx = \frac{\pi}{12}$$

Proof. Let $1 >> \delta > 0$. Consider the function $\frac{\log(1-z)}{z}$, which has the power series representation

$$\frac{\log(1-z)}{z} = -\frac{1}{z}\sum_{n=1}^{\infty}\frac{1}{n}z^n = \sum_{n=1}^{\infty}\frac{1}{n}z^{n-1} \ \forall |z| < 1$$

with the understanding that the singularity at $z=0$ is removable. Since the convergence is uniform on compact subsets, we may integrate over the contour $\gamma_\delta : [0, 1-\delta] \to \mathbb{C}, \gamma_\delta(t) = t$ term by term, Thus,

$$\int_{\gamma_\delta} \frac{\log(1-z)}{z} dz = \int_0^{1-\delta} \frac{\log(1-t)}{t} dt = \sum_{n=1}^{\infty} \frac{1}{n^2} (1-\delta)^n \to \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}]$$

since the function $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ is left-continuous at $x=1$. Hence,

$$\int_0^1 \frac{\log(1-t)}{t} dt = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

We now make the change of variable $t = e^{-2\pi x}$ to obtain

$$\frac{\pi^2}{6} = \int_0^\infty \frac{\log(1-e^{-2\pi x})}{e^{-2\pi x}} - 2\pi e^{-2\pi x} dx = -2\pi \int_0^\infty \log(1-e^{-2\pi x}) dx$$

which gives

$$\int_0^\infty \log\left(\frac{1}{1-e^{-2\pi x}}\right) dx = \int_0^\infty -\log\left(1-e^{-2\pi x}\right) dx = \frac{\pi}{12}$$

□

For $x \in \mathbb{R}^{>0}$, Stirling's formula (Ahlfors p. 203-4) for $\Gamma(z)$ tells us that

$$\Gamma(x) = \sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x} e^{J(x)}$$

where

$$J(x) = \frac{1}{\pi} \int_0^\infty \frac{x}{\eta^2 + x^2} \log\left(\frac{1}{1-e^{-2\pi\eta}}\right) d\eta$$

The preceding lemma tells us that

$$J(x) = \frac{1}{x} \cdot \frac{1}{\pi} \int_0^\infty \frac{x^2}{x^2+\eta^2} \log\left(\frac{1}{1-e^{-2\pi\eta}}\right) d\eta \le \frac{1}{x} \cdot \frac{1}{\pi} \int_0^\infty \log\left(\frac{1}{1-e^{-2\pi\eta}}\right) d\eta = \frac{1}{x} \cdot \frac{1}{\pi} \cdot \frac{\pi}{12} = \frac{1}{12x}$$

where we've used $0 < \frac{x^2}{x^2+\eta^2} \le 1 \; \forall \eta$. Set

$$\theta(x) = 12xJ(x)$$

It is obvious that $\theta(x) > 0$ and $\theta(x) < 1$ since $\frac{x^2}{x^2+\eta^2} < 1$ almost everywhere, and therefore the preceding inequality is strict. We thus conclude that

$$\Gamma(x) = \sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x} e^{\frac{\theta(x)}{12x}} \quad 0 < \theta(x) < 1$$

5.2.5 Exercise 3

Take $f(z) = e^{-z^2}$, and for $R >> 0$, define

$$\gamma_1 : [0, R] \to \mathbb{C}, \gamma_1(t) = t; \gamma_2 : [0, \frac{\pi}{4}] \to \mathbb{C}, \gamma_2(t) = Re^{it}; \gamma_3 : [0, R] \to \mathbb{C}, \gamma_3(t) = (R-t)e^{i\frac{\pi}{4}}$$

and let γ be the positively oriented closed curve defined by the γ_i.

$$\left| \int_{\gamma_2} f(z) dz \right| = \left| \int_0^{\frac{\pi}{4}} e^{-R\cos(2t) - iR\sin(2t)} Rie^{it} dt \right| \le \int_0^{\frac{\pi}{4}} e^{-R\cos(2t)} R dt$$

Since $\cos(2t)$ is nonnegative and $\cos(2t) \ge 2t$ (this is immediate from $\frac{d}{dt}\cos(2t) = -2\sin(2t) \ge -2$ on $[0, \frac{\pi}{4}]$) for $t \in [0, \frac{\pi}{4}]$, we have

$$\int_0^{\frac{\pi}{4}} e^{-R\cos(2t)} R dt \le \int_0^{\frac{\pi}{4}} e^{-2Rt} R dt = -\frac{1}{2}\left[e^{-R\frac{\pi}{2}} - 1\right] \to 0, R \to \infty$$

Since f is an entire function, by Cauchy's theorem,

$$0 = \int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz$$

and letting $R \to \infty$,

$$\int_0^\infty e^{-x^2} dx = \lim_{R\to\infty} e^{i\frac{\pi}{4}} \int_0^R e^{-(R-t)^2 e^{i\frac{\pi}{2}}} dt = \lim_{R\to\infty} e^{i\frac{\pi}{4}} \int_0^R e^{-i(R-t)^2} dt = e^{i\frac{\pi}{4}} \int_0^\infty e^{-iy^2} dy$$

where we make the substitution $y = R - t$. Substituting $\int_0^\infty e^{-x^2} dx = 2^{-1}\sqrt{\pi}$,

$$\int_0^\infty \cos(x^2)dx - i\int_0^\infty \sin(x^2)dx = \int_0^\infty e^{-ix^2} dx = e^{-i\frac{\pi}{4}} \frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{2\sqrt{2}} - i\frac{\sqrt{\pi}}{2\sqrt{2}}$$

Equating real and imaginary parts, we obtain the Fresnel integrals

$$\int_0^\infty \cos(x^2)dx = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

$$\int_0^\infty \sin(x^2)dx = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

Entire Functions

5.3.2 Exercise 1

We will show that the following two definitions of the **genus** of an entire function f are equivalent:

1. If

$$f(z) = z^m e^{g(z)} \prod_{n=1}^\infty \left(1 - \frac{z}{a_n}\right) e^{\sum_{j=1}^h \frac{1}{j}\left(\frac{z}{a_n}\right)^j}$$

   where h is the genus of the canonical product associated to (a_n), then the genus of f is $\max(\deg(g(z)), h)$. If no such representation exists, then f is said to be of infinite genus.

2. The genus of f is the minimal $h \in \mathbb{Z}^{\geq 0}$ such that

$$f(z) = z^m e^{g(z)} \prod_{n=1}^\infty \left(1 - \frac{z}{a_n}\right) e^{\sum_{j=1}^h \frac{1}{j}\left(\frac{z}{a_n}\right)^j}$$

   where $\deg(g(z)) \leq h$. If no such h exists, then f is said to be of infinite genus.

Proof. Suppose f has finite genus h_1 with respect to definition (1). If $h_1 = h$, then $\deg(g(z)) \leq h_1$. Hence, f is of a finite genus h_2 with respect to definition (2), and $h_2 \leq h_1$. Assume otherwise. By definition of the genus of the canonical product, the expression

$$\sum_{n=1}^\infty \sum_{j=h+1}^{h_1} \frac{1}{j}\left(\frac{z}{a_n}\right)^j = \sum_{j=h+1}^{h_1} \frac{1}{j}\left(\sum_{n=1}^\infty \frac{1}{a_n^j}\right) z^j$$

defines a polynomial of degree h_1. Hence, we may write

$$f(z) = z^m e^{g(z) - \sum_{n=1}^\infty \sum_{j=h+1}^{h_1} \frac{1}{j}\left(\frac{z}{a_n}\right)^j} \prod_{n=1}^\infty \left(1 - \frac{z}{a_n}\right) e^{\sum_{j=1}^{h_1} \frac{1}{j}\left(\frac{z}{a_n}\right)^j} = z^m e^{\tilde{g}(z)} \prod_{n=1}^\infty \left(1 - \frac{z}{a_n}\right) e^{\sum_{j=1}^{h_1} \frac{1}{j}\left(\frac{z}{a_n}\right)^j}$$

where we $\tilde{g}(z)$ is a polynomial of degree h_1. Hence, f is of finite genus h_2 with respect to definition (2) and $h_2 \leq h_1$.

Now suppose that f has finite genus h_2 with respect to definition (2). Reversing the steps of the previous argument, we attain that f has finite genus h_1 with respect to definition (1), and $h_1 \leq h_2$. It follows immediately that definitions (1) and (2) are equivalent if f has finite genus with respect to either (1) and (2), and by proving the contrapositives, we see that (1) and (2) are equivalent for all entire functions f. □

5.3.2 Exercise 2

lemma Let $a \in \mathbb{C}$ and $r > 0$. Then

$$\inf_{|z|=r} |z - |a|| = |r - |a|| \text{ and } \sup_{|z|=r} |z - |a|| = r + |a|$$

Proof. By the triangle inequality and reverse inequality, we have the double inequality

$$|r - |a|| = ||z| - |a|| \leq |z - |a|| \leq |z| + |a| = r + |a|$$

Hence, $\inf |z - |a|| \geq |r - |a||$ and $\sup |z - |a|| \leq r + |a|$. But these values are attained at $z = r$ and $z = -r$, respectively. □

By Weierstrass's extreme value theorem, $|f|$ and $|g|$ attain both their maximum and minimum on the circle $\{|z| = r\}$ at $z_{M,f}, z_{M,g}$ and $z_{m,f}, z_{m,g}$, respectively. The preceding lemma shows that $z_{M,g} = -r$ and $z_{m,g} = r$. Consider the expression

$$\left|\frac{f(z)}{g(z)}\right| = \left|\frac{z^m \prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right)}{z^m \prod_{n=1}^{\infty}\left(1 - \frac{z}{|a_n|}\right)}\right| = \prod_{n=1}^{\infty} \frac{|z - a_n|}{|z - |a_n||}$$

We have

$$\left|\frac{f(z_{M,f})}{g(z_{M,g})}\right| = \left|\frac{f(z_{M,f})}{g(-r)}\right| = \prod_{n=1}^{\infty} \frac{|re^{i\theta_{M,f}} - a_n|}{r + |a_n|} = \prod_{n=1}^{\infty} \frac{\left|re^{i(\theta_{M,f} - \arg(a_n))} - |a_n|\right|}{r + |a_n|}$$

$$\leq \prod_{n=1}^{\infty} \frac{\sup_{|z|=r} |z - |a_n||}{r + |a_n|} = \prod_{n=1}^{\infty} \frac{r + |a_n|}{r + |a_n|} = 1$$

Hence, $|f(z_{M,f})| \leq |g(z_{M,g})|$. Since

$$|z_{m,f} - a_n| = \left|re^{i(\theta_{m,f} - \arg(a_n))} - |a_n|\right| \geq |r - |a_n|| \ \ \forall n \in \mathbb{N}$$

we have that

$$\left|\frac{f(z_{m,f})}{g(z_{m,g})}\right| = \left|\frac{f(z_{m,f})}{g(r)}\right| = \prod_{n=1}^{\infty} \frac{|z_{m,f} - a_n|}{|r - |a_n||} \geq \prod_{n=1}^{\infty} \frac{|z_{m,f} - a_n|}{|z_{m,f} - a_n|} = 1$$

Hence, $|f(z_{m,f})| \geq |g(z_{m,g})|$.

5.5.5 Exercise 1

Let Ω be a fixed region and $\mathcal{F}$ be the family of holomorphic functions $f : \Omega \to \mathbb{C}$ with $\text{Re}(f(z)) > 0 \ \forall z \in \Omega$. I claim that $\mathcal{F}$ is normal. Consider the family of functions

$$\mathcal{G} = \left\{g : \Omega \to \mathbb{C} : g = e^{-f} \text{ for some } f \in \mathcal{F}\right\}$$

Since $\text{Re}(f(z)) > 0 \ \forall f \in \mathcal{F}$, we have

$$\left|e^{-f(z)}\right| = \left|e^{-\text{Re}(f(z)) - i\text{Im}(f(z))}\right| = \left|e^{-\text{Re}(f(z))}\right| \leq 1$$

Hence, $\mathcal{G}$ is uniformly bounded on compact subsets of Ω and is therefore a normal family. Fix a sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{F}$, and consider the sequence $g_n = e^{-f_n}$. (g_n) has a convergent subsequence (g_{n_k}) which converges to a holomorphic function g on compact sets (Weierstrass's theorem). Since g_{n_k} is nonvanishing for each k, g is either identically zero or nowhere zero by Hurwitz theorem. If g is identically zero, then it is immediate that f_{n_k} tends to ∞ uniformly on compact sets. Now, suppose that g is nowhere zero. $g(K) \subset \mathbb{D} \setminus \{0\}$ is compact by continuity. By the Open Mapping Theorem, for each $z \in K$, there exists $r > 0$ such that $D(g(z); r) \subset g(\Omega) \subset \mathbb{D} \setminus \{0\}$. The disks $D(g(z); 4^{-1}r)$ form an open cover of $g(K)$, so by compactness,

$$g(K) \subset \bigcup_{i=1}^{n} D(g(z_i); 4^{-1}r_i) \subset \bigcup_{i=1}^{n} \overline{D}(g(z_i); 2^{-1}r_i) \subset \bigcup_{i=1}^{n} D(g(z_i); r_i) \subset \mathbb{D} \setminus \{0\}$$

On each $D(g(z_i);r_i)$, we can choose a branch of the logarithm such that $\log(z)$ is holomorphic on $D(g(z_i);r_i)$, and in particular uniformly continuous on $\overline{D}(g(z_i);2^{-1}r_i)$. For each i, choose $\delta_i > 0$ such that

$$w, w' \in \overline{D}(g(z_i);r_i) \ \ |w - w'| < \delta_i \Rightarrow |\log(w) - \log(w')| < \epsilon$$

Set $\delta = \min_{1\leq i\leq n} \delta_i$, choose $k_0 \in \mathbb{N}$ such that $k \geq k_0 \Rightarrow |g_{n_k}(z) - g(z)| < \delta \ \forall z \in K$. Then for $1 \leq i \leq n$,

$$\forall k \geq k_0 \quad \left|\log\left(e^{-f_{n_k}(z)}\right) - \log(g(z))\right| < \epsilon \ \forall z \in g^{-1}\left(\overline{D}(g(z_i);2^{-1}r_i)\right)$$

It is not *a priori* true that $\log(e^{-f_{n_k}(z)}) = -f_{n_k}(z)$; the imaginary parts differ by an integer multiple of $2\pi i$. But the function given by $\frac{1}{2\pi i}\left[\log(e^{-f_{n_k}(z)}) + f_{n_k}(z)\right]$ is continuous and integer-valued on any open disk about each z_i in Ω, and therefore must be a constant $m \in \mathbb{Z}$ in that disk as a consequence of connectedness. Taking a new covering of $g(K)$, if necessary, such that $D(g(z_i);r_i)$ is contained in the image under g of such a disk (which we can do by the Open Mapping Theorem), we may assume that for each $z \in g^{-1}(D(g(z_i);r_i))$,

$$2\pi m_i = \lim_{k\to\infty}\left[\log\left(e^{-f_{n_k}(z)}\right) + f_{n_k}(z)\right] = \log(g(z)) + \lim_{k\to\infty} f_{n_k}(z)$$

Taking $k_0 \in \mathbb{N}$ larger if necessary, we conclude that

$$\forall k \geq k_0 \ \left|\log\left(e^{-f_{n_k}(z)}\right) - \log(g(z))\right| = |f_{n_k}(z) - [-\log(g(z)) + 2\pi m_i]| < \epsilon \ \forall z \in g^{-1}\left(\overline{D}(g(z_i);2^{-1}r_i)\right)$$

Since $K \subset \bigcup_{i=1}^n g^{-1}\left(\overline{D}(g(z_i);2^{-1}r_i)\right)$, we conclude from the uniqueness of limits that $f_{n_k}(z)$ converges to $\lim_{k\to\infty} f_{n_k}(z)$ uniformly on K.

Suppose in addition that $\{\text{Re}(f) : f \in \mathcal{F}\}$ is uniformly bounded on compact sets. I claim that $\mathcal{F}$ is then locally bounded. Let $K \subset \Omega$ be compact, and let $L > 0$ be such that $\text{Re}(f)(z) \leq L \ \forall z \in K \ \forall f \in \mathcal{F}$. Then

$$\left|e^{f(z)}\right| = e^{\text{Re}(f(z))} \leq e^L \ \forall z \in K \ \forall f \in \mathcal{F}$$

Hence, $\left\{g = e^f : f \in \mathcal{F}\right\}$ is a locally bounded family, and therefore its derivatives are locally bounded. Since $\text{Re}(f) > 0 \ \forall f \in \mathcal{F}$, we have that

$$|f'(z)| \leq \left|f'(z)e^{f(z)}\right| = |g'(z)|$$

which shows that $\{f' : f \in \mathcal{F}\}$ is a locally bounded family. Since K is compact, there exist $z_1, \cdots, z_n \in K$ and $r_1, \cdots, r_n > 0$ such that $K \subset \bigcup_{i=1}^n D(z_i; \frac{r_i}{2})$ and $D(z_i;r_i) \subset \Omega$. By Cauchy's theorem,

$$f(z) = \int_{[z_i,z]} f'(z)dz \ \forall z \in D\left(z_i;\frac{r_i}{2}\right) \Rightarrow |f(z)| \leq M_i r_i \ \forall z \in D\left(z_i;\frac{r_i}{2}\right)$$

where $[z, z_i]$ denotes the straight line segment, and M_i is a uniform bound for $\{f' : f \in \mathcal{F}\}$ on $D(z_i;2^{-1}r_i)$. Setting $M = \max_{1\leq i\leq n} M_i$ and $r = \max_{1\leq i\leq n} r_i$, we conclude that

$$|f(z)| \leq Mr \ \forall z \in K \ \forall f \in \mathcal{F}$$

Normal Families

5.5.5 Exercise 3

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire holomorphic function. Define a family of entire functions $\mathcal{F}$ by

$$\mathcal{F} = \{g : \mathbb{C} \to \mathbb{C} : g(z) = f(kz), k \in \mathbb{C}\}$$

Fix $0 \leq r_1 < r_2 \leq \infty$. I claim that $\mathcal{F}$ is normal (in the sense of Definition 3 p. 225) in the annulus $r_1 < |z| < r_2$ if and only if f is a polynomial.

Suppose $f = a_0 + a_1 z + \cdots + a_n z^n$ is a polynomial, where $a_n \neq 0$. By Ahlfors Theorem 17 (p. 226), it suffices to show that the expression

$$\rho(g) = \frac{2\,|g'(z)|}{1 + |g(z)|^2} \ g \in \mathcal{F}$$

is locally bounded. Since $g(z) = f(kz)$ for some $k \in \mathbb{C}$, it suffices to show that $\frac{2|f'(z)|}{1+|f(z)|^2}$ is bounded on $\mathbb{C}$. The function $F(z)$ given by

$$F(z) = \frac{2\left|f'\left(z^{-1}\right)\right|}{1+\left|f(z^{-1})\right|^2} = \frac{2\left|a_1 z^{2n} + 2a_2 z^{2n-1} + \cdots + na_n z^{n+1}\right|}{|z|^{2n} + \left|a_0 z^n + a_1 z^{n-1} + \cdots + a_n\right|^2}$$

is continuous in a neighborhood of 0 with $F(0) \neq +\infty$ since $a_n \neq 0$. Hence, $|F(z)| \leq M_1 \ \forall |z| \leq \delta$, which shows that

$$\frac{2\left|f'(z)\right|}{1+\left|f(z)\right|^2} \leq M_1 \ \forall |z| \geq \frac{1}{\delta}$$

$\frac{2|f'(z)|}{1+|f(z)|^2}$ is continuous on the compact set $\overline{D}(0; \frac{1}{\delta})$ and therefore bounded by some M_2. Taking $M = \max\{M_1, M_2\}$, we obtain the desired result.

Now suppose that $\mathcal{F}$ is normal in $r_1 < |z| < r_2$. If f is bounded, then we're done by Liouville's theorem. Assume otherwise. Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ be a sequence given by

$$f_n(z) = f(\kappa_n z) \text{ for some } \kappa_n \in \mathbb{C}$$

where $\kappa_n \to \infty, n \to \infty$. Since $\mathcal{F}$ is normal, (f_n) has a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ which either tends to ∞, uniformly on compact subsets of $\{r_1 < |z| < r_2\}$, or converges to some limit function g in likewise fashion. Fix $\delta > 0$ small and consider the compact subset $\{r_1 + \delta \leq |z| \leq r_2 - \delta\}$. If $f_{n_k} \to g$, then I claim that f is bounded on $\mathbb{C}$, which gives us a contradiction. Indeed, fix $z_0 \in \mathbb{C}$. Since (f_{n_k}) converges uniformly on $\{r_1 + \delta \leq |z| \leq r_2 - \delta\}$, (f_{n_k}) is uniformly bounded by some $M > 0$ on this set. Let $|\kappa_{n_k}(r_1 + \delta)| \geq |z_0|$. By the Maximum Modulus Principle, $|f(z)|$ is bounded on the disk $D(0; |\kappa_{n_k}(r_1+\delta)|)$ by some $|f(w)|$ for some w on the boundary. Hence,

$$|f(z_0)| \leq |f(w)| = |f_{n_k}(z)| \leq M \text{ for some } z \in \{|z| = r_1 + \delta\}$$

Since z_0 was arbitrary, we conclude that f is bounded.
I now claim that f has finitely many zeroes. Suppose not. Let $(a_n)_{n\in\mathbb{N}}$ be the sequence of zeroes of f ordered by increasing modulus, and consider the sequence of functions $f_n(z) = f_n(r^{-1}a_n z)$, where $r_1 < r < r_2$ is fixed. Our preceding work shows that (f_n) has a subsequence (f_{n_k}) which tends to ∞ on the compact set $\{|z| = r\}$. But this is a contradiction since $f_{n_k}(r) = 0 \ \forall k \in \mathbb{N}$.
If we can show that f has a pole at ∞, then we're done by Ahlfors Section 4.3.2 Exercise 2 (Problem Set 1). Let $f_n(z) = f(nz)$, and let $(f_{n_k})_{k\in\mathbb{N}}$ be a subsequence which tends to ∞ on compact sets. Let $M > 0$ be given. Fix $r_1 < r < r_2$. Then $f_{n_k} \to \infty$ uniformly on $\{|z| = r\}$, so there exists $k_0 \in \mathbb{N}$ such that for $k \geq k_0$, $|f_{n_k}(z)| > M \ \forall |z| = r$. Taking k_0 larger if necessary, we may assume that $|f(z)| > 0 \ \forall |z| \geq rn_{k_0}$. Let $z \in \mathbb{C}, |z| \geq rn_{k_0}$, and choose k so that $n_k r > |z|$. By the Minimum Modulus Principle, $|f|$ assumes its minimum on the boundary of the annulus $\{n_{k_0} r \leq |w| \leq rn_k\}$. But

$$\min\left\{\inf_{|w|=n_{k_0}r} |f(w)|, \inf_{|w|=n_k r} |f(w)|\right\} > M$$

and therefore,

$$|f(z)| \geq \inf_{n_{k_0}r \leq |w| \leq n_k r} |f(w)| > M$$

Since z was arbitrary, we conclude that $|f(z)| > M \ \forall |z| \geq rn_{k_0}$. Since $M > 0$ was arbitrary, we conclude that f has a pole at ∞.

5.5.5 Exercise 4

Let $\mathcal{F}$ be a family of meromorphic functions in a given region Ω, which is not normal in Ω. By Ahlfors Theorem 17 (p. 226), there must exist a compact set $K \subset \Omega$ such that the expression

$$\rho(f)(z) = \frac{2\left|f'(z)\right|}{1+\left|f(z)\right|^2} \ f \in \mathcal{F}$$

is not locally bounded on K. Hence, we can choose a sequence of functions $(f_n) \subset \mathcal{F}$ and of points $(z_n) \subset K$ such that

$$\frac{2\left|f_n'(z_n)\right|}{1+\left|f_n(z_n)\right|^2} \nearrow \infty, n \to \infty$$

Suppose for every $z \in \Omega$, there exists an open disk $D(z; r_z) \subset \Omega$ on which $\mathcal{F}$ is normal, equivalently $\rho(f)$ is locally bounded. Let $M_z > 0$ bound $\rho(f)$ on the closed disk $\overline{D}(z; 2^{-1}r_z)$. The collection $\left\{D(z; 2^{-1}r_z) : z \in K\right\}$ forms an open cover of K. By compactness, there exist finitely many disks $D(z_1; 2^{-1}r_1), \cdots, D(z_n; 2^{-1}r_n)$ such that

$$K \subset \bigcup_{i=1}^{n} D(z_i; 2^{-1}r_i) \text{ and } \forall i = 1, \cdots, n \;\; |\rho(f)(z)| \leq M_i \; \forall z \in \overline{D}(z_i; 2^{-1}r) \; \forall f \in \mathcal{F}$$

Setting $M = \max_{1 \leq i \leq n} M_i$, we conclude that

$$|\rho(f)(z)| \leq M \; \forall z \in K \; \forall f \in \mathcal{F}$$

This is obviously a contradiction since $\lim_{n \to \infty} \rho(f_n)(z_n) = +\infty$. We conclude that there must exist $z_0 \in \Omega$ such that $\mathcal{F}$ is not normal in any neighborhood of z_0.

Conformal Mapping, Dirichlet's Problem

The Riemann Mapping Theorem

6.1.1 Exercise 1

Lemma 6. *Let $f : \Omega \to \mathbb{C}$ be a holomorphic function on a symmetric region Ω (i.e. $\Omega = \overline{\Omega}$). Then the function $g : \Omega \to \mathbb{C}, g(z) = \overline{f(\overline{z})}$ is holomorphic.*

Proof. Writing $z = x + iy$, if $f(z) = u(x, y) + iv(x, y)$, where u, v are real, then $g(z) = u(x, -y) - iv(x, -y) = \bar{u}(x, y) + i\bar{v}(x, y)$. It is then evident that g is continuous and u, v have C^1 partials. We verify the Cauchy-Riemann equations.

$$\frac{\partial \bar{u}}{\partial x}(x, y) = \frac{\partial u}{\partial x}(x, -y); \ \frac{\partial \bar{u}}{\partial y}(x, y) = -\frac{\partial u}{\partial y}(x, -y)$$
$$\frac{\partial \bar{v}}{\partial x}(x, y) = -\frac{\partial v}{\partial x}(x, -y); \ \frac{\partial \bar{v}}{\partial y}(x, y) = \frac{\partial v}{\partial y}(x, -y)$$

The claim follows immediately from the fact that u, v satisfy the Cauchy-Riemann equations. □

Let $\Omega \subset \mathbb{C}$ be simply connected symmetric region, $z_0 \in \Omega$ be real, and $f : \Omega \to \mathbb{D}$ be the unique conformal map satisfying $f(z_0) = 0, f'(z_0) > 0$ (as guaranteed by the Riemann Mapping Theorem). Define $g(z) = \overline{f(\overline{z})}$. Then $g : \Omega \to \mathbb{D}$ is holomorphic by the lemma and bijective, being the composition of bijections; hence, g is conformal. Furthermore, $g(z_0) = 0$ since $z_0, f(z_0) \in \mathbb{R}$. Since

$$0 < f'(z_0) = \frac{\partial u}{\partial x}(z_0) = \frac{\partial u}{\partial x}(\overline{z_0}) = g'(z_0)$$

we conclude by uniqueness that $f = g$. Equivalently, $\overline{f(z)} = f(\overline{z}) \ \forall z \in \Omega$.

6.1.1 Exercise 2

Suppose now that Ω is symmetric with respect to z_0 (i.e. $z \in \Omega \iff 2z_0 - z \in \Omega$). I claim that f satisfies

$$f(z) = 2f(z_0) - f(2z_0 - z) = -f(2z_0 - z)$$

Define $g : \Omega \to \mathbb{D}$ by $g(z) = -f(2z_0 - z)$. Clearly, g is conformal, being the composition of conformal maps, and $g(z_0) = 0$. Furthermore, by the chain rule, $g'(z_0) = f'(z_0) > 0$. We conclude from the uniqueness statement of the Riemann Mapping Theorem that $g(z) = f(z) \ \forall z \in \Omega$.

Elliptic Functions

Weierstrass Theory

7.3.2 Exercise 1

Let f be an even elliptic function periods ω_1, ω_2. If f is constant then there is nothing to prove, so assume otherwise. First, suppose that 0 is neither a zero nor a pole of f. Observe that since f is even, its zeroes and poles occur in pairs. Since f is elliptic, f has the same number of poles as zeroes. So, let $a_1, \cdots, a_n$, and $b_1, \cdots, b_n$ denote the incongruent zeroes and poles of f in some fundamental parallelogram P_a, where $a_i \not\equiv -a_j \mod M, b_i \not\equiv -b_j \mod M\ \forall i,j$ and where we repeat for multiplicity. Define a function g by

$$g(z) = f(z)\left(\prod_{k=1}^{n} \frac{\wp(z) - \wp(a_k)}{\wp(z) - \wp(b_k)}\right)^{-1}$$

and where $\wp$ is the Weierstrass p-function with respect to the lattice generated by ω_1, ω_2. I claim that g is a holomorphic elliptic function. Since $\wp(z) - \wp(a_k)$ and $\wp(z) - \wp(b_k)$ have double poles at each $z \in M$ for all k, g has a removable singularity at each $z \in M$. For each k, $\wp(z) - \wp(b_k)$ has the same poles as $\wp$ and is therefore an elliptic function of order 2. Since $b_k \neq 0$ and $\wp$ is even, it follows that $\wp(z) - \wp(b_k)$ has zeroes of order 1 at $z = \pm b_k$. From our convention for repeating zeroes and poles, we conclude that g has a removable singularity at $\pm b_k$. The argument that g has removable singularity at each a_k is completely analogous. Clearly,

$$g(z + \omega_1) = g(z + \omega_2) = g(z) \text{ for } z \notin a_i + M \cup b_i + M \cup M$$

so by continuity, we conclude that g is a holomorphic elliptic function with periods ω_1, ω_2 and is therefore equal to a constant C. Hence,

$$f(z) = C \prod_{k=1}^{n} \frac{\wp(z) - \wp(a_k)}{\wp(z) - \wp(b_k)}$$

Since f is even, its Laurent series about the origin only has nonzero terms with even powers. So if f vanishes or has a pole at the origin, the order is $2m, m \in \mathbb{N}$. Suppose that f vanishes with order $2m$. The function given by

$$\tilde{f}(z) = f(z) \cdot \wp(z)^m$$

is elliptic with periods ω_1, ω_2. $\tilde{f}$ has a removable singularity at $z = 0$, since $\wp(z)^k$ has a pole of order $2k$ at $z = 0$. Hence, we are reduced to the previous case of elliptic function, so applying the preceding argument, we conclude that

$$\tilde{f}(z) = C \prod_{k=1}^{n} \frac{\wp(z) - \wp(a_k)}{\wp(z) - \wp(b_k)} \Rightarrow f(z) = \frac{C}{\wp(z)^m} \prod_{k=1}^{n} \frac{\wp(z) - \wp(a_k)}{\wp(z) - \wp(b_k)}$$

If f has a pole of order $2m$ at the origin, then the function given by

$$\tilde{f}(z) = \frac{f(z)}{\wp(z)^m}$$

is elliptic with periods ω_1, ω_2 and has a removable singularity at the origin. From the same argument, we conclude that

$$f(z) = C\wp(z)^m \prod_{k=1}^{n} \frac{\wp(z) - \wp(a_k)}{\wp(z) - \wp(b_k)}$$

7.3.2 Exercise 2

Let f be an elliptic function with periods ω_1, ω_2. By Ahlfors Theorem 5 (p. 271), f has the same number of zeroes and poles counted with multiplicity. Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ denote the incongruent zeroes and poles of f, respectively, where we repeat for multiplicity. By Ahlfors Theorem p. 271, $\sum_{k=1}^n b_k - a_k \in M$, so replacing a_1 by $a_1' = a_1 + \sum_{k=1}^n b_k - a_k$, we may assume without loss of generality that $\sum_{k=1}^n b_k - a_k = 0$. Define a function g by

$$g(z) = f(z) \left(\prod_{k=1}^n \frac{\sigma(z-a_k)}{\sigma(z-b_k)} \right)^{-1}$$

where σ is the entire function (Ahlfors p. 274) given by

$$\sigma(z) = z \prod_{\omega \neq 0} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{w} + \frac{1}{2}\left(\frac{z}{w}\right)^2}$$

g has removable singularities at $a_i + M, b_i + M$ for $1 \leq i \leq n$. I claim that g is elliptic with periods ω_1, ω_2. Recall (Ahlfors p. 274) that σ satisfies

$$\sigma(z+\omega_1) = -\sigma(z) e^{-\eta_1\left(z + \frac{\omega_1}{2}\right)} \text{ and } \sigma(z+\omega_2) = -\sigma(z) e^{-\eta_2\left(z+\frac{\omega_2}{2}\right)} \ \forall z \in \mathbb{C}$$

where $\eta_2\omega_1 - \eta_1\omega_2 = 2\pi i$ (Legendre's relation). Hence, for $z \not\equiv b_i + M, a_i + M$,

$$g(z+\omega_1) = f(z+\omega_1) \left(\prod_{k=1}^n \frac{\sigma(z-a_k+\omega_1)}{\sigma(z-b_k+\omega_1)} \right)^{-1} = f(z) \left(\prod_{k=1}^n \frac{-\sigma(z-a_k) e^{\eta_1\left(z-a_k+\frac{\omega_1}{2}\right)}}{-\sigma(z-b_k) e^{\eta_1\left(z-b_k+\frac{\omega_1}{2}\right)}} \right)^{-1}$$

$$= e^{\eta_1 \sum_{k=1}^n a_k - b_k} f(z) \left(\prod_{k=1}^n \frac{\sigma(z-a_k)}{\sigma(z-b_k)} \right)^{-1} = g(z)$$

By continuity, we conclude that $g(z+\omega_1) = g(z) \ \forall z \in \mathbb{C}$. Analogously, for $z \not\equiv b_i + M, a_i + M$,

$$g(z+\omega_2) = f(z+\omega_2) \left(\prod_{k=1}^n \frac{\sigma(z-a_k+\omega_2)}{\sigma(z-b_k+\omega_2)} \right)^{-1} = f(z) \left(\prod_{k=1}^n \frac{-\sigma(z-a_k) e^{\eta_2\left(z-a_k+\frac{\omega_2}{2}\right)}}{-\sigma(z-b_k) e^{\eta_2\left(z-b_k+\frac{\omega_2}{2}\right)}} \right)^{-1}$$

$$= e^{\eta_2 \sum_{k=1}^n a_k - b_k} f(z) \left(\prod_{k=1}^n \frac{\sigma(z-a_k)}{\sigma(z-b_k)} \right)^{-1} = g(z)$$

By continuity, we conclude that $g(z+\omega_2) = g(z) \ \forall z \in \mathbb{C}$. Since g is an entire elliptic function, it is constant by Ahlfors Theorem 3 (p. 270). We conclude that for some $C \in \mathbb{C}$,

$$f(z) = C \prod_{k=1}^n \frac{\sigma(z-a_k)}{\sigma(z-b_k)}$$

7.3.3 Exercise 1

Fix a rank-2 lattice $M \subset \mathbb{C}$ and $u \notin M$. Then

$$\wp(z) - \wp(u) = -\frac{\sigma(z-u)\sigma(z+u)}{\sigma(z)^2\sigma(u)^2}$$

Proof. I first claim that the RHS is periodic with respect to M. Let ω_1, ω_2 be generators of M and let $\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$. For $z \notin M$,

$$-\frac{\sigma(z+\omega_1-u)\sigma(z+\omega_1+u)}{\sigma(z+\omega_1)^2\sigma(u)^2} = -\frac{\sigma(z-u)e^{\eta_1(z-u+\frac{\omega_1}{2})}\sigma(z+u)e^{\eta_1(z+u+\frac{\omega_1}{2})}}{\sigma(z)^2 e^{2\eta_1(z+\frac{\omega_1}{2})}\sigma(u)^2} = -\frac{\sigma(z-u)\sigma(z+u)e^{2\eta_1(z+\frac{\omega_1}{2})}}{\sigma(z)^2\sigma(u)^2 e^{2\eta_1(z+\frac{\omega_1}{2})}}$$

$$= -\frac{\sigma(z-u)\sigma(z+u)}{\sigma(z)^2\sigma(u)^2} = f(z)$$

The argument for ω_2 is completely analogous. The RHS has zeroes at $\pm u$ and a double pole at 0. Hence, by the same reasoning used above, we see that

$$\wp(z) - \wp(u) = -C\frac{\sigma(z-u)\sigma(z+u)}{\sigma(z)^2\sigma(u)^2} \text{ for some } C \in \mathbb{C}$$

To find conclude that $C = 1$, we first note that $\wp(z) - \wp(u)$ has a coefficient of 1 for the z^{-2} term in its Laurent expansion. If we show that the Laurent expansion of the $f(z)$ also has a coefficient of 1 for the z^{-2}, then it follow from the uniqueness of Laurent expansions that $C = 1$.

$$-\frac{\sigma(z-u)\sigma(z+u)}{\sigma(z)^2\sigma(u)^2} = -\frac{(z^2-u^2)\prod_{\omega\neq 0}\left(1-\frac{z-u}{\omega}\right)e^{\frac{z-u}{\omega}+\frac{1}{2}\left(\frac{z-u}{\omega}\right)^2}\prod_{\omega\neq 0}\left(1-\frac{z+u}{\omega}\right)e^{\frac{z+u}{\omega}+\frac{1}{2}\left(\frac{z+u}{\omega}\right)^2}}{z^2\sigma(u)^2\left(\prod_{\omega\neq 0}\left(1-\frac{z}{\omega}\right)e^{\frac{z}{\omega}+\frac{1}{2}\left(\frac{z}{\omega}\right)^2}\right)^2}$$

$$= \underbrace{-\frac{\prod_{\omega\neq 0}\left(1-\frac{z-u}{\omega}\right)e^{\frac{z-u}{\omega}+\frac{1}{2}\left(\frac{z-u}{\omega}\right)^2}\prod_{\omega\neq 0}\left(1-\frac{z+u}{\omega}\right)e^{\frac{z+u}{\omega}+\frac{1}{2}\left(\frac{z+u}{\omega}\right)^2}}{\sigma(u)^2\left(\prod_{\omega\neq 0}\left(1-\frac{z}{\omega}\right)e^{\frac{z}{\omega}+\frac{1}{2}\left(\frac{z}{\omega}\right)^2}\right)^2}}_{g_1(z)}$$

$$+\frac{1}{z^2}\cdot\underbrace{\frac{u^2\prod_{\omega\neq 0}\left(1-\frac{z-u}{\omega}\right)e^{\frac{z-u}{\omega}+\frac{1}{2}\left(\frac{z-u}{\omega}\right)^2}\prod_{\omega\neq 0}\left(1-\frac{z+u}{\omega}\right)e^{\frac{z+u}{\omega}+\frac{1}{2}\left(\frac{z+u}{\omega}\right)^2}}{\sigma(u)^2\left(\prod_{\omega\neq 0}\left(1-\frac{z}{\omega}\right)e^{\frac{z}{\omega}+\frac{1}{2}\left(\frac{z}{\omega}\right)^2}\right)^2}}_{g_2(z)}$$

Observe that both $g_1(z)$ and $g_2(z)$ are holomorphic in a neighborhood of 0 since we have eliminated the double pole at 0. Hence, the coefficient of the z^{-2} in the Laurent expansion of $f(z)$ is given by $g_2(0)$. But since σ is an odd function, it is immediate that $g_2(0) = 1$. □

7.3.3 Exercise 2

With the hypotheses of the preceding problem,

$$\frac{\wp'(z)}{\wp(z)-\wp(u)} = \zeta(z-u) + \zeta(z+u) - 2\zeta(z)$$

Proof. For $z \neq u + M$, we can choose a branch of the logarithm holomorphic in a neighborhood of $\wp(z) - \wp(u)$. Taking the derivative of the log of both sides and using the chain rule,

$$\frac{\wp'(z)}{\wp(z)-\wp(u)} = \frac{\partial}{\partial z}\left[\log(-\sigma(z-u)) + \log(\sigma(z-u)) - \log(\sigma(u)^2\sigma(z)^2)\right]$$

$$= \frac{\sigma'(z-u)}{\sigma(z-u)} + \frac{\sigma'(z+u)}{\sigma(z+u)} - \frac{2\sigma'(z)}{\sigma(z)} = \zeta(z-u) + \zeta(z+u) - 2\zeta(z)$$

where we've used $\frac{\sigma'(w)}{\sigma(w)} = \zeta(w)\ \forall w \in \mathbb{C}$ (Ahlfors p. 274). □

7.3.3 Exercise 3

With the same hypotheses as above, for $z \neq -u + M$,

$$\zeta(z+u) = \zeta(z) + \zeta(u) + \frac{1}{2}\frac{\wp'(z)-\wp'(u)}{\wp(z)-\wp(u)}$$

Proof. Since the last term has a removable singularity at $z = u + M$, by continuity, we may also assume that $z \neq u + M$. First, observe that by replacing switching u and z in the argument for the last identity, we have that

$$\frac{\wp'(u)}{\wp(z)-\wp(u)} = -\left[\zeta(u-z) + \zeta(z+u) - 2\zeta(u)\right] = \zeta(z-u) - \zeta(z+u) + 2\zeta(u)$$

where we've used the fact that $\sigma(z)$ is odd and therefore $\zeta(z) = \frac{\sigma'(z)}{\sigma(z)}$ is also odd. Hence,

$$\frac{\wp'(z) - \wp' u}{\wp(z) - \wp(u)} = (\zeta(z-u) + \zeta(z+u) - 2\zeta(z)) - (\zeta(z-u) - \zeta(z+u) + 2\zeta(u)) = 2\zeta(z+u) - 2\zeta(z) - 2\zeta(u)$$

The stated identity follows immediately. □

7.3.3 Exercise 4

By Ahlfors Section 7.3.3 Exercise 3,

$$\zeta(z+u) = \zeta(z) + \zeta(u) + \frac{1}{2}\left(\frac{\wp'(z) - \wp'(u)}{\wp(z) - \wp(u)}\right)$$

Differentiating both sides with respect to z and using $-\zeta'(w) = \wp(w)\ \forall w \in \mathbb{C} \setminus M$, we obtain

$$-\wp(z+u) = -\wp(z) + \frac{1}{2}\left(\frac{\wp''(z)}{\wp(z) - \wp(u)} - \frac{(\wp'(z) - \wp'(u))\wp'(z)}{(\wp(z) - \wp(u))^2}\right)$$

We seek an expression for $\wp''(z)$ in terms of $\wp(z)$. For $z \neq \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1 + \omega_2}{2} + M$,

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_2 \Rightarrow 2\wp'(z)\wp''(z) = 12\wp(z)^2\wp'(z) - g_2\wp'(z) \Rightarrow \wp''(z) = 6\wp(z)^2 - \frac{g_2}{2}$$

We conclude from continuity that $\wp''(z) = 6\wp(z)^2 - \frac{g_2}{2}$. Substituting this identity in,

$$-\wp(z+u) = -\wp(z) + \frac{1}{2}\left(\frac{6\wp(z)^2 - \frac{g_2}{2}}{\wp(z) - \wp(u)} - \frac{(\wp'(z) - \wp'(u))\wp'(z)}{(\wp(z) - \wp(u))^2}\right)$$

Applying the same arguments as above except taking u to be variable, we obtain that

$$-\wp(z+u) = -\wp(u) + \frac{1}{2}\left(-\frac{6p(u)^2 - \frac{g_2}{2}}{p(z) - p(u)} + \frac{(\wp'(z) - \wp'(u))\wp'(u)}{(\wp(z) - \wp(u))^2}\right)$$

Hence,

$$-2\wp(z+u) = -\wp(z) + -\wp(u) + \frac{1}{2}\left(\frac{6(\wp(z)^2 - \wp(u)^2)}{\wp(z) - \wp(u)} - \frac{(\wp'(z) - \wp'(u))^2}{(\wp(z) - \wp(u))^2}\right)^2 = 2\wp(z) + 2\wp(u) - \frac{1}{2}\left(\frac{\wp'(z) - \wp'(u)}{\wp(z) - \wp(u)}\right)^2$$

$$\Rightarrow \wp(z+u) = -\wp(z) - \wp(u) + \frac{1}{4}\left(\frac{\wp'(z) - \wp'(u)}{\wp(z) - \wp(u)}\right)^2$$

7.3.3 Exercise 5

Using the identity obtained in the previous exercise, we have by the continuity of $\wp$ that

$$\wp(2z) = \lim_{u \to z} \wp(z+u) = \lim_{u \to z}\left[-\wp(z) - \wp(u) + \frac{1}{4}\left(\frac{\wp'(z) - \wp'(u)}{\wp(z) - \wp(u)}\right)^2\right]$$

$$= \lim_{u \to z}\left[-\wp(u) - \wp(z) + \frac{1}{4}\left(\frac{\frac{\wp'(z) - \wp'(u)}{z-u}}{\frac{\wp(z) - \wp(u)}{z-u}}\right)^2\right] = -2\wp(z) + \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2$$

where we use the continuity of $w \mapsto w^2$ to obtain the last expression.

7.3.3 Exercise 7

Fix $u, v \notin M$ such that $|u| \neq |v|$, and define a function $f : \mathbb{C} \setminus M \to \mathbb{C}$ by

$$f(z) = \det \begin{pmatrix} \wp(z) & \wp'(z) & 1 \\ \wp(u) & \wp'(u) & 1 \\ \wp(v) & -\wp'(v) & 1 \end{pmatrix} = -\wp'(z)(\wp(u) - \wp(v)) + \wp'(u)(\wp(z) - \wp(v)) + \wp'(v)(\wp(z) - \wp(u))$$

$$= \underbrace{(\wp'(u) + \wp'(v))}_{A}\wp(z) + \underbrace{(\wp(v) - \wp(u))}_{B}\wp'(z) + \underbrace{-(\wp'(u)\wp(v) + \wp'(v)\wp(u))}_{C}$$

where we use Laplace expansion for determinants. By our choice of u, v and the fact that the Weierstrass function is elliptic of order 2, $B \neq 0$. Hence, $f(z)$ is an elliptic function of order 3 with poles at the lattice points of M. Since the determinant of any matrix with linearly dependent rows is zero, f has zeroes at $u, -v$. Since f has order 3, it has a third zero z, and by Abel's Theorem (Ahlfors p. 271 Theorem 6),

$$u - v + z \equiv 0 \mod M \Rightarrow z = v - u$$

We conclude that

$$\det \begin{pmatrix} \wp(z) & \wp'(z) & 1 \\ \wp(u) & \wp'(u) & 1 \\ \wp(u+z) & -\wp'(u+z) & 1 \end{pmatrix} = 0$$

7.3.5 Exercise 1

Since λ is invariant under $\Gamma(2)$ and $\Gamma \setminus \Gamma(2)$ is generated by the linear fractional transformations $\tau \mapsto \tau + 1$ and $\tau \mapsto -\tau^{-1}$, it suffices to show that $J(\tau+1) = J(\tau)$ and $J(-\tau^{-1}) = J(\tau)$. Recall that λ satisfies the functional equations

$$\lambda(\tau+1) = \frac{\lambda(\tau)}{\lambda(\tau) - 1} \text{ and } \lambda\left(-\frac{1}{\tau}\right) = 1 - \lambda(\tau)$$

So,

$$J(\tau+1) = \frac{4}{27}\frac{(1 - \lambda(\tau+1) + \lambda(\tau+1)^2)^3}{\lambda(\tau+1)^2(1-\lambda(\tau+1))^2} = \frac{4}{27}\frac{(1 - \lambda(\tau)(\lambda(\tau)-1)^{-1} + \lambda(\tau)^2(\lambda(\tau)-1)^{-2})^3}{\lambda(\tau)^2(\lambda(\tau)-1)^{-2}\left(1 - \lambda(\tau)(\lambda(\tau)-1)^{-1}\right)^2} \cdot \frac{(\lambda(\tau)-1)^6}{(\lambda(\tau)-1)^6}$$

$$= \frac{4}{27}\frac{\left((\lambda(\tau)-1)^2 - \lambda(\tau)(\lambda(\tau)-1) + \lambda(\tau)^2\right)^3}{\lambda(\tau)^2(\lambda(\tau)-1)^2} = \frac{4}{27}\frac{(1 - \lambda(\tau) + \lambda(\tau)^2)^3}{\lambda(\tau)^2(\lambda(\tau)-1)^2} = J(\tau)$$

and

$$J\left(-\frac{1}{\tau}\right) = \frac{4}{27}\frac{\left(1 - (1-\lambda(\tau)) + (1-\lambda(\tau))^2\right)^3}{(1-\lambda(\tau))^2(1-(1-\lambda(\tau)))^2} = \frac{4}{27}\frac{\left(1 - \lambda(\tau) + \lambda(\tau)^2\right)^3}{\lambda(\tau)^2(1-\lambda(\tau))^2} = J(\tau)$$

Observe that

$$J(\tau) = \frac{4}{27}\frac{\left(1 - \lambda(\tau) + \lambda(\tau)^2\right)^3}{\lambda(\tau)^2(1-\lambda(\tau))^2} = \frac{4}{27}\frac{(\lambda(\tau) - e^{i\frac{\pi}{3}})^3(\lambda(\tau) - e^{-i\frac{\pi}{3}})^3}{\lambda(\tau)^2(1-\lambda(\tau))^2}$$

So, $J(\tau)$ assumes the value 0 on $\lambda^{-1}\left(\{e^{\pm i\frac{\pi}{3}}\}\right)$. Since λ is a bijection on $\overline{\Omega} \cup \Omega'$, $J(\tau)$ has two zeroes, each of order 3. We proved in Problem Set 8 that

$$g_2 = -4(e_1e_2 + e_1e_3 + e_2e_3) = 0$$

for $\tau = e^{i\frac{2\pi}{3}}$. So using the identity for $J(\tau)$ proved below, $J(e^{i\frac{2\pi}{3}}) = 0$. Using the invariance of $J(\tau)$ under Γ, we see that $J(e^{i\frac{\pi}{3}}) = 0$.
$J(\tau)$ assumes the value 1 on $\lambda^{-1}(\{\lambda_1, \cdots, \lambda_6\})$, where the λ_i are the roots of degree 6 the polynomial

$$p(z) = 4(1 - z + z^2)^3 - 27z^2(1-z)^2$$

It is easy to check that

$$e_3 = \wp\left(\frac{1+i}{2}; i\right) = -\wp\left(\frac{i+1}{2}; i\right) = -e_3 \Rightarrow e_3 = 0$$

Since $e_1 + e_2 + e_3 = 0$ (see below for argument), we have $e_1 = -e_2$ and therefore

$$\lambda(i) = \frac{e_3 - e_2}{e_1 - e_2} = \frac{1}{2}$$

Since each point in $\mathbb{H}^+$ is congruent modulo 2 to a point in $\overline{\Omega} \cup \Omega'$, λ maps this fundamental conformally onto $\mathbb{C} \setminus \{0, 1\}$, and $J(\tau)$ is invariant under Γ, we conclude $J(\tau)$ assumes the value 1 at $\tau = i, 1 + i, \frac{i+1}{2}$. I claim that these are these are the only possible points up to modulo 2 congruence. Suppose $J(\tau) = 1$ for $\tau \notin \left\{i, 1 + i, \frac{i+1}{2}\right\}$. If we let $S_1, \cdots, S_6$ denote the complete set of mutually incongruent transformations, then since $\tau \notin \left\{e^{i\frac{\pi}{3}}, e^{i\frac{2\pi}{3}}\right\}$ (otherwise $J(\tau) = 0$), $S_1\tau, \cdots, S_6\tau \in \overline{\Omega} \cup \Omega'$ are distinct, hence the $\lambda(S_i\tau)$ are distinct roots of $p(z)$, and we obtain that $p(z)$ has more than 6 roots, a contradiction. Moreover, this argument shows that the polynomial $p(z)$ has three roots, which by inspection, we see are given by $\left\{-1, \frac{1}{2}, 2\right\}$.
I claim that $J(\tau)$ assumes the value 1 with order 2 at $\tau = i, 1 + i, \frac{i+1}{2}$. We need to show that the zeroes of $p(z)$ are each of order 2. Indeed, one can verify that

$$p(z) = 4(1 - z + z^2)^3 - 27z^2(1 - z)^2 = (z - 2)^2(2z - 1)^2(z + 1)^2$$

Substituting $\lambda = \frac{e_3 - e_2}{e_1 - e_2}$, we have

$$J(\tau) = \frac{4}{27}\frac{\left(1 - (e_3 - e_2)(e_1 - e_2)^{-1} + (e_3 - e_2)^2(e_1 - e_2)^{-2}\right)^3}{(e_3 - e_2)^2(e_1 - e_2)^{-2}(e_1 - e_3)^2(e_1 - e_2)^{-2}}$$

$$= \frac{4}{27}\frac{\left((e_1 - e_2)^2 - (e_3 - e_2)(e_1 - e_2) + (e_3 - e_2)^2\right)^3}{(e_3 - e_2)^2(e_1 - e_3)^2(e_1 - e_2)^2}$$

$$= \frac{4}{27}\frac{\left(e_1^2 - 2e_1e_2 + e_2^2 - e_3e_1 + e_3e_2 + e_2e_1 - e_2^2 + e_3^2 - 2e_3e_2 + e_2^2\right)^3}{(e_3 - e_2)^2(e_1 - e_3)^2(e_1 - e_2)^2}$$

Since

$$4z^3 - g_2z - g_3 = 4(z - e_1)(z - e_2)(z - e_3) = 4(z^2 - (e_1 + e_2)z + e_1e_2)(z - e_3) = 4(e_1 + e_2 + e_3)z^2 + \cdots$$

we have that $e_1 + e_2 + e_3 = 0$ and so,

$$0 = (e_1 + e_2 + e_3)^2 = e_1^2 + e_2^2 + e_3^2 + 2e_1e_2 + 2e_1e_3 + 2e_2e_3 \Rightarrow e_1^2 + e_2^2 + e_3^3 = -2\left(e_1e_2 + e_1e_3 + e_2e_3\right)$$

Substituting this identity in,

$$J(\tau) = \frac{4}{27}\frac{\left(-2(e_1e_2 + e_2e_3 + e_1e_3) - (e_1e_2 + e_2e_3 + e_1e_3)\right)^3}{(e_3 - e_2)^2(e_1 - e_3)^2(e_1 - e_2)^2} = -4\frac{\left(e_1e_2 + e_2e_3 + e_1e_3\right)^3}{(e_3 - e_2)^2(e_1 - e_3)^2(e_1 - e_2)^2}$$