12 Lecture 12: Holomorphic functions

For the remainder of this course we will be thinking hard about how the following theorem allows one to explicitly evaluate a large class of Fourier transforms. This will enable us to write down explicit solutions to a large class of ODEs and PDEs.

The Cauchy Residue Theorem:

Let $C \subset \mathbb{C}$ be a simple closed contour. Let $f : \mathbb{C} \to \mathbb{C}$ be a complex function which is holomorphic along C and inside C except possibly at a finite number of points $a_1, \ldots, a_m$ at which f has a pole. Then, with C oriented in an anti-clockwise sense,

$$\int_C f(z)\,dz \;=\; 2\pi i \sum_{k=1}^{m} \operatorname{res}(f, a_k), \tag{12.1}$$

where $\operatorname{res}(f, a_k)$ is the residue of the function f at the pole $a_k \in \mathbb{C}$.

You are probably not yet familiar with the meaning of the various components in the statement of this theorem, in particular the underlined terms and what is meant by the contour integral $\int_C f(z)\,dz$, and so our first task will be to explain the terminology. The Cauchy Residue theorem has wide application in many areas of pure and applied mathematics, it is a basic tool both in engineering mathematics and also in the purest parts of geometric analysis. The idea is that the right-side of (12.1), which is just a finite sum of complex numbers, gives a simple method for evaluating the contour integral; on the other hand, sometimes one can play the reverse game and use an 'easy' contour integral and (12.1) to evaluate a difficult infinite sum (allowing $m \to \infty$). More broadly, the theory of functions of a complex variable provides a considerably more powerful calculus than the calculus of functions of two real variables ('Calculus II').

As listed on the course webpage, a good text for this part of the course is:
H A Priestley, Introduction to Complex Analysis (2nd Edition) (OUP)

We start by considering complex functions and the sub class of holomorphic functions.

12.1 Complex functions

Although as (real) vector spaces

$$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R},\ i^2 = -1\}$$

and $\mathbb{R}^2$ are indistinguishable, the complex numbers $\mathbb{C}$ are crucially different from $\mathbb{R}^2$ because $\mathbb{C}$ comes with a natural commutative multiplication structure, as well as addition. That is,

$$\mathbb{C} \;=\; \mathbb{R}^2 \;+\; \text{"structure of complex multiplication"}. \tag{12.2}$$

Moreover,

$$\mathbb{C}^* = \mathbb{C}\backslash\{0\}$$

is a abelian multiplicative group – meaning that complex multiplication is commutative and (for non-zero numbers) invertible. Specifically, for $z = a + ib$, $w = \alpha + i\beta \in \mathbb{C}$ one has

$$z + w = (a+\alpha) + i(b+\beta), \qquad zw = (a\alpha - b\beta) + i(a\beta + b\alpha) = wz,$$

$$z^{-1} = \frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2} = \frac{\overline{z}}{|z|^2} \qquad \text{for } z \in \mathbb{C}^*.$$

Here we have used complex conjugation

$$\mathbb{C} \to \mathbb{C}, \qquad z = a + ib \;\longmapsto\; \overline{z} := a - ib$$

and the identity $|z|^2 = a^2 + b^2$; note, in particular, that $z = a + ib = 0$ if and only if $a = 0$ and $b = 0$.

It is often convenient to use 'polar coordinates'

$$z = r\,e^{i\,\theta} \tag{12.3}$$

where we define (or use power series to equate)

$$e^{i\,\theta} \;:=\; \cos\theta \;+\; i\,\sin\theta. \tag{12.4}$$

From the double-angle formulae we then have

$$e^{i\,\theta}e^{i\,\phi} = e^{i\,(\theta+\phi)}, \qquad \text{and so } \frac{1}{e^{i\,\theta}} = e^{-i\,\theta}, \;\; (e^{i\,\theta})^m = e^{i\,m\theta}, \; m \in \mathbb{R}. \tag{12.5}$$

We will be interested in complex functions — meaning maps

$$f : \mathbb{C} \longrightarrow \mathbb{C}, \qquad z \mapsto f(z), \tag{12.6}$$

so

$$z = x + iy \mapsto f(z) = u(x,y) + iv(x,y) \quad \text{for some } \; u, v : \mathbb{R}^2 \to \mathbb{R}^1.$$

Here are some examples of the type of complex function with which we shall be working:

$$f(z) = z \tag{12.7}$$

$$f(z) = z^2 \quad f(z) = z^m \text{ with } m \in \mathbb{Z} \tag{12.8}$$

$$f(z) = \overline{z} \tag{12.9}$$

$$f(z) = |z|^2 := z\overline{z}. \tag{12.10}$$

$$f(z) = z + \overline{z}. \tag{12.11}$$

$$f(z) = (z + \overline{z})^2. \tag{12.12}$$

$$f(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0, \quad a_j \in \mathbb{C}, \tag{12.13}$$

$$f(z) = \frac{\beta_{-m}}{(z-2)^m} + \frac{\beta_{-m+1}}{(z-2)^{m-1}} + \cdots + \frac{\beta_{-1}}{(z-2)} + \beta_0 + b_1 z + e^z, \quad \beta_j \in \mathbb{C}, \tag{12.14}$$

$$f(z) \;=\; e^z \;\stackrel{\text{defn}}{:=}\; e^x \cdot e^{iy} \;=\; e^x (\cos y + i \sin y), \tag{12.15}$$

where $z = x + iy$. Notice that

$$e^z e^w = e^{z+w}, \quad z, w \in \mathbb{C}, \quad \text{and so} \quad e^z e^{-z} = 1. \tag{12.16}$$

$$f(z) = \sin z := \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \tag{12.17}$$

$$f(z) = \cos z := \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \tag{12.18}$$

$$f(z) = \sinh z := \frac{1}{2}\left(e^{z} - e^{-z}\right) \tag{12.19}$$

$$f(z) = \cosh z := \frac{1}{2}\left(e^{z} + e^{-z}\right) \tag{12.20}$$

It is important to know the following identities, or be able to quickly derive them:

$$\sin iz = i \sinh z \quad \text{or} \quad i \sin z = \sinh iz \tag{12.21}$$

and

$$\cos iz = \cosh z \quad \text{or} \quad \cos z = \cosh iz \tag{12.22}$$

Note these reduce to the 'usual' formulae when $z = x \in \mathbb{R}$. Also note that, as usual,

$$\cos(z + w) = \cos z \cos w - \sin z \sin w, \quad \sin(z + w) = \sin z \cos w + \cos z \sin w. \tag{12.23}$$

However, notice that unlike the case of real variable trigonometric functions it is not the case that $\cos(z)$ and $\sin(z)$ are bounded (a complex function f is bounded if there exists a positive real number $M > 0$ such that $|f(z)| \leq M$ for all z).

Clearly, as is already evident from some of the examples above, we may compose complex functions just as with real functions. That is, from

$$f : \mathbb{C} \longrightarrow \mathbb{C} \qquad \text{and} \qquad g : \mathbb{C} \longrightarrow \mathbb{C} \tag{12.24}$$

we may form

$$f \circ g : \mathbb{C} \longrightarrow \mathbb{C} \qquad \text{or} \qquad g \circ f : \mathbb{C} \longrightarrow \mathbb{C}. \tag{12.25}$$

In general, these will not be the same. If it so happens that f is bijective (one-to-one and onto) then there exists a function we denote $f^{-1} : \mathbb{C} \longrightarrow \mathbb{C}$ and called the inverse function to f and which is characterized by

$$f \circ f^{-1} = \iota \qquad \text{and} \qquad f^{-1} \circ f : \iota. \tag{12.26}$$

Be careful not to confuse f^{-1} (if it exists!) with $\frac{1}{f}$ (if it exists!). (For example, if $f(z) = 1 - z$, then $1/f(z) = 1/(1-z)$ while $f^{-1}(z) = 1 - z$.) In fact, 'reciprocal' functions like, for example,

$$f(z) = \frac{1}{z-a} \quad \text{and} \quad g(z) = \frac{1}{z^2-4} = \frac{1}{4}\left(\frac{1}{z-2} - \frac{1}{z+2}\right) \tag{12.27}$$

will be playing a central role in everything that follows. Notice that these reciprocal functions are not defined everywhere on $\mathbb{C}$, but rather on $\mathbb{C}$ minus those points where the denominator function vanishes.

Of course ,we can also add and multiply functions together which, unlike taking the reciprocal, can be done without further thought

$$f + g : \mathbb{C} \longrightarrow \mathbb{C} \qquad \text{and} \qquad gf = fg : \mathbb{C} \longrightarrow \mathbb{C}. \tag{12.28}$$

and also define the quotient

$$z \longmapsto \frac{f(z)}{g(z)} \quad \text{provided } g(z) \neq 0. \tag{12.29}$$

That is, we may regard f/g as a function on $\mathbb{C}$ which has *singularities* at those points $z_0 \in \mathbb{C}$ where $g(z_0) = 0$ (meaning the function is not defined at those points).

12.2 Holomorphic functions

Within the space of all functions $f : \mathbb{C} \to \mathbb{C}$ there is a distinguished subspace of *holomorphic functions*, often also called *analytic functions*. Being holomorphic is just a *local property*, meaning that whether a function is *holomorphic at a point* $a \in \mathbb{C}$ depends only on the value of f at a and, for some small real number $\varepsilon > 0$, and its behaviour in a small disc

$$B_\varepsilon(a) : \{z \in \mathbb{C} \mid |z - a| < \varepsilon\}$$

around a. (By definition, $B_\varepsilon(a)$ consists of those complex numbers whose distance from a is less than ε.)

Working definition: *A function is holomorphic at* $a \in \mathbb{C}$ *if it is independent of* $\overline{z}$ *near* a *and has no singularity at* $z = a$ *(meaning it is well defined at all points near* a *and is differentiable (smooth) in* z*) there.*

In practise, those are the properties we look for in order to identify whether function is holomorphic at a given point: it must be a function of z alone and must be differentiable, the latter meaning (in practise) that if you replace z by a real variable x then you recognize the resulting function as differentiable in the usual (real variable) sense.

Specifically, in the above examples of functions $f : \mathbb{C} \to \mathbb{C}$, each of the functions in (12.9), (12.10), (12.11), (12.12) is *not holomorphic* at any $z \in \mathbb{C}$, but all the others are holomorphic everywhere in $\mathbb{C}$ except (12.14) which is holomorphic at all points of $\mathbb{C}\backslash\{2\}$ – that is, it is not holomorphic at $z = 2$, because it has a singularity there, but it is holomorphic everywhere else. In particular, for any given $b \in \mathbb{C}$ the exponential function

$$f(z) = e^{bz}$$

is holomorphic at all $z \in \mathbb{C}$. That immediately implies, from (12.17) – (12.20), that all the trigonometric and hyperbolic functions

$$f(z) = \sin z, \quad f(z) = \cos z, \quad f(z) = \sinh z, \quad f(z) = \cosh z, \tag{12.30}$$

are all holomorphic at all $z \in \mathbb{C}$. The implication is immediate because of the following properties which tell us we can build many holomorphic functions just by knowing a few simple ones:

$$f,\ g\ ,: \mathbb{C} \to \mathbb{C}\ \textbf{holomorphic}\ \text{at}\ a \in \mathbb{C} \ \Rightarrow\ f.g,\ f{+}g,\ f{\circ}g\ \textbf{holomorphic}\ \text{at}\ a \in \mathbb{C}. \tag{12.31}$$

and

$$\frac{f}{g}\ \textbf{holomorphic}\ \text{at}\ a \in \mathbb{C}\ \ \textbf{provided}\ g(a) \neq 0. \tag{12.32}$$

Thus, as another example, because $f(z) = z$ is holomorphic (everywhere) then so is any polynomial

$$f(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0, \quad a_j \in \mathbb{C}. \tag{12.33}$$

On the other hand,

$$f(z) = \frac{e^z}{z-b} \tag{12.34}$$

is holomorphic at all points *except* at $z = b$.

<u>*When*</u> *we have a function which is holomorphic at* $a \in \mathbb{C}$ then its derivative

$$f'(a) := \left.\frac{\partial f}{\partial z}\right|_{z=a} \in \mathbb{C}$$

at a is defined and we may compute it in the usual way — as a partial derivative with respect to z. All the usual identities hold:

$$\frac{\partial}{\partial z} z^n = n z^{n-1} \qquad (\text{if } n < 0 \text{ then for } z \neq 0).$$

$$\text{For any fixed } \lambda \in \mathbb{C}, \quad \frac{\partial}{\partial z}\, e^{\lambda z} = \lambda\, e^z$$

and hence from (12.17) – (12.20)

$$\frac{\partial}{\partial z}\cos z = -\sin z, \quad \frac{\partial}{\partial z}\sin z = \cos z,$$

$$\frac{\partial}{\partial z}\cosh z = \sinh z, \quad \frac{\partial}{\partial z}\sinh z = \cosh z.$$

'Hence' because all the usual properties of (partial) differentiation hold: if $f,\ g\ : \mathbb{C} \to \mathbb{C}$ are holomorphic at $z \in \mathbb{C}$ then

$$(f+g)'(z) = f'(z) + g'(z), \quad (\lambda f)'(z) = \lambda f'(z),$$

$$(fg)'(z) = f(z)g'(z) + f'(z)g(z), \quad (f \circ g)^{'}(z) = g^{'}(z)\, f^{'}(g(z)),$$

and also $(f(z)/g(z))' = (g(z)f'(z) - f(z)g'(z))/(g^2(z))$ provided $g(z) \neq 0$.

A more mathematically rigorous definition of holomorphic: *Let* $a \in \mathbb{C}$ *and let* $\varepsilon > 0$ *be a positive real number. A function is holomorphic at* $a \in \mathbb{C}$ *if there exists an* $\varepsilon > 0$ *such that there is a power series expansion*

$$f(z) = \sum_{n=0}^{\infty} c_n\, (z-a)^n \quad \text{valied for all } z \text{ with } |z-a| < \varepsilon. \tag{12.35}$$

Thus the expansion must hold for all z *in an 'open disc' of radius* ε *centred at* a*, that is, for the set of points which have distance less than* ε *from* a*.*

In fact, when this holds it is just the complex 'Taylor series' expansion: the coefficients are given by

$$c_n = \frac{1}{n!}\, f^{(n)}(a), \qquad \text{where } f^{(n)}(a) := \left.\frac{\partial^n f}{\partial z^z}\right|_{z=a}. \tag{12.36}$$

For example, the exponential is can be expanded around $a = 0$ into the Taylor power series

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!}\, z^n \tag{12.37}$$

This particular expansion, in fact, holds for all z, i.e. we can take ε arbitrarily large. We can likewise compute its expansion (12.35) around any other $a \in \mathbb{C}$ using the multiplicative property (12.16) to see that

$$e^z \;=\; e^a e^{z-a} \;\overset{(12.37)}{=}\; e^a \cdot \sum_{n=0}^{\infty} \frac{1}{n!}\,(z-a)^n \;=\; \sum_{n=0}^{\infty} \underbrace{\frac{e^a}{n!}}_{=\,c_n}\,(z-a)^n. \tag{12.38}$$

As another example,

$$f(z) := \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \text{valied for all } z \text{ with } |z| < 1. \tag{12.39}$$

That is, (12.35) holds for $f(z) = 1/(1-z)$ at $a = 0$ with $\varepsilon = 1$; that is, the expansion is valid for all z which distance less than 1 from the origin. This expansion follows by the 'same' proof as for real variable functions — one has

$$1 - z^{n+1} = (1-z)(1 + z + \cdots + z^n)$$

so that

$$\frac{1}{1-z} = 1 + z + \cdots + z^n \;+\; \frac{z^{n+1}}{1-z}. \tag{12.40}$$

But, writing z in polars $z = Re^{i\theta}$ we have

$$\left|\frac{z^{n+1}}{1-z}\right| \;\leq\; \frac{R^{n+1}}{\underbrace{1-|R|}_{>0}} \quad \underbrace{\longrightarrow 0 \quad \text{as } n \to \infty}_{\text{provided } R<1}. \tag{12.41}$$

Hence letting $n \to \infty$ in (12.40) gives (12.39).

Exercise: Justify the estimate in (12.41).

Thus (12.39) shows (rigorously) that $1/(1-z)$ is holomorphic at the origin $a = 0$.

We can immediately deduce from (12.39) that there is a power series expansion around any $a \in \mathbb{C}\backslash\{1\}$ — indeed, as **you** can see (12.39) implies the power series expansion

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{1}{(1-a)^{n+1}}\,(z-a)^n \quad \text{valied for all } |z-a| < |1-a|, \tag{12.42}$$

that is, valid provided z is closer to a that a is to 1 – which makes sense, does it not?

Of course, it is 'obvious' from (12.32) that $f(z) = \frac{1}{1-z}$ is holomorphic everywhere except at $z = 1$ because it is the quotient of two functions (1 and $1-z$) which really are obviously holomorphic everywhere, so the only points where f will fail to be holomorphic are where the denominator has zeroes, i.e. at $z = 1$.

12.2.1 Some revision (from Calculus I)— Roots of unity:

Let n be a positive integer. We want to find the n roots in $\mathbb{C}$ of the equation $z^n = 1$. To this end write $z = r\mathrm{e}^{i\theta},\ r = |z|,\ \theta = arg(z)$. Substitution gives $r^n(\mathrm{e}^{i\theta})^n = 1$ and therefore $r^n\mathrm{e}^{ni\theta} = (1)\mathrm{e}^{i(0+2k\pi)}$, where k is any integer. It follows that $r^n = 1$, so that $r = 1$ (since r is real), and that $n\theta = 2k\pi$. We can choose $k = 0, 1, 2, 3, \ldots (n-1)$ to generate the required n roots; other values of k merely give repetitions, as one easily checks. We conclude that the n solutions of $z^n = 1$ are given by

$$z = z_k = \mathrm{e}^{i(2k\pi/n)}, \quad k = 0, 1, 2, 3, \ldots, (n-1).$$

The n values $z_k,\ \ k = 0, 1, 2, 3, \ldots, (n-1)$ are referred to as the n-th roots of unity. $k = 0$ gives the obvious real root $z = z_0 = 1$. Geometrically the n-th roots of unity lie on the unit circle centre 0; the angular separation between consecutive roots is clearly $2\pi/n$. As an illustration consider the following example.

For example, the roots of the equation $z^7 = 1$ are given by

$$z = z_k = \mathrm{e}^{i(2k\pi/7)} = \cos(2k\pi/7) + i\sin(2k\pi/7), k = 0, \pm1, \pm2, \pm3.$$

(It's convenient to choose these values of k rather than $k = 0, 1, 2, 3, 4, 5, 6.$ to generate the 7 roots) Now

$$(z^7 - 1) = (z-1)(z-z_1)(z-z_{-1})(z-z_2)(z-z_{-2})(z-z_3)(z-z_{-3}).$$

Since $(z-z_k)(z-z_{-k}) = z^2 - z(z_k + z_{-k}) + z_k z_{-k} = z^2 - 2z\cos(2k\pi/7) + 1$ we deduce that

$$\frac{z^7-1}{z-1} = (z^2-2z\cos(2\pi/7)+1)(z^2-2z\cos(4\pi/7)+1)(z^2-2z\cos(6\pi/7)+1),\ z \neq 1.$$

The left-hand side is equal to $1 + z + z^2 + \cdots + z^6$ and if we now let $z \to 1$ we obtain the formula

$$7 = 2^3(1-\cos(2\pi/7))(1-\cos(4\pi/7))(1-\cos(6\pi/7)) = 2^3 2^3 \sin^2(\pi/7)\sin^2(2\pi/7)\sin^2(3\pi/7).$$

Equivalently

$$\sin^2(\pi/7)\sin^2(2\pi/7)\sin^2(3\pi/7) = 7/64.$$

This result can obviously be generalized by applying the same considerations to the equation $z^{(2n+1)} = 1$.

13 Lecture 13: Elements of the Cauchy Residue Theorem

Here we are going to put in place some more of the elements that make up the statement of the Cauchy-Residue Theorem – as stated as the beginning of the last lecture. Remember that the point of all this is to enable us to compute Fourier transforms explicitly and hence provide exact solutions to PDEs.

13.1 The Cauchy-Riemann equations.

Recall that $\phi : \mathbb{C} \to \mathbb{C}$ is said to be *holomorphic* at $a \in \mathbb{C}$ (note: this is also often alternatively referred to as being "analytic at a") if there is a power series expansion

$$\phi(z) = \sum_n \beta_n \, (z-a)^n, \qquad \beta_n = \frac{\phi^{(n)}(a)}{n!}, \qquad \text{for all } z \text{ `near' } a. \tag{13.1}$$

More precisely, 'near' means that there exists an $\varepsilon > 0$ such that (13.5) holds for

$$z \in D_\varepsilon(a) := \{w \in \mathbb{C} \ : \ |w-a| < \varepsilon\}.$$

At any rate, (13.5) says that ϕ has an expansion valid for z sufficiently near to a

$$\phi(z) \ = \ \beta_0 \ + \ \beta_1 \, (z-a) \ + \ \beta_2 \, (z-a)^2 \ + \ \beta_3 \, (z-a)^3 \ + \ \cdots \tag{13.2}$$

in *positive* powers of $z - a$.

Another characterization of holomorphic is as follows:

Suppose that

$$f : \mathbb{C} \to \mathbb{C}, \qquad z = x + iy \mapsto f(z) = u(x,y) + iv(x,y),$$

is differentiable at $a \in \mathbb{C}$ *— meaning that the partial derivatives* $u_x = \partial u/\partial x$, u_y, v_x v_y *exist and are continuous. Then* f *is holomorphic at* $a \in \mathbb{C}$ *if and only if in some small enough disc* $D_\varepsilon(a)$ *centred at* $a \in \mathbb{C}$ *one has*

$$\frac{\partial f}{\partial \overline{z}} = 0. \tag{13.3}$$

An equivalent way to state (13.3) is

$$u_x = v_y \quad \text{and} \quad u_y = -v_x. \tag{13.4}$$

So (13.3) says that f is independent of $\overline{z}$ in D_ε, which, since it is differentiable and hence has no singularities, is what we said holomorphic intuitively means.

The equivalent form (13.4) (to (13.3)) are called the Cauchy Riemann equations.

To see why they are equivalent we can apply the Chain rule to the change of coordinates

$$(x, y) \mapsto (z = x + iy,\ \overline{z} = x - iy),$$

to obtain

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial z}{\partial x}\frac{\partial}{\partial z} + \frac{\partial \overline{z}}{\partial x}\frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}},\\ \frac{\partial}{\partial y} &= \frac{\partial z}{\partial y}\frac{\partial}{\partial z} + \frac{\partial \overline{z}}{\partial y}\frac{\partial}{\partial \overline{z}} = i\frac{\partial}{\partial z} - i\frac{\partial}{\partial \overline{z}},\end{aligned}$$

which gives

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

And hence (13.3) is the equality

$$\begin{aligned}0 = \frac{\partial f}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u(x,y) + iv(x,y))\\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\end{aligned}$$

which yields (13.4) on equating the real and imaginary parts to zero.

Example: *Find, in terms of $z = x + iy$, the most general holomorphic function whose real part is $e^x \sin y$.*

Solution: *Set $f(z) = u + iv$ with $u = e^x \sin y$. The Cauchy-Riemann equations state that*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

which gives

$$e^x \sin y = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -e^x \cos y.$$

The first of these equations gives $v = -e^x \cos y + f_1(x)$ and substituting this in the second gives

$$-e^x \cos y + f_1'(x) = -e^x \cos y, \qquad f_1(x) = C, \quad C \in \mathbb{R}.$$

Hence

$$f(z) = e^x \sin y + i(-e^x \cos y + C) = -ie^z + iC, \quad C \in \mathbb{R}.$$

Example: *Find, using the Cauchy-Riemann equations, the most general analytic function $f : \mathbb{C} \to \mathbb{C}$ whose real part is given by $u(x,y) = x - e^{-y}\sin x$, $\forall z = x+iy \in \mathbb{C}$. Express your answer in terms of z.*

Solution: *Since f is holomorphic is satisfies the Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

from which we derive

$$\frac{\partial v}{\partial y} = 1 - e^{-y}\cos x \quad (*)$$

$$\frac{\partial v}{\partial x} = -e^{-y}\sin x \quad (**)$$

Equation (*) then gives

$$v(x,y) = y + e^{-y}\cos x + g(x)$$

and substitution in equation (**) shows that g must satisfy

$$-e^{-y}\sin x + g'(x) = -e^{-y}\sin x, \qquad g'(x) = 0, \quad g(x) = C$$

where C is real. Hence

$$f(z) = x + iy - e^{-y}\sin x + ie^{-y}\cos x + iC = z + ie^{iz} + iC.$$

Example: *Show that if a holomorphic function has constant real part, then the function is constant*

Solution: *With $z = x + iy$ we have $f(z) = c + iv(x,y)$ for some real constant c. The Cauchy-Riemann equations therefore imply that*

$$\frac{\partial v}{\partial y} = 0, \qquad \frac{\partial v}{\partial x} = 0,$$

which says that v is independent of both x and y. Hence $v(x,y) = c'$ is a real constant and thus $f(z) = c + ic'$ is likewise constant.

13.2 Poles and residues

If $\phi : \mathbb{C} \to \mathbb{C}$ is holomorphic $a \in \mathbb{C}$ (note: this is also often alternatively referred to as being "analytic at a") then there is a power series expansion

$$\phi(z) = \sum_{n\geq 0} \beta_n \,(z-a)^n, \qquad \beta_n = \frac{\phi^{(n)}(a)}{n!}, \qquad \text{for all } z \text{ 'near' } a. \tag{13.5}$$

That is, ϕ has an expansion valid for z sufficiently near to a

$$\phi(z) \;=\; \beta_0 \;+\; \beta_1\,(z-a) \;+\; \beta_2\,(z-a)^2 \;+\; \beta_3\,(z-a)^3 \;+\; \cdots \tag{13.6}$$

in *positive* powers of $z-a$.

On the other hand, if $f:\mathbb{C}\to\mathbb{C}$, $z\mapsto f(z)$, has an *isolated singularity* at $a\in\mathbb{C}$ (meaning that f is not holomorphic at a but it is holomorphic at those points $z\in\mathbb{C}$ with $0<|z-a|<\epsilon$ some $\epsilon>0$) and the singularity at a looks like

$$f(z) = \frac{\beta_{-m}}{(z-a)^m} \;+\; \ldots \;+\; \frac{\beta_{-1}}{(z-a)} \;+\phi(z), \quad 0<|z-a|<\epsilon, \tag{13.7}$$

with ϕ holomorphic for $|z-a|<\epsilon$, then f is said to have a **pole of order m at a**. Note that the complex numbers occuring in (13.7)

$$\beta_{-r} = \beta_{-r}(a)$$

will depend on the point $a\in\mathbb{C}$. If (13.7) holds with $m=1$, i.e. if for some $\varepsilon>0$

$$f(z) = \frac{\beta_{-1}(a)}{(z-a)} \;+\phi(z), \quad 0<|z-a|<\epsilon, \tag{13.8}$$

with ϕ holomorphic at $a\in\mathbb{C}$, then f is said to have a **simple pole at a**.

If f has a pole at a (of some order) then the **residue of f at** a is defined by

$$\mathrm{res}(f,a) \;=\; \beta_{-1}(a). \tag{13.9}$$

(13.7) can be equivalently written

$$f(a+h) = \frac{\beta_{-m}(a)}{h^m} \;+\; \ldots \;+\; \frac{\beta_{-1}(a)}{h} \;+\phi(a+h), \quad 0<|h|<\epsilon, \tag{13.10}$$

which can sometimes be easier for computing (13.9); note also that (13.5) may be similarly written

$$\phi(a+h) = \sum_{n\geq 0}\beta_n(a)\,h^n, \qquad \beta_n(a) = \frac{\phi^{(n)}(a)}{n!}, \qquad \text{for all `}\mathit{small}\text{' } h. \tag{13.11}$$

13.2.1 Computing residues

In order to use the Cauchy Residue Theorem effectively we need to have some methods for computing residues. For particularly simple functions one can do this directly, but more generally it is usually easier to resort to one of the following formulae for computing residues.

Suppose that

$$f(z) = \frac{\phi(z)}{z-a} \quad \textit{where } \phi \textit{ is holomorphic at } a. \tag{13.12}$$

Then

$$\mathrm{res}(f,a) \;=\; \phi(a). \tag{13.13}$$

We can use this to easily compute, for example, the following residues (which can also be computed directly from the expansion (13.7) quite easily). So clearly we have

$$\text{if } f(z) = \frac{e^z}{z-4} \text{ then } \mathrm{res}(f,4) \;=\; e^4,$$

for example. More subtle is the function

$$f(z) = \frac{1}{(z-5)(z-1)}.$$

This has two simple-poles, one at 1 and one at 5 – that's because it's the quotient of two polynomials in z, and polynomials in z which are holomorphic everywhere, so the only singular points can arise from the zeroes of the polynomial in the denominator.

<u>The pole at 1:</u>

Direct computation: We have

$$f(z) = -\frac{1}{4}\frac{1}{(z-1)} + \underbrace{\frac{1}{4}\frac{1}{(z-5)}}_{\text{holomorphic at } z=1}. \tag{13.14}$$

The crucial point here is that this $\phi(z) = \frac{1}{4}\frac{1}{(z-5)}$ *is* holomorphic *at* $z=1$ — in fact, it is holomorphic everywhere except at $z=5$. So (13.14) is of the form (13.8) from which we read-off

$$\mathrm{res}\left(\frac{1}{(z-5)(z-1)}, 1\right) = -\frac{1}{4}. \tag{13.15}$$

Alternatively, we may use (13.13): write

$$f(z) = \frac{\phi(z)}{z-1} \quad \text{with} \quad \phi(z) = \frac{1}{z-5}.$$

Again, this $\phi(z)$ *is* holomorphic *at* $z=1$ so we can apply (13.13) and obtain

$$\mathrm{res}\left(\frac{1}{(z-5)(z-1)}, 1\right) \;=\; \phi(1) = \frac{1}{1-5} = -\frac{1}{4}. \tag{13.16}$$

The pole at 5: Exercise – use a similar argument to show that

$$\mathrm{res}\left(\frac{1}{(z-5)(z-1)}, 5\right) \;=\; \frac{1}{4}. \tag{13.17}$$

An easy generalization of (13.13) is:

Suppose

$$f(z) = \frac{\phi(z)}{(z-a)^m}, \quad \phi \;\; \textit{holomorphic at } a, \tag{13.18}$$

then

$$\mathrm{res}(f, a) \;=\; \frac{\phi^{(m-1)}(a)}{(m-1)!}. \tag{13.19}$$

For example,

$$\text{if} \;\; f(z) = \frac{\sin(3z)}{(z-2i)^3}$$

then we can apply (13.19) with $m = 3$, $a = 2i$ and $\phi(z) = \sin(3z)$. Since, then,

$$\phi^{(2)}(z) = -9\sin(3z)$$

we have $\phi^{(2)}(2i) = -9\sin(6i) = 9i\sinh(6)$ and hence

$$\mathrm{res}(f, 2i) \;=\; \frac{9i}{2}\sinh(6).$$

But not all functions with a pole at $a \in \mathbb{C}$ look like (13.18) —for example

$$f(z) = \cot(\pi z)$$

has a simple pole at each integer, but is not of the form (13.18). But in this case, at least, we can use the following residue formula:

Suppose that

$$f(z) \;=\; \frac{\phi(z)}{\psi(z)} \tag{13.20}$$

with ϕ and ψ holomorphic at $a \in \mathbb{C}$, and that

$$\psi(a) = 0 \quad \text{and} \quad \psi^{'}(a) \neq 0.$$

Then f has a simple *pole at $z = a$ and*

$$\operatorname{res}(f, a) \;=\; \frac{\phi(a)}{\psi'(a)}. \tag{13.21}$$

Notice that the requirement that $\psi^{'}(a) \neq 0$ means that this formula does not apply to (13.18) if $m > 1$.

Example For

$$f(z) = \cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)}$$

we can apply the formula with $\phi(z) = \cos(\pi z)$ and $\psi(z) = \sin(\pi z)$; for, both these functions are holomorphic everywhere and so the only poles of f are where $\sin(\pi z) = 0$, which occurs when $z = n \in \mathbb{Z}$ is an integer and then $\psi'(n) = \pi\cos(\pi n) \neq 0$ and we have

$$\operatorname{res}(\cot(\pi z), n) \;=\; \frac{1}{\pi}. \tag{13.22}$$

14 Lecture 14: Computing more residues

In this lecture we saw how to derive the formulae below for residues and saw some more examples.

14.1 Formulae for computing residues

Suppose that

$$f(z) = \frac{\phi(z)}{z-a} \quad \textit{where } \phi \textit{ is holomorphic at } a. \tag{14.1}$$

Then

$$\mathrm{res}(f,a) \;=\; \phi(a). \tag{14.2}$$

Suppose

$$f(z) = \frac{\phi(z)}{(z-a)^m}, \quad \phi \;\; \textit{holomorphic at } a, \tag{14.3}$$

then

$$\mathrm{res}(f,a) \;=\; \frac{\phi^{(m-1)}(a)}{(m-1)!}. \tag{14.4}$$

Suppose that

$$f(z) \;=\; \frac{\phi(z)}{\psi(z)} \tag{14.5}$$

with ϕ and ψ holomorphic at $a \in \mathbb{C}$, and that

$$\psi(a) = 0 \quad \text{and} \quad \psi^{'}(a) \neq 0.$$

Then f has a simple *pole at $z = a$ and*

$$\mathrm{res}(f,a) \;=\; \frac{\phi(a)}{\psi'(a)}. \tag{14.6}$$

Example To identify where $f(z) = \frac{\sinh(z)}{z-3}$ has a pole, note that both $\phi(z) = \sinh(z)$ and $\psi(z) = z-3$ are holomorphic for all z and hence the quotient $\phi(z)/\psi(z)$ is holomorphic except at any point where $\psi(z) = 0$. For the case here, there is therefore a pole at $3 \in \mathbb{C}$. To compute its residue we may use (14.2) to see

$$\mathrm{res}(f,3) \;=\; \phi(3) = \sinh(3).$$

We might, on the other hand, alternatively use (14.6), though (14.2) is simpler in this case. Alternatively, we can compute directly

$$f(3+h) = \frac{\sinh(3+h)}{h} \quad = \quad \frac{1}{h} \cdot \frac{1}{2}(e^3 e^h - e^{-3} e^{-h})$$

and use $e^{\pm h} = \sum_{n \geq 0} (\pm h)^n / n!$ to see that the coefficient of $1/h$ is $\frac{1}{2}(e^3 - e^{-3})$, confirming what we have just computed using the formulae.

Note, though, if we instead want to know the residue of

$$g(z) = \frac{\sinh(z)}{(z-3)^2}$$

then neither (14.2) nor (14.6) can be used, but (14.4) can. So, what is

$$\text{res}\left(\frac{\sinh(z)}{(z-3)^2},\ 3\right) \quad ?$$

Example For

$$f(z) = \frac{1}{z^4+1}$$

we can apply the formula with $\phi(z) = 1$ and $\psi(z) = z^4 + 1$. Both these functions are holomorphic everywhere and so the only poles of f are where $z^4 + 1 = 0$, that is, where $z^4 = e^{i\pi} = e^{(2k+1)\pi i}$ with k an integer. There are at most four distinct solutions (being a polynomial of order four) and these occur for $k = 0, 1, 2, 3$; that is, at

$$e^{\frac{\pi}{4}i}, \quad e^{\frac{3\pi}{4}i}, \quad e^{\frac{5\pi}{4}i}, \quad e^{\frac{7\pi}{4}i}.$$

We have $\psi'(e^{\frac{(2k+1)}{4}\pi i}) = 4e^{\frac{3}{4}(2k+1)\pi i} \neq 0$, so we can apply the formula to get

$$\text{res}\left(\frac{1}{z^4+1}, e^{\frac{(2k+1)}{4}\pi i}\right) \quad = \quad \frac{1}{4}\, e^{-\frac{3}{4}(2k+1)\pi i}. \tag{14.7}$$

For instance,

$$\text{res}\left(\frac{1}{z^4+1}, e^{\frac{\pi}{4}i}\right) \quad = \quad \frac{1}{4}\, e^{-\frac{3}{4}\pi i} = -\frac{1}{4\sqrt{2}}(1+i). \tag{14.8}$$

Exercise — compute in the form $a + ib$ the other three poles

Exercise: state where the poles of the function

$$f(z) = \frac{e^{iz}}{z^2 + a^2}$$

occur (there are two) and compute the residue at each point. See the Assignment sheets for more examples.

15 Evaluating integrals with the CRT

The Cauchy Residue Theorem:

Let $C \subset \mathbb{C}$ be a simple closed contour. Let $f : \mathbb{C} \to \mathbb{C}$ be a complex function which is holomorphic along C and inside C except possibly at a finite number of points $a_1, \ldots, a_m$ at which points f has poles. Then, with C oriented in an anti-clockwise sense,

$$\int_C f(z)\,dz \;=\; 2\pi i \sum_{k=1}^{m} \operatorname{res}(f, a_k), \tag{15.1}$$

where $\operatorname{res}(f, a_k)$ is the residue of the function f at the point $a_k \in \mathbb{C}$. If f has no poles inside C (f is holomorphic inside C) then $\int_C f(z)\,dz = 0$.

15.1 The right-hand side of the Cauchy Residue Formula (15.1): adding-up residues

If $\phi(z)$ is analytic at a, then from (15.1) and formula (14.2) of online lecture 14 we have

$$\begin{aligned} \int_C \frac{\phi(z)}{z-a}\,dz &= 2\pi i \operatorname{res}\left(\frac{\phi(z)}{z-a}, a\right) \qquad (15.2)\\ &= \begin{cases} 2\pi i\,\phi(a), & \text{if } a \text{ is inside } C \\ 0, & \text{if } a \text{ is outside } C. \end{cases} \end{aligned}$$

Let

$$C(a, R) \;=\; \text{circle centred at } a \in \mathbb{C} \text{ and with radius } R > 0.$$

15.2 Example

Since the functions e^z and z are holomorphic every where the only pole of $f(z) = \frac{e^z}{z}$ is at $z = 0$ with, by (15.2), residue equal to $e^0 = 1$. Hence

$$\int_{C(0,1)} \frac{e^z}{z}\,dz = 2\pi i. \tag{15.3}$$

15.3 Example

Evaluate

$$\int_\gamma \frac{e^z}{(z-1)(z-3)}\,dz,$$

taken round the circle γ given by $|z| = 2$ in the positive (anti-clockwise) sense. What is the value of the integral taken around the circle $|z| = 1/2$ in the positive sense?

We have $f(z) = \frac{\phi(z)}{z-a}$ with $\phi(z) = \frac{e^z/(z-3)}{z-1}$ and $a = 1$; i.e. this $\phi(z) = e^z/(z-3)$ is analytic everywhere *inside* the curve $|z| = 2$ — and, in particular, at $z = 1$. Hence from (15.2)

$$\int_\gamma \frac{e^z}{(z-3)(z-1)}\,dz = 2\pi i\,\phi(1) = -\pi i\,e.$$

If the integral is taken around $|z| = 1/2$ then the CIF says that

$$\int_{|z|=1/2} \frac{e^z}{(z-3)(z-1)}\,dz = 0$$

as the integrand is analytic everywhere inside the contour (i.e. there are no poles).

Likewise, from (15.1) and (14.4) of online lecture 14 we can compute

$$\begin{aligned}\int_C \frac{\phi(z)}{(z-a)^m}\,dz &= 2\pi i\,\mathrm{res}\left(\frac{\phi(z)}{(z-a)^m}, a\right)\\ &= \begin{cases} 2\pi i\,\frac{\phi^{(m-1)}(a)}{(m-1)!}, & \text{if } a \text{ is inside } C\\ 0, & \text{if } a \text{ is outside } C.\end{cases}\end{aligned}$$

15.4 Example

Evaluate the contour integral

$$\int_C \frac{\sin(3z)}{(z-2i)^2}\,dz. \tag{15.4}$$

With $\phi(z) = \sin(3z)$, $a = 2i$, $m = 2$, we have

$$\int_C \frac{\sin(3z)}{(z-2i)^2}\,dz = 2\pi i\,3\cosh 6 = 6\pi i\cosh 6.$$

On the other hand, if g analytic at $a \in \mathbb{C}$ while

$$\psi(a) = 0 \quad \text{and} \quad \psi'(a) \neq 0,$$

from (15.1) and (14.6) of online lecture 14 we infer that if a is *inside* the contour C then

$$\int_C \frac{g(z)}{\psi(z)}\,dz = 2\pi i\,\frac{g(a)}{\psi'(a)}. \tag{15.5}$$

If there are two points $a_1, a_2 \in \mathbb{C}$ which are *inside* the contour C and which are poles of this type, then

$$\int_C \frac{g(z)}{\psi(z)}\,dz = 2\pi i\,\frac{g(a_1)}{\psi'(a_1)} + 2\pi i\,\frac{g(a_2)}{\psi'(a_2)}, \tag{15.6}$$

and so on.

15.5 Example

Evaluate

$$\int_{C(0,5/2)} \cot(\pi z)\,dz.$$

We saw in an earlier Insertion that $\cot(\pi z)$ has a simple pole at each integer $n \in \mathbb{Z}$ with residue

$$\mathrm{res}(\cot(\pi z), n) = \frac{1}{\pi}$$

(independent of n). Now, the contour $C(0, 5/2)$ is the circle centred at the origin with radius $5/2$, and hence it contains within it *only* the integers $-2, -1, 0, 1, 2$ and hence these are the only poles of $\cot(\pi z)$ which contribute to the integral. That is, using the extension of (15.6) to the case of 5 poles of that type we obtain

$$\int_{C(0,5/2)} \cot(\pi z)\,dz$$

$$= 2\pi i\,\{\mathrm{res}(\cot(\pi z), -2) + \mathrm{res}(\cot(\pi z), -1) + \mathrm{res}(\cot(\pi z), 0) + \mathrm{res}(\cot(\pi z), 1) + \mathrm{res}(\cot(\pi z), 2)\}$$

$$= 2\pi i\left(\frac{1}{\pi} + \frac{1}{\pi} + \frac{1}{\pi} + \frac{1}{\pi} + \frac{1}{\pi}\right)$$

$$= 10\,i.$$

15.6 Example

Evaluate
$$\int_{C_R} \frac{1}{z^4+1}\, dz,$$
where C_R is the union of the line segment $[-R, R]$ on the x-axis and the semicircle centre at 0 and radius $R > 0$ in the upper-half plane.

Write the integrand as
$$f(z) = \frac{g(z)}{\phi(z)}, \qquad g(z) = 1, \quad \phi(z) = 1 + z^4.$$

Since g and ϕ are analytic everywhere ('entire') the poles of f are precisely the zeroes of ϕ — that is, at $e^{\pm i\pi/4}, e^{\pm i3\pi/4}$. But only $e^{i\pi/4}, e^{i3\pi/4}$ are inside C_R and so only these two poles contribute to the integral, i.e. by the CRF
$$\int_{C_R} \frac{1}{z^4+1}\, dz = 2\pi i \left\{ \operatorname{res}(f(z), e^{i\pi/4}) + \operatorname{res}(f(z), e^{i3\pi/4}) \right\}.$$

We have $\phi'(z) = 4z^3$ and this is non-zero at the poles, i.e. $\phi'(e^{i\pi/4})$ and $\phi'(e^{i3\pi/4})$ are non-zero. Hence the poles are simple and using (15.5) we have
$$\operatorname{res}(f(z), e^{i\pi/4}) = \frac{1}{4e^{i3\pi/4}} = \frac{e^{-3i\pi/4}}{4}, \qquad \operatorname{res}(f(z), e^{3i\pi/4}) = \frac{1}{4e^{i9\pi/4}} = \frac{e^{-i\pi/4}}{4}.$$

Hence
$$\int_{C_R} \frac{1}{z^4+1}\, dz = \frac{\pi}{\sqrt{2}}.$$

15.7 Example

Evaluate
$$\int_S \frac{e^{iz}}{z^2+a^2}\, dz,$$
where S is the square with vertices at $\pm a \pm 2ia$.

Write the integrand as
$$f(z) = \frac{g(z)}{\phi(z)}, \qquad g(z) = e^{iz}, \quad \phi(z) = z^2 + a^2.$$

Since g and ϕ are entire, the poles of f are precisely the zeroes of ϕ — that is, at $\pm ia$. Since both poles are are inside S we have by the CRF and (15.5)
$$\int_S \frac{e^{iz}}{z^2+a^2}\, dz = 2\pi i \left\{ \operatorname{res}(f(z), ia) + \operatorname{res}(f(z), -ia) \right\}$$

$$
\begin{aligned}
&= 2\pi i \left\{ \frac{e^{-a}}{2ia} + \frac{e^{a}}{-2ia} \right\}. \\
&= -2\pi \frac{\sinh a}{a}.
\end{aligned}
$$

16 Lecture 16: Integration along a contour

A **path** γ is a smooth function

$$\gamma : [a, b] \to \mathbb{C}, \quad t \mapsto \gamma(t) = x(t) + iy(t)$$

for some $a \leq b$. Here, 'smooth' means $\gamma'(t) = x'(t) + iy'(t)$ exists for each t. We say that γ is **closed** if $\gamma(a) = \gamma(b)$ (it is like a loop of string). γ is said to be **simple** if it does not cross itself i.e $\gamma(t) = \gamma(t')$ implies $t = t'$. γ is said to be **closed and simple** if $\gamma(t) = \gamma(t')$ implies $t = t'$ for $t \in (a, b)$ and also $\gamma(a) = \gamma(b)$.

It can be important to distinguish between a path and the geometric curve in $\mathbb{C}$ it traces out; for example, the paths

$$\gamma_1(t) = e^{2\pi it}, \ 0 \leq t < 1 \quad \text{and} \quad \gamma_2(t) = e^{-4\pi it}, \ 0 \leq t \leq 1,$$

both describe the unit circle in $\mathbb{C}$ (or $\mathbb{R}^2$) of points with distance 1 from the origin, but the first path goes around the circle just once (anti-clockwise) while the second goes around four times and in the opposite sense (clockwise). Thus the second path is not 'simple'. The path $\gamma(t) = e^{-5\pi it}$ with $0 \leq t \leq 1$, on the other hand, also traces out the unit circle but γ is neither closed nor simple. We may, nevertheless, sometimes refer to γ as a 'curve'. The point is that if a path is simple then it defines a coordinate for the geometric curve it describes and we can use that to evaluate integrals along paths/curves.

Note that choosing a simple path $\gamma : [a, b] \to \mathbb{C}$ describing a curve $C \subset \mathbb{C}$ determines a direction along the curve C — one can start at a and end at b, or vice-versa. Given a choice of simple path γ describing C, the reverse path $\widetilde{\gamma}$ describes C in the opposite sense by

$$\widetilde{\gamma} : [a, b] \to \mathbb{C}, \quad \widetilde{\gamma}(t) = \gamma(a + b - t).$$

A **contour** is a piecewise smooth simple path. That is, it is a union of paths which do not cross each other except possibly at the end points where they may match-up — for example, (the perimeter of) a square is the union of four straight lines coinciding pairwise at the corners. More formally, if $\gamma : [a, b] \to \mathbb{C}$ is a contour we can partition $[a, b]$ by points $a_0, a_1, \ldots, a_{m-1}, a_m$ (for some m) such that $a = a_0 < a_1 < a_2 < \ldots < a_{m-1} < a_m = b$ and for which $\gamma : [a_i, a_{i+1}] \to \mathbb{C}$ is smooth for $0 \leq i \leq m - 1$. We denote the restriction of γ to $[a_i, a_{i+1}]$ by γ_i and write in an obvious notation $\gamma = \gamma_0 + \gamma_1 + \cdots + \gamma_{m-1}$. we may refer to the image $C = \gamma([a, b]) \subset \mathbb{C}$ as the geometric contour described by C — note any geometric contour $C \subset \mathbb{C}$ can be described by infinitely many different paths – that is, a path γ is a choice of function, or coordinate, to describe C and there are many such choices (just as there are for an navigational atlas); the point is that γ is a means to evaluate integrals along C, but the numerical answer is independent of the particular choice of γ. The choice of $\gamma = \gamma_0 + \gamma_1 + \cdots + \gamma_{m-1}$ is called a **parametrization** of the geometric contour $C \subset \mathbb{C}$.

16.1 Integration along a contour

We begin by noting that if $g : [a, b] \to \mathbb{C}$ is a continuous complex valued function such that $g(t) = g_1(t) + ig_2(t)$, where g_1, g_2 are real valued on $[a, b]$, then

$$\int_a^b g(t)\,dt = \int_a^b g_1(t)\,dt + i\int_a^b g_2(t)\,dt.$$

Note that it is immediate from the definition that

$$\int_a^b g'(t)\,dt = g(b) - g(a) \tag{16.1}$$

since we know this holds for the real valued functions g_1 and g_2.

Now let $\gamma : [a, b] \to \mathbb{C}$ be a smooth curve and suppose that $f : \mathbb{C} \to \mathbb{C}$ is a function which is continuous in a region containing the path traced out by γ. We wish to define the integral of f along the curve γ,

$$\int_\gamma f(z)\,dz$$

A natural way to proceed is to partition $[a, b]$ as above by points $t_0, t_1, \ldots t_{n-1}, t_n$ such that $a = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = b$. The points $z_j = \gamma(t_j)$ define a polygon with vertices at $z_0, z_1, \ldots, z_n$. We may form the sum

$$\sum_{j=1}^{n} f(t_j)(z_j - z_{j-1}) = \sum_{j=1}^{n} f(t_j)\underbrace{\frac{\gamma(t_j) - \gamma(t_{j-1})}{t_j - t_{j-1}}}_{\gamma'(t_j)\;+\;o(t)}\underbrace{(t_j - t_{j-1})}_{\delta t_j}.$$

(Note, f could be evaluated at any point $s_j \in [t_{j-1}, t_j]$. As we take finer and finer partitions we can take the limit of these sums and as the length of the longest interval tends to zero, it tends to $\int_a^b f(\gamma(t))\gamma'(t)\,dt$.

We will take this integral as our *definition* of $\int_\gamma f(z)\,dz$. To be precise:

Definition 16.1 *Let $\gamma : [a,b] \to \mathbb{C}$ be a smooth curve and suppose that f is a function which is continuous in a region containing the path of γ. Then*

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

Since $z = \gamma(t)$ it may perhaps appeal to our intuition if we write the integral in the equivalent notation

$$\int_a^b f(z(t))\frac{dz}{dt}\,dt, \quad z(t) = \gamma(t).$$

If, in the usual notation, we write $f(z) = u(x,y) + iv(x,y)$, $z(t) = \gamma(t)$ we have

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_a^b \Big\{u\big(x(t),y(t)\big) + iv\big(x(t),y(t)\big)\Big\}(x'(t) + iy'(t))\,dt \\ &= \int_a^b \Big\{u\big(x(t),y(t)\big)x'(t) - v\big(x(t),y(t)\big)y'(t)\Big\}\,dt \\ &+ i\int_a^b \Big\{v\big(x(t),y(t)\big)x'(t) + u\big(x(t),y(t)\big)y'(t)\Big\}\,dt.\end{aligned}$$

If γ is a contour we make the definition:

Definition 16.2

$$\int_\gamma f(z)\,dz = \sum_{j=1}^n \int_{\gamma_j} f(z)\,dz,$$

where the γ_j are the smooth parts of γ. We also often write this indicating only the path C (resp. C_j) traced out by γ (resp γ_j)

$$\int_C f(z)\,dz = \sum_{j=1}^n \int_{C_j} f(z)\,dz.$$

In this case it is assumed that γ (resp. γ_j) is a parametrization of the path C (resp. C_j).

It will be useful to note the following basic properties: Let γ be a contour. For constants $\alpha, \beta \in \mathbb{C}$

- $$\int_\gamma (\alpha f(z) + \beta g(z))\, dz = \alpha \int_\gamma f(z)\, dz + \beta \int_\gamma g(z)\, dz, \tag{16.2}$$

- $$\int_{\widetilde{\gamma}} f(z)\, dz = -\int_\gamma f(z)\, dz, \tag{16.3}$$

  where $\widetilde{\gamma}$ is the reverse curve to γ

- Let
  $$\gamma : [a, b] \longrightarrow \mathbb{C}, \quad t \mapsto \gamma(t), \qquad \mu : [c, d] \longrightarrow \mathbb{C}, \quad s \mapsto \mu(s),$$
  be two parametrizations of a path $C \subset \mathbb{C}$. Then if they both have the same direction
  $$\int_\gamma f(z)\, dz = \int_\mu f(z)\ dz. \tag{16.4}$$
  That is, $\int_C f(z)\, dz$ is independent of the choice of parametrization (up to sign).

16.2 Some Examples

16.2.1 Example

Let us evaluate $\int_C 2z\ dz$ where C is the portion of the circle with centre at 1 and of radius 2 with $\mathrm{Re}(z) \geq 0$ and $\mathrm{Im}(z) \geq 0$

We have to choose a parametrization of C and an easy one is

$$\gamma(t) = 1 + 2e^{it}, \quad 0 \leq t \leq \pi.$$

So
$$\int_C 2z\ dz = \int_0^\pi (1 + 2e^{it})\, 2ie^{it}\ dz = -8,$$
where for the final equality we make use of (16.1).

16.2.2 Example

Evaluate $\int_\gamma \overline{z}\, dz$, where $\gamma = \gamma_1 + \gamma_2$, γ_1 being the line segment from 1 to 0 and γ_2 being the line segment from 0 to $2 + 2i$.

In this case the contour consists of two lines and we need to parametrize them separately by, for example,

$$\gamma_1 : [0, 1] \to \mathbb{C}, \quad \gamma_1(t) = 1 - t,$$

and

$$\gamma_2 : [0,1] \to \mathbb{C}, \qquad \gamma_2(t) = t(2+2i)$$

So

$$\int_C \overline{z}\, dz = \int_{\gamma_1} \overline{z}\, dz \; + \; \int_{\gamma_2} \overline{z}\, dz.$$

We have in each case

$$\int_{\gamma_j} \overline{z}\, dz = \int_0^1 \overline{\gamma_j(t)}\, \gamma_j'(t)\; dt$$

from which we find $\int_{\gamma_1} \overline{z}\, dz = -1/2$ and $\int_{\gamma_2} \overline{z}\, dz = 4$, so that

$$\int_\gamma \overline{z}\, dz = \frac{7}{2}.$$

16.2.3 Example

Evaluate $\int_C z^2\; dz$, where $C = C_1 + C_2$ with C_1 the line segment from -2 to 2 on the real axis, and C_2 the semi-circle of radius 2 and centre 0 in the upper-half plane from 2 to -2.

The contour consists of two pieces which we need to parametrize separately. Parametrize C_1 by, for example,

$$\gamma_1 : [-2,2] \to \mathbb{C}, \qquad \gamma_1(t) = t,$$

and C_2 by, for example,

$$\gamma_2 : [0,\pi] \to \mathbb{C}, \qquad \gamma_2(t) = 2e^{it}.$$

So

$$\int_C z^2\; dz = \int_{-2}^2 t^2\; dt \; + \; \int_0^\pi 4e^{2it} \cdot 2ie^{it}\; dt = 0.$$

Make sure you can see why.

16.2.4 Useful parametrization formulae

Here are some general formulae one can always use (in case no better choice of parametrization is obvious):

To parametrize a straight line which begins at $a \in \mathbb{C}$ and ends at $b \in \mathbb{C}$:

$$\gamma : [0,1] \to \mathbb{C}, \qquad \gamma(t) = a(1-t) + tb = a + t(b-a). \tag{16.5}$$

To parametrize a circle $C(a,r)$ centred at $a \in \mathbb{C}$ and of radius $r > 0$:

$$\gamma : [0,2\pi] \to \mathbb{C}, \qquad \gamma(t) = a + r\, e^{it}. \tag{16.6}$$

Likewise, one can parametrize an arc of the circle by suitably restricting t — what happens, for example, if we restrict t to $[\pi/2, 3\pi/2]$?

16.2.5 A particularly important Example

This example is one of the key elements that goes into the Cauchy Residue theorem. Let $C(a, r)$ be the circle of radius $r > 0$ with centre at $a \in \mathbb{C}$. Then

$$\int_{C(a,r)} \frac{1}{(z-a)^n}\, dz \quad = \quad \begin{cases} 0 & n \neq 1 \\ 2\pi i & n = 1. \end{cases} \tag{16.7}$$

Let us emphasize

$$\frac{1}{2\pi i} \int_{C(a,r)} \frac{1}{(z-a)}\, dz \quad = \quad 1 \tag{16.8}$$

for *any* $r > 0$. This says, in fact, that this integral is measuring a 'topological' (look it up) property of the complex plane with a hole at a (in the sense that the function being integrated is only holomorphic on $\mathbb{C}\backslash\{a\}$, i.e. it is in some sense detecting there is a hole in the punctured plane $\mathbb{C}\backslash\{a\}$ – it does not matter which particular contour you use to detect it (as long as a is inside the contour).

Exercise: use the parametrization (16.6) to see (16.7).

17 Lecture 17: FTC and the CRT

Here, we will outline the argument for the Cauchy Residue Theorem (CRT), based on a certain Fundamental Theorem of Calculus (FTC) for contour integrals.

In practise, we evaluate real integrals analytically by 'reverse differentiation', using the FTC for real integrals. A similar result is true for certain types of contour integration:

Theorem 17.1 *Let $f : \mathbb{C} \to \mathbb{C}$ be a complex function, and let C be a contour beginning at $p \in \mathbb{C}$ and ending at $q \in \mathbb{C}$. If $f = F'$ is the derivative of a function F which is holomorphic at each point of C (recall here $F'(z) := \partial F/\partial z$) then*

$$\int_\gamma f(z)\,dz = F\big(q\big) - F\big(p\big). \tag{17.1}$$

In particular, if C is a closed contour and $f = F'$ then

$$\int_C f(z)\,dz = 0. \tag{17.2}$$

(Recall, a 'closed contour' is a contour which looks something like a circle, or a loop of string, it has no end points.)

Non-examinable: To see this, assume we have a smooth parametrisation of C

$$\gamma : [a,b] \to C \subset C, \quad t \mapsto \gamma(t), \quad p = \gamma(a),\ q = \gamma(b).$$

Then along the contour $f(z) = F'(z)$ and so

$$\begin{aligned}\int_C f(z)\,dz &:= \int_a^b f(\gamma(t))\,\dot\gamma(t)\ dt \\ &= \int_a^b F'(\gamma(t))\,\dot\gamma(t)\ dt \\ &= \int_a^b \frac{d}{dt}F(\gamma(t))\ dt \\ &= F(\gamma(b)) - F(\gamma(a)) \\ &= F(q) - F(p),\end{aligned}$$

where $\dot\gamma = d\gamma/dt$ and the Chain rule for derivatives is used in the third equality.

The FTC can greatly simplify the evaluation and analysis of contour integrals.

Example: As a simple example, let $f(z) = (z-a)^m$, where m is an integer. If C is a smooth curve which does not pass through $a \in C$ and which starts at p and ends at q then

$$\int_C (z-a)^m \, dz = \frac{1}{m+1}\left[q^{m+1} - p^{m+1}\right] \quad \text{provided } m \neq -1 \tag{17.3}$$

since $F(z) = \frac{1}{m+1}(z-a)^{m+1}$ is a primitive for $(z-a)^m$ on $\mathbb{C}$ if $m \geq 0$ and on $\mathbb{C}\backslash\{a\}$ if $m < -1$. In particular if $C = C_{\text{closed}}$ is a closed contour then

$$\int_{C_{\text{closed}}} (z-a)^m \, dz = 0 \quad \text{provided } m \neq -1 \tag{17.4}$$

On the other hand, when $m = -1$ and C_{closed} is a simple (no self-crossings) closed contour then

$$\int_{C_{\text{closed}}} (z-a)^{-1} \, dz = \begin{cases} 2\pi i, & \text{if a is inside the contour} \\ 0, & \text{if a is outside the contour.} \end{cases} \tag{17.5}$$

We can write (17.5) for the case of a circle $C(a,r)$ centred at a of radius $r > 0$ as

$$\int_{C(a,r)} (z-a)^{-1} \, dz = 2\pi i. \tag{17.6}$$

Exercise: How about $\int_{C(b,r)} (z-a)^{-1} \, dz$ where $b \neq a$? (Your answer will depend on $|a-b|$).

The identities (17.5) and (17.6) hold because although on proper subsectors of $\mathbb{C}$ the function $(z-a)^{-1}$ does have a primitive

$$(z-a)^{-1} = \frac{\partial}{\partial z} \operatorname{Log}(z-a) \quad \text{'locally'}$$

there does not exist a function $F(z)$ defined on along all of the circle $C(a,r)$ with derivative equal to $(z-a)^{-1}$; hence the FTC Theorem 17.1 does not apply, the actual evaluation $2\pi i$ was done in the previous lecture. The logarithm Log (z-a) above is only defined along proper sub-arcs of the circle $C(a,r)$, not all the way around (in fact, logarithms are amazing functions and give rise to some extraordinary invariants of punctured space, allowing us to say mathematically why, for example, a sphere is different from flat space or from a torus).

In fact, (17.5) follows from the simpler fact (17.6) because of the following fundamental theorem:

Theorem 17.2 *(Cauchy's Theorem) Let $f : \mathbb{C} \to \mathbb{C}$ be a complex function, and let $C_{\text{closed}} \subset \mathbb{C}$ be a simple closed contour. Then if f is holomorphic along and at all points inside C_{closed}, then*

$$\int_{C_{\text{closed}}} f(z) \, dz = 0. \tag{17.7}$$

The proof of this theorem essentially involves showing that if f is holomorphic along and at all points inside C_{closed} that f then has a primitive; there exists a holomorphic function F in the same region with $F'(z) = f(z)$. We can then apply (17.2) of the FTC to infer the result. The proof is given in full detail in the course 6CCM322A Complex Analysis which is a 3rd year mathematics option.

You need to know the statement of the above theorems, but not the proofs. Cauchy's Theorem 17.2 combined with (17.6) yields the Cauchy Residue Theorem 17.3 (CRT), as is explained below.

Theorem 17.3 *(Cauchy Residue Theorem) Let $C_{\text{closed}} \subset \mathbb{C}$ be a simple closed contour which is positively oriented (anti-clockwise). Let $f : \mathbb{C} \to \mathbb{C}$ be a complex function which is holomorphic along C and inside C_{closed} except possibly at a finite number of points $a_1, \ldots, a_m$ inside C_{closed} at which points f has poles. Then*

$$\int_{C_{\text{closed}}} f(z)\, dz \;=\; 2\pi i \sum_{k=1}^{m} \operatorname{res}(f, a_k), \tag{17.8}$$

where $\operatorname{res}(f, a_k)$ *is the residue of the function f at the point a_k.*

The identity (17.8) is sometimes referred to as the Cauchy residue formula (CRF).

Example: Suppose C_{closed} encloses a point $a \in \mathbb{C}$, and let $\phi(z)$ be a function which holomorphic along and inside C_{closed} (including at $a \in \mathbb{C}$) then the only pole of $f(z) = \frac{\phi(z)}{z-a}$ is at $a \in \mathbb{C}$ with (from equation (15.9) of online lecture 15 "Poles and residues") residue $\phi(a)$. Hence (17.8) yields

$$\int_{C_{\text{closed}}} \frac{\phi(z)}{z-a}\, dz \;=\; 2\pi i\, \phi(a). \tag{17.9}$$

in terms of $\phi(a)$. (This is known as the *Cauchy integral formula.*) For example, we can apply (17.9) to $\int_{C(0,1)} \frac{e^z}{z}\, dz$ with $\phi(z) = e^z$ and $a = 0$ to obtain

$$\int_{C(0,1)} \frac{e^z}{z}\, dz = 2\pi i. \tag{17.10}$$

Moreover, if we then evaluate this integral using the parametrisation

$$\gamma : [0, 2\pi] \to C(0,1) \subset \mathbb{C}, \quad t \mapsto e^{it},$$

and equate the real and imaginary parts of the result to the real and imaginary parts of the right-hand side of (17.10) (respectively equal to 0 and 2π) then we obtain the interesting evaluation of real integrals

$$\int_0^{2\pi} e^{\cos(t)} \cos(\sin(t))\, dt = 2\pi, \qquad \int_0^{2\pi} e^{\cos(t)} \sin(\sin(t))\, dt = 0. \tag{17.11}$$

Exercise: Make sure you can derive (17.11).

17.1 Proof of (17.8): Non-examinable

First, let us see how Theorem 17.2 combined with (17.6) yields (17.5).

If a is *outside* the contour C_{closed} then $f(z) = (z-a)^{-1}$ is holomorphic along C_{closed} and at all points inside it, hence Theorem 17.2 says that $\int_{C_{\text{closed}}} (z-a)^{-1}\, dz = 0$ and we are done.

If a is *inside* C_{closed} we use the following deformation argument. We have, then, the following situation:

So we drill a hole in C_{closed} and construct the following 'key hole' contour Γ_ε for some small $\varepsilon > 0$:

The point a is outside of the closed contour Γ_ε and so $f(z) = (z-a)^{-1}$ is holomorphic along and at all points inside Γ_ε so that Theorem 17.2 implies

$$\int_{\Gamma_\varepsilon} (z-a)^{-1}\,dz = 0 \quad \forall\,\varepsilon > 0. \tag{17.12}$$

We next let $\varepsilon = 0$ and infer from (17.12) by continuity that also

$$\int_{\Gamma_0} (z-a)^{-1}\,dz = 0 \tag{17.13}$$

where Γ_0 is the contour

$$\Gamma_0 = C_{\text{closed}} \bigcup L_+ \bigcup C_-(a,r) \bigcup L_- \tag{17.14}$$

and $C_-(a,r)$ is the same circle as $C(a,r)$ but traversed negatively (clockwise)

From (17.13) and (17.14) (and definition (17.2)) we have

$$0 = \int_{C_{\text{closed}}} (z-a)^{-1}\,dz + \int_{L_+} (z-a)^{-1}\,dz + \int_{C_-(a,r)} (z-a)^{-1}\,dz + \int_{L_-} (z-a)^{-1}\,dz \tag{17.15}$$

But from the identity (17.3) of online lecture notes 17

$$\int_{L_+} (z-a)^{-1}\,dz = -\int_{L_-} (z-a)^{-1}\,dz$$

and hence from (17.15) that

$$\int_{C_{\text{closed}}} (z-a)^{-1}\,dz = \int_{C(a,r)} (z-a)^{-1}\,dz \tag{17.16}$$

again using (17.3) to note that $\int_{C_-(a,r)} = -\int_{C(a,r)}$.

Hence we have used Theorem 17.2 (in (17.12)) to see that (17.6) implies (17.5).

Now consider (17.8) in the case where $k = 1$; there is a single pole $a_1 = a$. Then exactly the same argument as the one we have just used but with $(z-a)^{-1}$ replaced by $f(z)$ shows that

$$\int_{C_{\text{closed}}} f(z)\,dz \;=\; \int_{C(a,r)} f(z)\,dz. \tag{17.17}$$

This holds for *any* $r > 0$ (so this is really a topological fact), so we may take r sufficiently small that it is inside a disc small enough for the Laurent expansion of f at a to hold (see earlier lectures)

$$f(z) = \frac{\beta_{-m}}{(z-a)^m} \;+\; \ldots \;+\; \frac{\beta_{-1}}{(z-a)} \;+\phi(z) \tag{17.18}$$

with ϕ holomorphic everywhere inside the disc. So we can substitute this into the right-hand side of (17.17) to obtain

$$\int_{C_{\text{closed}}} f(z)\,dz \;=\; \sum_{j=0}^{m-1} \beta_{-m+j} \int_{C(a,r)} (z-a)^{-m+j}\,dz \;+\; \int_{C(a,r)} \phi(z)\,dz. \tag{17.19}$$

But $\int_{C(a,r)} \phi(z)\,dz = 0$ by Theorem 17.2 while $\int_{C(a,r)}(z-a)^{-m+j}\,dz$ is only non-zero when $-m+j = -1$ in which case it is equal to $2\pi i$. That is,

$$\int_{C_{\text{closed}}} f(z)\,dz \;= \beta_{-1}\,2\pi\,i, \tag{17.20}$$

which is (17.8) for the case of a single pole.

The case for multiple poles is easily obtained by interating the above argument, this is left to you as an exercise.

18 Evaluating real integrals using the Cauchy Residue theorem

The Cauchy Residue Theorem states:

Let $C \subset \mathbb{C}$ be a simple closed contour. Let $f : \mathbb{C} \to \mathbb{C}$ be a complex function which is holomorphic along C and inside C except possibly at a finite number of points $a_1, \ldots, a_m$ at which points f has poles. Then, with C oriented in an anti-clockwise sense,

$$\int_C f(z)\,dz \;=\; 2\pi i \sum_{k=1}^{m} \operatorname{res}(f, a_k), \tag{18.1}$$

where $\operatorname{res}(f, a_k)$ is the residue of the function f at the point $a_k \in \mathbb{C}$.

This provides us with a powerful method of computing real integrals which would be impossible to evaluate using standard real integration techniques on $\mathbb{R}$.

In fact, every time we evaluate a contour integral using (18.1) we evaluate two real integrals. For, if $\gamma : [a, b] \to C \subset \mathbb{C}$ is a parametrisation of C then since

$$\int_C f(z)\,dz \;=\; \int_a^b f(\gamma(t))\,\gamma'(t)\;dt \tag{18.2}$$

while $f(\gamma(t))\,\gamma'(t) = \alpha(t) + i\beta(t)$ for some real valued functions $\alpha,\, \beta : [a, b] \to \mathbb{R}$ so that

$$\int_C f(z)\,dz \;=\; \int_a^b \alpha(t)\;dt \;+\; i\int_a^b \beta(t)\;dt. \tag{18.3}$$

On the other hand, we can also write the right-hand side $2\pi i \sum_{k=1}^{m} \operatorname{res}(f, a_k)$ of (18.1) as

$$2\pi i \sum_{k=1}^{m} \operatorname{res}(f, a_k) = A + iB \quad \text{for some real numbers } A, B \in \mathbb{R}, \tag{18.4}$$

and hence we obtain the evaluations

$$\int_a^b \alpha(t)\;dt \;=\; A \tag{18.5}$$

and

$$\int_a^b \beta(t)\;dt \;=\; B. \tag{18.6}$$

18.0.1 Example:

Parametrise C_{ab} the ellipse $\frac{x^2}{a^2}+\frac{x^2}{b^2}=1$, where $a,b\in(0,\infty)$, by $\gamma(t)=a\cos t+ib\sin t$, $0\le t\le 2\pi$. Then the Cauchy residue theorem tells us that

$$\int_{C_{ab}} \frac{1}{z}\,dz = 2\pi i. \tag{18.7}$$

On the other hand, using the parametrization we have

$$\begin{aligned}\int_C \frac{1}{z}\,dz &= \int_0^{2\pi} \frac{1}{a\cos t + ib\sin t}\,(-a\sin t + ib\cos t)\,d\theta \\ &= \text{Real Part} \; + \; i\int_0^{2\pi} \frac{ab}{a^2\cos^2 t + b^2\sin^2 t}\,dt. \qquad (*)\end{aligned}$$

Equating the imaginary parts of this with that of (18.7) we obtain the evaluation

$$\int_0^{2\pi} \frac{1}{a^2\cos^2 t + b^2\sin^2 t}\,dt = \frac{2\pi}{ab}.$$

What does equatng the real parts yield? (You will need to compute the "Real Part" in (*) to find out!

18.0.2 Exercise:

By considering the contour integral $\int_{C(0,1)} \frac{1}{z^2+4z+1}\,dz$ show that

$$\int_0^{2\pi} \frac{1}{2+\cos(t)}\,dt = \frac{2\pi}{\sqrt{3}}.$$

18.1 Evaluating integrals $\int_{-\infty}^{\infty} f(x)\ dx$ over the whole real line

Specifically this will provide us with a method of evaluating Fourier transforms and hence obtaining explicit solutions to a given ODE or PDE.

The idea is to compute $\int_{C_R} f(z)\ dz$ over a simple closed contour, typically, of the form

$$C_R = [-R, R] \cup A_R$$

where f is holomorphic on and inside C_R except possibly at poles $a_1, \ldots, a_{m_R}$ inside C_R, and then allow $R \to \infty$. Since

$$\begin{aligned}\int_{C_R} f(z)\ dz &= \int_{[-R,R]} f(z)\ dz + \int_{A_R} f(z)\ dz \\ &= \int_{-R}^{R} f(x)\ dx + \int_{A_R} f(z)\ dz,\end{aligned}$$

the aim is to choose A_R so that

$$\lim_{R\to\infty} \int_{A_R} f(z)\ dz = 0 \tag{18.8}$$

and hence that by the Cauchy Residue Theorem infer that

$$\int_{-\infty}^{\infty} f(x)\ dx = \lim_{R\to\infty} 2\pi i \sum_{k=1}^{m_R} \operatorname{res}(f, a_k). \tag{18.9}$$

Note that the right-hand side must in this case be a real number. This method depends on the convergence of all the limits and the existence of the integrals and so forth — but it works for large classes of functions.

18.1.1 Estimating contour integrals

The following fact is useful in showing properties like (18.8).

We know that for continuous real valued functions $f : [a, b] \to \mathbb{R}$ that

$$\left| \int_a^b f(x)\, dx \right| \le \int_a^b |f(x)|\, dx.$$

This extends to complex integrals in the following way.

Theorem 18.1 *Let C be a contour in $\mathbb{C}$. Then*

$$\left| \int_C f(z)\, dz \right| \le \int_a^b \left| f\big(\gamma(t)\big) \right| \cdot |\gamma'(t)|\ dt, \tag{18.10}$$

where $\gamma : [a,b] \to C \subset \mathbb{C}$ *is a parametrisation of* C*.*

Note that if $|f(z)| \leq M$*,* $\forall\, z \in C$ *(*f *is bounded by* M *along* C*, this implies* $|\int_C f(z)\,dz| \leq ML_C$*, where* L_C *is the length of* C*.*

Proof (non-examinable): To prove this result we first note that if $\int_C f = 0$ *then there is nothing to prove. We therefore suppose that* $\int_C f \neq 0$ *and let* $\theta = \arg \int_C f$*. (Recall if* $w = re^{i\theta}$ *then* $\arg(w) = \theta \in [0, 2\pi)$*.) We then have* $\int_C f = e^{i\theta}\left|\int_C f\right|$, so that

$$\begin{aligned}\left|\int_C f\right| &= e^{-i\theta}\int_C f = \int_a^b e^{-i\theta} f\big(\gamma(t)\big)\gamma'(t)\,dt \\ &= \int_a^b \mathrm{Re}\big(e^{-i\theta} f\big(\gamma(t)\big)\gamma'(t)\big)\,dt + i\int_a^b \mathrm{Im}\big(e^{-i\theta} f\big(\gamma(t)\big)\gamma'(t)\big)\,dt\end{aligned}$$

The integral on the left is a real number, as is the first integral on the right (being the integral of a real function), so it follows that the second integral on the right must be zero. We then have

$$\left|\int_C f\right| = \int_a^b \mathrm{Re}\big(e^{-i\theta} f\big(\gamma(t)\big)\gamma'(t)\big)\,dt \leq \int_a^b \big|\big(e^{-i\theta} f\big(\gamma(t)\big)\gamma'(t)\big)\big|\,dt$$

since $|\mathrm{Re} z| \leq |z|$ *for any complex number* z*. We therefore obtain* (18.10)*. The last part follows at once since if* $|f| \leq M|$ *on* γ *then*

$$\left|\int_C f(z)\,dz\right| \leq \int_a^b M|\gamma'(t)|\,dt = M\int_a^b |\gamma'(t)|\,dt = ML(\gamma).$$

18.1.2 Example:

By considering the contour integral

$$\int_{C_R} \frac{e^{iz}}{z - ia}\,dz,$$

where $C_R = [-R, R] \cup A_R$ *and* A_R *is the semicircle centre at* 0 *and radius* $R > a$ *in the upper-half plane, show that*

$$\int_{-\infty}^{\infty} \frac{x\sin x + a\cos x}{x^2 + a^2}\,dx = 2\pi e^{-a}$$

and

$$\int_{-\infty}^{\infty} \frac{x\cos x - a\sin x}{x^2 + a^2}\,dx = 0.$$

To see this, we first note that $f(z) = e^{iz}/(z - ia)$ has a single simple pole at $z = ia$ with residue e^{-a}. So an application of Cauchy's residue theorem to the contour C_R now gives

$$\int_{C_R} \frac{e^{iz}}{z - ia}\, dz = 2\pi i e^{-a}. \qquad \text{Why?}$$

Writing out the contour integral using the parametrization $\gamma(x) = x$ of $[-R, R]$, we therefore have

$$\int_{-R}^{R} \frac{e^{ix}}{x - ia}\, dx + \int_{A_R} \frac{e^{iz}}{z - ia}\, dz = (2\pi e^{-a})\, i.$$

Assuming that $\int_{A_R} f(z)\, dz \to 0$ as $R \to \infty$ then, letting $R \to \infty$, we obtain

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x - ia}\, dx = (2\pi e^{-a})\, i. \tag{18.11}$$

we obtain the asserted identities by equating the real part of the left-hand side of (18.11) with the real part of the right-hand side of (18.11), and the imaginary part of the left-hand side of (18.11) with the imaginary part of the right-hand side of (18.11).

Exercise: Make sure you can do this.

It remains to see that $\int_{A_R} f(z)\, dz \to 0$ as $R \to \infty$ and next time we will why (18.10) tells us that that does indeed hold true.

19 Using the Cauchy Residue Theorem and the Fourier Transform to compute exact solutions to ODEs

At the end of last time we were contemplating the contour integral

$$\int_{C_R} \frac{e^{iz}}{z - ia}\, dz \tag{19.1}$$

where $C_R = [-R, R] \cup A_R$ and A_R is the semicircle centre at 0 and radius $R > a > 0$ in the upper-half plane.

$f(z) = e^{iz}/(z - ia)$ has a single simple pole at $z = ia$ with residue e^{-a} (since a is real and positive the pole is inside C_R), so the Cauchy residue theorem gives

$$\int_{C_R} \frac{e^{iz}}{z - ia}\, dz = 2\pi i e^{-a}. \tag{19.2}$$

Writing out the contour integral using the parametrization $\gamma(x) = x$ of $[-R, R]$, we therefore have

$$\int_{-R}^{R} \frac{e^{ix}}{x - ia}\, dx + \int_{A_R} \frac{e^{iz}}{z - ia}\, dz = (2\pi e^{-a})\, i. \tag{19.3}$$

We claim that that $\int_{A_R} f(z)\, dz \to 0$ as $R \to \infty$. To see this, we use the basic estimate (proof in last lecture notes):

$$\left| \int_\gamma f(z)\, dz \right| \le \int_a^b |f(\gamma(t))| \cdot |\gamma'(t)|\, dt, \tag{19.4}$$

where γ is a contour.

Precisely, from the parametrization $\gamma_2(x) = Re^{i\theta}$ of A_R we have using (19.4)

$$\left| \int_{A_R} f(z)\, dz \right| = \left| \int_0^\pi \frac{e^{i[R\cos\theta + iR\sin\theta]} Rie^{i\theta}\, d\theta}{Re^{i\theta} - ia} \right| \le \int_0^\pi \frac{Re^{-R\sin\theta}\, d\theta}{|Re^{i\theta} - ia|}.$$

From the inequality $||\alpha| - |\beta|| \le |\alpha - \beta|$ we have

$$|Re^{i\theta} - ia| \ge ||Re^{i\theta}| - |-ia|| = R - a,$$

using $R > a$ for the final equality. Therefore

$$\left| \int_{A_R} f(z)\, dz \right| \le \int_0^\pi \frac{R}{R - a} e^{-R\sin\theta}\, d\theta = \frac{2R}{R - a} \int_0^{\pi/2} e^{-R\sin\theta}\, d\theta.$$

Since $\sin\theta \geq 2\theta/\pi \ \forall\theta \in [0, \pi/2]$ we thus obtain

$$\left|\int_{A_R} f(z)\, dz\right| \leq \frac{2R}{R-a}\int_0^{\pi/2} e^{-2R\theta/\pi}\, d\theta = \frac{\pi}{R-a}(1-e^{-R}) \to 0 \text{ as } \ R \to \infty. \quad (19.5)$$

Consequently, letting $R \to \infty$ in (19.3) we conclude that

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x-ia}\ dx = (2\pi e^{-a})\, i. \quad (19.6)$$

Now contemplate the ODE (with a as before)

$$-\frac{df}{dx} + af(x) = g(x) \quad (19.7)$$

on $\mathbb{R}^1$ with boundary conditions at infinity

$$f(x) \to 0 \quad \text{and} \quad g(x) \to 0 \text{ as } |x| \to \infty.$$

The object is to determine f for a given g. Given the boundary conditions, we may use the Fourier Transform method to obtain the solution

$$f(x) = \int_{y=-\infty}^{\infty} k(x,y)g(y)\ dy \quad (19.8)$$

with

$$k(x,y) = \frac{1}{2\pi i}\int_{\xi=-\infty}^{\infty} \frac{e^{i(y-x)\xi}}{\xi - ia}\ d\xi. \quad (19.9)$$

This is almost the same as the left hand side of (19.6): it has the form

$$\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{x-ia}\ dx, \qquad \alpha \in \mathbb{R}^1, \quad (19.10)$$

times a constant (α is equal to $y-x$ in the case of interest). We just need to check if including α changes anything that led to (19.6). Just as before, to evaluate (19.10) we replace ξ by z and consider the contour integral

$$I_R(\alpha) := \int_{C_R} \frac{e^{i\alpha z}}{z-ia}\, dz. \quad (19.11)$$

We saw in the lecture that provided $\alpha > 0$, then everything goes through as before to give

$$\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{x-ia}\ dx = 2\pi i\, e^{-a\alpha} \qquad \text{if } \alpha > 0. \quad (19.12)$$

In particular, by setting $\alpha = y - x$ we obtain that

$$k(x,y) = e^{a(x-y)} \qquad \text{if } y > x. \quad (19.13)$$

The reason for the restriction on α is that in the estimate to show the integral along the semicircle goes to zero as $R \to \infty$, one has, as before,

$$\left| \int_{A_R} \frac{e^{i\alpha z}}{z - ia} \, dz \right| \le \int_0^\pi \frac{R}{R-a} e^{-R\alpha \sin\theta} \, d\theta = \frac{2R}{R-a} \int_0^{\pi/2} e^{-R\alpha \sin\theta} \, d\theta.$$

Crucially, since $\alpha R \sin\theta \ge 0$ then $e^{-R\alpha \sin\theta}$ has modulus never greater than 1 and exponentially decreasing as $R \to \infty$ for θ not zero or π. But if $\alpha < 0$ then $-\alpha R \sin\theta \ge 0$ and so $e^{-R\alpha \sin\theta}$ then has modulus exponentially increasing as $R \to \infty$ for θ not zero or π – so the integral cannot converge to zero (or any other number) as $R \to \infty$.

One can also deduce (19.12) by making the substitution αz for z and αa for a. Exercise!

The problem with $\alpha < 0$ can be readily solved by instead considering the contour which is C_R reflected in the real axis, so with the semicircle below the real axis. Next time we will do this next time to conclude that

$$k(x,y) = \begin{cases} e^{a(x-y)} & \text{if } y > x, \\ 0 & \text{if } y < x, \end{cases} \tag{19.14}$$

and thus from (19.8) that the final solution to (19.7) is

$$f(x) = \int_x^\infty e^{a(x-y)} g(y) \, dy = e^{ax} \int_x^\infty e^{-ay} g(y) \, dy. \tag{19.15}$$

Exercise: Check that this solution really does satisfy (19.7) – i.e. compute the derivative of (19.15). Check also what condition on g is needed in order that (19.15) satisfies the boundary condition $f(x) \to 0$ as $|x| \to \infty$.

What changes if we take $a < 0$?

20 More on computing exact solutions to ODEs using the Cauchy Residue Theorem and the Fourier Transform

Last time we were investigating the ODE (with $a \in \mathbb{R}$)

$$-\frac{df}{dx} + af(x) = g(x) \tag{20.1}$$

on $\mathbb{R}^1$ with boundary conditions at infinity

$$f(x) \to 0 \quad \text{and} \quad g(x) \to 0 \text{ as } |x| \to \infty.$$

The object, recall, is to determine f for a given g. We may then use the Fourier Transform method to obtain the solution

$$f(x) = \int_{y=-\infty}^{\infty} k(x,y) g(y)\; dy \tag{20.2}$$

with

$$k(x,y) = \frac{1}{2\pi i} \int_{\xi=-\infty}^{\infty} \frac{e^{i(y-x)\xi}}{\xi - ia}\; d\xi. \tag{20.3}$$

To evaluate (20.3) we considered the contour integral

$$I_R(\alpha) := \int_{C_R} \frac{e^{i\alpha z}}{z - ia}\, dz, \tag{20.4}$$

where $C_R = [-R, R] \cup A_R$ and A_R is the semicircle centre at 0 of radius $R > a > 0$ in the upper-half plane, to obtain

$$\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{x - ia}\, dx = 2\pi i\, e^{-a\alpha} \qquad \underline{\text{provided that}}\;\; \alpha > 0, \tag{20.5}$$

and so by setting $\alpha = y - x$ that

$$k(x,y) = e^{a(x-y)} \qquad \underline{\text{provided that}}\;\; y > x. \tag{20.6}$$

The reason for the restriction on α is that in the estimate to show the integral along the semicircle goes to zero as $R \to \infty$, one has

$$\left| \int_{A_R} \frac{e^{i\alpha z}}{z - ia}\, dz \right| \le \int_0^{\pi} \frac{R}{R-a} e^{-R\alpha \sin\theta}\, d\theta = \frac{2R}{R-a} \int_0^{\pi/2} e^{-R\alpha \sin\theta}\, d\theta. \tag{20.7}$$

Crucially, since $\alpha R \sin\theta \ge 0$ is a non-negative real number then $e^{-R\alpha\sin\theta}$ has modulus never greater than 1 and exponentially decreasing (for θ not zero or π)

as $R \to \infty$. But if $\alpha < 0$ then $-\alpha R \sin\theta \geq 0$ and so $e^{-R\alpha\sin\theta}$ then has modulus exponentially increasing as $R \to \infty$ for θ not zero or π – so the integral cannot converge to zero (or any other number) as $R \to \infty$.

To get around this we observe that $\sin(\phi) \leq 0$ for $\pi \leq \phi \leq 2\pi$, and hence that $\alpha R \sin\phi \geq 0$ (since also $\alpha < 0$) meaning the estimate (20.7) again holds but now with $\alpha < 0$. But taking $\pi \leq \phi \leq 2\pi$ means we are travelling along the semicircle of radius R which lies below the real axis; that is, to deal with $\alpha < 0$ we must consider the contour integral

$$J_R(\alpha) := \int_{B_R} \frac{e^{i\alpha z}}{z - ia}\, dz \tag{20.8}$$

where $B_R = [-R, R] \cup D_R$ and D_R is the semicircle centre at 0 and radius $R > a > 0$ in the lower-half plane — traversed anti-clockwise.

But now ia is not inside the contour B_R and so the Cauchy theorem gives

$$\int_{B_R} \frac{e^{i\alpha z}}{z - ia}\, dz = 0. \tag{20.9}$$

On the other hand, using the parametrisation we obtain an estimate very similar to (20.7) showing that

$$\int_{D_R} \frac{e^{i\alpha z}}{z - ia}\, dz \longrightarrow 0 \quad \text{as} \quad R \to \infty \qquad \underline{\text{provided that}} \quad \alpha < 0. \tag{20.10}$$

Can you provide the details? (They are almost the same as for $\alpha \geq 0$ — solution to follow shortly, possibly in the video for this lecture.)

Putting it altogether, then,

$$\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{x - ia}\, dx = 0 \qquad \text{if } \alpha < 0. \tag{20.11}$$

We hence end up with the answer

$$k(x, y) = \begin{cases} e^{a(x-y)} & \text{if } y > x, \\ 0 & \text{if } y < x, \end{cases} \tag{20.12}$$

and thus from (20.2) that the final solution to (20.14) is

$$f(x) = \int_x^{\infty} e^{a(x-y)} g(y)\, dy = e^{ax} \int_x^{\infty} e^{-ay} g(y)\, dy. \tag{20.13}$$

20.1 Solving a 4th order ODE

The example we just looked at shows that our methods coincide with a solution we could equally well deduce using direct elementary methods (how?)

Here, we are going to start seeing how to find the general solution to the ODE

$$\frac{d^4 f}{dx^4} + f = g \tag{20.14}$$

provided that

$$\frac{d^k f}{dx^k} \to 0 \text{ and } g \to 0 \quad \text{as } |x| \to \infty \text{ for } k = 0, 1, 2, 3. \tag{20.15}$$

Thus the function $g : \mathbb{R}^1 \to \mathbb{R}^1$ is assumed to be given, and the objective is to determine $f : \mathbb{R}^1 \to \mathbb{R}^1$.

This is a much harder case, not amenable to elementary methods, but our general FT + CRT methods mean it is technically no more difficult than the previous case (obtaining the estimate as $R \to \infty$ needed, in fact, is easier).

20.2 Applying Fourier transform

The assumption (20.15) means that we can apply the Fourier transform to the ODE (20.14). Recall that the Fourier transform $\widehat{f}(\xi) = \mathsf{F}(f)(\xi) := \int_{-\infty}^{\infty} e^{i\xi t} f(t)\, dt$ is linear $\mathsf{F}(\lambda f + \mu g)(\xi) = \lambda \mathsf{F}(f)(\xi) + \mu \mathsf{F}(g)(\xi)$ and

$$\mathsf{F}(f')(\xi) = -i\xi\, \mathsf{F}(f)(\xi) \quad \text{provided that } f(x) \to 0 \quad \text{as} \quad |x| \to \infty, \tag{20.16}$$

and hence $\mathsf{F}(f^{(m)})(\xi) = (-i\xi)^m \mathsf{F}(f)(\xi)$ provided $f^{(k)}(x) \to 0$ as $|x| \to \infty$ for $k = 0, 1, \dots, m-1$. Applied to (20.14) linearity yields

$$\mathsf{F}(f^{(4)})(\xi) + \mathsf{F}(f)(\xi) = \mathsf{F}(g)(\xi).$$

and because of (20.15) and (20.16) this is equivalent to the algebraic equation

$$\widehat{f}(\xi) = \frac{\widehat{g}(\xi)}{p_4(\xi)},$$

where

$$p_4(\xi) = \xi^4 + 1.$$

Since p_4 has no real zeroes this is well defined and we can apply Fourier inversion to obtain

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\xi} \widehat{f}(\xi)\, d\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\xi} \frac{\widehat{g}(\xi)}{p_4(\xi)}\, d\xi.$$

Substituting $\widehat{g}(\xi) = \int_{-\infty}^{\infty} e^{i\xi y} g(y)\, dy$ and rearranging yields

$$f(x) = \int_{-\infty}^{\infty} k(x, y)\, g(y)\, dy$$

where

$$k(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i(y-x)\xi}}{\xi^4 + 1}\, d\xi. \tag{20.17}$$

It remains, then, to evaluate the integral on the right-side of (20.17). To evaluate this real integral previous examples suggest that we may get somewhere by evaluating the contour integral

$$I_R(\alpha) = \int_{C_R} \frac{e^{i\alpha z}}{z^4 + 1}\, dz$$

around the contour C_R in the z-plane which consists of the portion of the real axis from $-R$ to R, together with the semi-circular arc γ_R parametrized by $\gamma_R(\theta) = Re^{i\theta}, \ \ 0 \leq \theta \leq \pi$.

We assume that

$$\alpha \in \mathbb{R}^1. \tag{20.18}$$

For $R > 1$ $f(z) = \frac{e^{i\alpha z}}{z^4+1}$ has two poles inside C_R at $z_0 = e^{i\pi/4}$ and $z_1 = e^{i\,3\pi/4}$ and one may use the Cauchy residue theorem to thus evaluate the integral.

Exercise: Do the evaluation. (We will see the details in the next lecture.)

On the other hand,

$$I_R(\alpha) = \int_{-R}^{R} \frac{e^{i\alpha\xi}}{\xi^4 + 1}\, d\xi + \int_{A_R} \frac{e^{i\alpha z}}{z^4 + 1}\, dz$$

with A_R the arc component of the contour. Using the estimate (18.10) from online Lecture notes 18, we have

$$\begin{aligned}
\left| \int_{A_R} \frac{e^{i\alpha z}}{z^4 + 1}\, dz \right| &= \left| \int_0^{\pi} \frac{e^{i\alpha Re^{i\theta}}}{R^4 e^{i4\theta} + 1}\, Re^{i\theta}\, d\theta \right| \\
&\leq R \int_0^{\pi} \frac{|e^{i\alpha Re^{i\theta}}|}{|R^4 e^{i4\theta} + 1|}\, d\theta \\
&= R \int_0^{\pi} \frac{e^{-\alpha R \sin(\theta)}}{|R^4 e^{i4\theta} + 1|}\, d\theta.
\end{aligned}$$

By the triangle inequality $|R^4 e^{i4\theta} + 1| \geq ||R^4 e^{i4\theta}| - 1| = |R^4 - 1| = R^4 - 1$ for $R > 1$. Since also $0 < e^{-\alpha R \sin(\theta)} \leq 1$ for $\alpha \geq 0$ we obtain therefore

$$\left| \int_{A_R} \frac{e^{i\alpha z}}{z^4 + 1}\, dz \right| \leq \frac{R\pi}{R^4 - 1} \ \ \to 0 \quad \text{as } R \to \infty \ \ \text{if } \alpha \geq 0. \tag{20.19}$$

We are therefore left with

$$\text{provided } \alpha \geq 0 \qquad \lim_{R\to\infty} \int_{C_R} \frac{e^{i\alpha z}}{z^4+1}\, dz = \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{i\alpha \xi}}{\xi^4+1}\, d\xi \ + \underbrace{\lim_{R\to\infty} \int_{A_R} \frac{e^{i\alpha z}}{z^4+1}\, dz}_{=0 \text{ by } (20.19)}$$

$$= \int_{-\infty}^{\infty} \frac{e^{i\alpha \xi}}{\xi^4+1}\, d\xi.$$

That is,

$$\text{provided } \alpha \geq 0 \qquad \int_{-\infty}^{\infty} \frac{e^{i\alpha \xi}}{\xi^4+1}\, d\xi = 2\pi i \quad \times \quad \text{the sum of the residues.} \qquad (20.20)$$

Exercise: Use your evaluation, above, of the residues to give the formula for $k(x, y)$ when $y \geq x$.

We'll carry on with this next time...

21 Lecture 21: An application to ordinary differential equations

In this lecture we are going to use a contour integral to help solve the 4th ODE we began to look at at the end f the pervious lecture. This method we are going to use is, in fact, very general and can be applied to partial differential equations on $\mathbb{R}^m$ (recall that ODEs refer to differential equations on $\mathbb{R}^1$). As a warm up to this you are encouraged to re-read Lecture 5: “ODEs and the FT”.

21.1 The ODE

Here, we are going to see how to find the general solution to the ODE

$$\frac{d^4 f}{dx^4} + f = g \tag{21.1}$$

provided that

$$\frac{d^k f}{dx^k} \to 0 \text{ and } g \to 0 \quad \text{as } |x| \to \infty \text{ for } k = 0, 1, 2, 3. \tag{21.2}$$

Thus the function $g : \mathbb{R}^1 \to \mathbb{R}^1$ is assumed to be given, and the objective is to determine $f : \mathbb{R}^1 \to \mathbb{R}^1$.

21.2 Applying Fourier transform

The assumption (21.2) means that we can apply the Fourier transform to the ODE (21.15). Recall that the Fourier transform $\widehat{f}(\xi) = \mathsf{F}(f)(\xi) := \int_{-\infty}^{\infty} e^{i\xi t} f(t)\, dt$ is linear $\mathsf{F}(\lambda f + \mu g)(\xi) = \lambda \mathsf{F}(f)(\xi) + \mu \mathsf{F}(g)(\xi)$ and

$$\mathsf{F}(f')(\xi) = -i\xi\, \mathsf{F}(f)(\xi) \quad \text{provided that } f(x) \to 0 \quad \text{as} \quad |x| \to \infty, \tag{21.3}$$

and hence $\mathsf{F}(f^{(m)})(\xi) = (-i\xi)^m\, \mathsf{F}(f)(\xi)$ provided $f^{(k)}(x) \to 0$ as $|x| \to \infty$ for $k = 0, 1, \ldots, m-1$. Applied to (21.15) linearity yields

$$\mathsf{F}(f^{(4)})(\xi) + \mathsf{F}(f)(\xi) = \mathsf{F}(g)(\xi).$$

and because of (21.2) and (21.3) this is equivalent to the algebraic equation

$$\widehat{f}(\xi) = \frac{\widehat{g}(\xi)}{p_4(\xi)},$$

where

$$p_4(\xi) = \xi^4 + 1.$$

Since p_4 has no real zeroes this is well defined and we can apply Fourier inversion to obtain

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\xi} \widehat{f}(\xi)\, d\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\xi} \frac{\widehat{g}(\xi)}{p_4(\xi)}\, d\xi.$$

Substituting $\widehat{g}(\xi) = \int_{-\infty}^{\infty} e^{i\xi y} g(y)\, dy$ and rearranging yields

$$f(x) = \int_{-\infty}^{\infty} k(x,y)\, g(y)\, dy$$

where

$$k(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i(y-x)\xi}}{\xi^4 + 1}\, d\xi. \tag{21.4}$$

It remains, then, to evaluate the integral on the right-side of (21.4).

21.3 Using the Cauchy residue theorem

To evaluate this real integral previous examples suggest that we may get somewhere by evaluating the contour integral

$$I_R(\alpha) = \int_{C_R} \frac{e^{i\alpha z}}{z^4 + 1}\, dz$$

around the contour C_R in the z-plane which consists of the portion of the real axis from $-R$ to R, together with the semi-circular arc γ_R parametrized by $\gamma_R(\theta) = Re^{i\theta}, \quad 0 \le \theta \le \pi$.
We assume that

$$\alpha \in \mathbb{R}^1. \tag{21.5}$$

For $R > 1$ $f(z) = \frac{e^{i\alpha z}}{z^4+1}$ has two poles inside C_R at $z_0 = e^{i\pi/4}$ and $z_1 = e^{i\,3\pi/4}$. The Cauchy residue theorem says that

$$I_R(\alpha) = 2\pi i(\mathrm{res}(f, z_0) + \mathrm{res}(f, z_1)), \tag{21.6}$$

where the right-side refers to the residues. By a standard formula

$$\mathrm{res}(f, z_k) = \frac{e^{i\alpha z_k}}{4z_k^3}.$$

So

$$\mathrm{res}(f, z_0) + \mathrm{res}(f, z_1) = \frac{e^{i\frac{\alpha}{\sqrt{2}}(1+i)}}{4z_0^3} + \frac{e^{i\frac{\alpha}{\sqrt{2}}(-1+i)}}{4z_1^3}$$

$$= \frac{1}{4}(e^{-i\frac{3\pi}{4}}\, e^{i\frac{\alpha}{\sqrt{2}}} + e^{-i\frac{\pi}{4}}\, e^{-i\frac{\alpha}{\sqrt{2}}})\, e^{-\frac{\alpha}{\sqrt{2}}}$$

$$= -\frac{i}{2\sqrt{2}}\left\{\sin\left(\frac{\alpha}{\sqrt{2}}\right) + \cos\left(\frac{\alpha}{\sqrt{2}}\right)\right\} e^{-\frac{\alpha}{\sqrt{2}}}$$

$$= -\frac{i}{2}\,\sin\left(\frac{\alpha}{\sqrt{2}} + \frac{\pi}{4}\right) e^{-\frac{\alpha}{\sqrt{2}}}.$$

Hence by (21.6)

$$I_R(\alpha) = \pi\, e^{-\frac{\alpha}{\sqrt{2}}}\,\sin\left(\frac{\alpha}{\sqrt{2}} + \frac{\pi}{4}\right). \tag{21.7}$$

Notice that this holds for *any* $R > 1$.

On the other hand,

$$I_R(\alpha) = \int_{-R}^{R} \frac{e^{i\alpha\xi}}{\xi^4 + 1}\, d\xi + \int_{A_R} \frac{e^{i\alpha z}}{z^4 + 1}\, dz$$

with A_R the arc component of the contour. Using the estimate (19.10) from online Lecture notes 19, we have

$$\left|\int_{A_R} \frac{e^{i\alpha z}}{z^4 + 1}\, dz\right| = \left|\int_0^{\pi} \frac{e^{i\alpha Re^{i\theta}}}{R^4 e^{i4\theta} + 1}\, Re^{i\theta}\, d\theta\right|$$

$$\leq R\int_0^{\pi} \frac{|e^{i\alpha Re^{i\theta}}|}{|R^4 e^{i4\theta} + 1|}\, d\theta$$

$$= R\int_0^{\pi} \frac{e^{-\alpha R\sin(\theta)}}{|R^4 e^{i4\theta} + 1|}\, d\theta.$$

By the triangle inequality $|R^4 e^{i4\theta} + 1| \geq ||R^4 e^{i4\theta}| - 1| = |R^4 - 1| = R^4 - 1$ for $R > 1$. Since also $0 < e^{-\alpha R\sin(\theta)} \leq 1$ for $\alpha \geq 0$ we obtain therefore

$$\left|\int_{A_R} \frac{e^{i\alpha z}}{z^4 + 1}\, dz\right| \leq \frac{R\pi}{R^4 - 1} \to 0 \quad \text{as } R \to \infty \;\text{ if } \alpha \geq 0. \tag{21.8}$$

We are therefore left with for $\alpha \geq 0$

$$\pi\, e^{-\frac{\alpha}{\sqrt{2}}}\,\sin\left(\frac{\alpha}{\sqrt{2}} + \frac{\pi}{4}\right) \overset{(21.7)}{=} \lim_{R\to\infty} \int_{C_R} \frac{e^{i\alpha z}}{z^4 + 1}\, dz$$

$$= \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{i\alpha\xi}}{\xi^4 + 1}\, d\xi \; + \; \underbrace{\lim_{R\to\infty} \int_{A_R} \frac{e^{i\alpha z}}{z^4 + 1}\, dz}_{=0 \text{ by } (21.8)}$$

$$= \int_{-\infty}^{\infty} \frac{e^{i\alpha\xi}}{\xi^4 + 1}\, d\xi.$$

That is,

$$\text{provided } \alpha \geq 0 \qquad \int_{-\infty}^{\infty} \frac{e^{i\alpha\xi}}{\xi^4+1}\, d\xi = \pi\, e^{\frac{-\alpha}{\sqrt{2}}} \sin\left(\frac{\alpha}{\sqrt{2}} + \frac{\pi}{4}\right). \tag{21.9}$$

From (21.4) we want to apply this with $\alpha = y - x$. But $y - x$ can take on any real value, so we need to extend (21.9) to the case $\alpha \leq 0$. To do that we make use of the fact that we have only used two of the four poles of f. So we now evaluate

$$I_{D_R}(\alpha) = \int_{D_R} \frac{e^{i\alpha z}}{z^4+1}\, dz$$

around the contour D_R in the z-plane which consists of the portion of the real axis from R to $-R$, together with the semi-circular arc B_R parametrized by $\gamma_R(\theta) = Re^{i\theta}$, $\pi \leq \theta \leq 2\pi$, (note this is positively oriented (anticlockwise) which means the direction along the $[-R, R]]$ is reversed!). We now assume that

$$\alpha \leq 0. \tag{21.10}$$

We now repeat the above process with D_R instead of C_R. For $R > 1$ $f(z) = \frac{e^{i\alpha z}}{z^4+1}$ has two poles inside D_R at $z_2 = e^{i5\pi/4}$ and $z_3 = e^{i\,7\pi/4}$. The Cauchy residue theorem says that

$$I_{D_R}(\alpha) = 2\pi i(\operatorname{res}(f, z_2) + \operatorname{res}(f, z_3)). \tag{21.11}$$

As before, $\operatorname{res}(f, z_k) = \frac{e^{i\alpha z_k}}{4z_k^3}$ so

$$\begin{aligned}
\operatorname{res}(f, z_2) + \operatorname{res}(f, z_3) &= \frac{e^{-i\frac{\alpha}{\sqrt{2}}(1+i)}}{4(z_2)^3} + \frac{e^{i\frac{\alpha}{\sqrt{2}}(1-i)}}{4(z_3)^3} \\
&= i\,\frac{e^{\frac{\alpha}{\sqrt{2}}}}{2}\left(\cos\left(\frac{\alpha}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} - \sin\left(\frac{\alpha}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}\right) \\
&= i\,\frac{e^{\frac{\alpha}{\sqrt{2}}}}{2}\sin\left(-\frac{\alpha}{\sqrt{2}} + \frac{\pi}{4}\right).
\end{aligned}$$

Hence by (21.11)

$$I_{D_R}(\alpha) \;=\; -\pi\, e^{\frac{\alpha}{\sqrt{2}}} \sin\left(-\frac{\alpha}{\sqrt{2}} + \frac{\pi}{4}\right).$$

On the other hand,

$$I_{D_R}(\alpha) = -\int_{-R}^{R} \frac{e^{i\alpha\xi}}{\xi^4+1}\, d\xi + \int_{B_R} \frac{e^{i\alpha z}}{z^4+1}\, dz$$

with B_R the arc component of the contour in the lower-half plane. Note the minus sign in front of $\int_{-R}^{R}$ is there to take account of the fact that we traverse the interval

$[-R, R]$ in reverse: by starting at R and ending at $-R$. Then, similarly as before, we obtain

$$\text{provided } \alpha \le 0 \qquad \int_{-\infty}^{\infty} \frac{e^{i\alpha\xi}}{\xi^4+1}\, d\xi = \pi\, e^{\frac{\alpha}{\sqrt{2}}}\, \sin\left(-\frac{\alpha}{\sqrt{2}} + \frac{\pi}{4}\right). \tag{21.12}$$

Exercise: Make sure *you* can derive (21.12)!

But since

$$-|\alpha| = \begin{cases} -\alpha, & \text{if } \alpha \ge 0, \\ \alpha, & \text{if } \alpha \le 0, \end{cases}$$

then we can write down (21.9) and (21.12) in one go as

$$\underline{\text{For any } \alpha \in \mathbb{R}^1} \qquad \int_{-\infty}^{\infty} \frac{e^{i\alpha\xi}}{\xi^4+1}\, d\xi = \pi\, e^{\frac{-|\alpha|}{\sqrt{2}}}\, \sin\left(\frac{|\alpha|}{\sqrt{2}} + \frac{\pi}{4}\right). \tag{21.13}$$

We now obtain from (21.4) and (21.13) that the general solution to (21.15) is given by

$$f(x) = \int_{-\infty}^{\infty} k(x,y)\, g(y)\, dy$$

with

$$k(x,y) = \frac{1}{2}\, e^{-\frac{|x-y|}{\sqrt{2}}}\, \sin\left(\frac{|x-y|}{\sqrt{2}} + \frac{\pi}{4}\right).$$

21.4 Exercises:

Using the same contours as above:

[1] Find the general solution to the ODE

$$-\frac{d^2 f}{dx^2} + f = g \tag{21.14}$$

provided that f, $f' \to 0$ and $g \to 0$ as $|x| \to \infty$.

[1] Find the general solution to the ODE

$$\frac{d^4 f}{dx^4} - 2\frac{d^2 f}{dx^2} + f = g \tag{21.15}$$

provided that $f^{(k)} \to 0$, $k = 0, 1, 2, 3$, and $g \to 0$ as $|x| \to \infty$.

Solutions will be posted on the web page shortly.

Lecture Notes for the course

PDEs and Complex Variable

CCM211A / CCM211B

2014-15

1 The Transport Equation

Let's consider the homogeneous (meaning $v(x,y) = 0$) constant coefficient *Transport Equation*

$$a\, u_x + b\, u_y = 0. \tag{1.1}$$

Assume $a \neq 0$. To simplify, divide through by a to rewrite (1.1) as

$$u_x + c\, u_y = 0 \quad \text{with} \;\; c := \frac{b}{a} \neq 0. \tag{1.2}$$

(1.2) can be solved using the Chain Rule. To see this, consider a smooth curve passing through a given point $(x,y) \in \mathbb{R}^2$

$$(a,b) \mapsto \mathbb{R}^2, \quad t \mapsto \mathbf{r}(t) = (x(t), y(t)) \quad \text{with} \quad \mathbf{r}(0) = (x(0), y(0)) = (x,y),$$

where we assume $a < 0 < b$, and the composite function

$$(a,b) \mapsto \mathbb{R}^1, \quad \mapsto u(\mathbf{r}(t)) = u(x(t), y(t)). \tag{1.3}$$

The Chain rule says that

$$\frac{d}{dt}\, u(x(t), y(t)) = x'(t) u_x(x(t), y(t)) + y'(t) u_y(x(t), y(t)). \tag{1.4}$$

Consider specifically the case

$$\mathbf{r}(t) = (x + t, y + ct). \tag{1.5}$$

Then (1.4) says

$$\frac{d}{dt}\, u(x(t), y(t))\bigg|_{t=0} = u_x(x,y) + c u_y(x,y). \tag{1.6}$$

Hence, (1.6) may be written

$$\frac{d}{dt}\, u(x(t), y(t))\bigg|_{t=0} = 0 \tag{1.7}$$

That is, if u solves the transport equation then the function $t \mapsto u(x(t), y(t))$ is constant along the curve (1.5). In other words for any two times t_0 and t_1 we have

$$u(x + t_0, y + ct_0) = u(x + t_1, y + ct_1).$$

Choosing, at a given point (x,y), the values $t_0 = 0$ and $t_1 = -x$ gives us

$$u(x,y) = u(0, y - cx). \tag{1.8}$$

This says that u is determined by evaluation along the line $x = 0$, the y-axis, in which the variable is replaced by $y - cx$; that is, u is a function of 1-variable w with $w = y - cx$, so

$$u(x, y) = f(y - cx), \qquad f : \mathbb{R}^1 \to \mathbb{R}^1,\ w \mapsto f(w), \tag{1.9}$$

for any differentiable function f of one variable. We could also write this family of solutions as

$$u(x, y) = h(bx - ay), \qquad \text{any } h : \mathbb{R}^1 \to \mathbb{R}^1,\ w \mapsto h(w). \tag{1.10}$$

An equivalent method for solving the PDE is to to make the change of variable $(x, y) \mapsto (\xi, \eta)$ with $\xi = bx - ay, \eta = ax + by$ and the Chain Rule in 2 variables

$$\frac{\partial u}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial u}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial u}{\partial \eta}, \qquad \frac{\partial u}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial u}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial u}{\partial \eta},$$

reduces (1.1) to

$$\frac{\partial u}{\partial \eta} = 0, \qquad u = u(\xi, \eta),$$

which has solution $u(\xi, \eta) = f(\xi)$ some $f : \mathbb{R}^1 \to \mathbb{R}^1$, which in (x, y) coordinates is (1.10) once more.

1.1 For a specific solutions, impose impose 'boundary' conditions

To get a specific solution we have to specify how the solution behaves along a suitable curve in $\mathbb{R}^2$, just like in the cases with $a \neq 0$ discussed above.

For example, consider

$$\begin{cases} 2u_x + 3u_y = 0 \\ u(x, 0) = \sin(x^2). \end{cases} \tag{1.11}$$

Here, then, $a = 2$ and $b = 3$. From (1.9) we know that a general solution of $2u_x + 3u_y = 0$ has the form

$$u(x, y) = f\left(y - \frac{3}{2}x\right), \qquad \text{any } f : \mathbb{R}^1 \to \mathbb{R}^1,\ w \mapsto f(w).$$

(Check that it does!) The condition $u(x, 0) = \sin\left(x^2\right)$ therefore says that

$$f\left(-\frac{3}{2}x\right) = \sin\left(x^2\right).$$

Hence f is the function $f(w) = \sin((-2w/3)^2) = \sin(4w^2/9)$. Thus the solution to (1.15) is

$$u(x,y) = \sin\left(x^2 - \frac{4}{3}xy + \frac{4}{9}y^2\right). \tag{1.12}$$

We might generalize by specifying the boundary condition along the line $y = \alpha x$ for some given $\alpha \in \mathbb{R}$, so that

$$\begin{cases} au_x + bu_y = 0 \\ u(x, \alpha x) = h(x). \end{cases} \tag{1.13}$$

We know any solution has the form $u(x,y) = f(bx - ay)$ for some function f of one variable. The condition $u(x, \alpha x) = h(x)$ says that $f(bx - \alpha ax) = h(x)$ and hence the unique solution to (1.13) is

$$u(x,y) = h\left(\frac{bx - ay}{b - \alpha a}\right) \quad \text{provided} \quad \alpha \neq \frac{b}{a} = c. \tag{1.14}$$

For example,

$$\begin{cases} 2u_x + 3u_y = 0 \\ u(x, -5x) = \sinh(x - 4). \end{cases} \tag{1.15}$$

has solution (**check it!**) $u(x,y) = \sinh\left(\frac{1}{13}(3x - 2y) - 4\right)$.

1.1.1 Uniqueness of (1.14)

This is the 'same' proof as we saw for the case $b = 0$: if $u_1, u_2 : \mathbb{R}^2 \to \mathbb{R}^1$ are both solutions of (1.13) then

$$z(x,y) := u_1(x,y) - u_2(x,y)$$

satisfies the PDE

$$\begin{cases} az_x + bz_y = 0 \\ z(x, \alpha x) = 0 \end{cases} \tag{1.16}$$

(Why?)

We know that $z(x,y) = f(bx - ay)$ some $f : \mathbb{R} \to \mathbb{R}$. In order that it satisfy the condition $z(x, \alpha x) = 0$, we have that $0 = z(x, \alpha x) = f(\beta x)$, with $\beta := b - \alpha a$, and hence f must be identically zero. That is, $z(x,y) = 0$ for all $(x,y) \in \mathbb{R}^2$. That is

$$u_1 = u_2.$$

2 2nd order PDEs

A homogeneous, linear, constant coefficient, partial differential equation PDE of second order on $\mathbb{R}^2$ is an equality of the form

$$Lu = 0, \qquad u : \mathbb{R}^2 \to \mathbb{C},\ (x,y) \mapsto u(x,y), \tag{2.1}$$

with

$$L = a_{11}\frac{\partial^2}{\partial x^2} + 2a_{01}\frac{\partial^2}{\partial x \partial y} + a_{22}\frac{\partial^2}{\partial x^2} + b_1\frac{\partial}{\partial x} + b_2\frac{\partial}{\partial y} + c \tag{2.2}$$

for some *constants* a_{ij}, b_k, c.

Equivalently, (2.1) can be written with $u = u(x,y)$

$$a_{11}\frac{\partial^2 u}{\partial x^2} + 2a_{01}\frac{\partial^2 u}{\partial x \partial y} + a_{22}\frac{\partial^2 u}{\partial y^2} + b_1\frac{\partial u}{\partial x} + b_2\frac{\partial u}{\partial y} + cu = 0, \tag{2.3}$$

or, in alternative notation,

$$a_{11}u_{xx} + 2a_{01}u_{xy} + a_{22}u_{yy} + b_1u_x + b_2u_y + cu = 0. \tag{2.4}$$

2.0.2 The wave equation

Let's take a look at the *wave equation*, which is the special case $a_{11} = -a_{22} \neq 0$ and all the other coefficients are zero, so that

$$\partial_t^2 u(x,t) = \partial_x^2 u(x,t). \tag{2.5}$$

This can also be written

$$Lu = 0 \quad \text{with} \quad L = \partial_t^2 - \partial_x^2. \tag{2.6}$$

L can be factorized into two first-order 'transport' operators

$$L := \partial_t^2 - \partial_x^2 \ := \ (\partial_t - \partial_x)(\partial_t + \partial_x) \tag{2.7}$$

$$:= \ (\partial_t + \partial_x)(\partial_t - \partial_x) \tag{2.8}$$

From (2.7) it follows that any $z = z(x,t)$ which solves the transport equation

$$0 = (\partial_t + \partial_x)\, z = z_t + z_x$$

will automatically define a solution for (2.5), and likewise, from (2.8) so will any $w = w(x,t)$ which solves the transport equation $0 = (\partial_t - \partial_x)\, w = w_t - w_x$. But we already know these two 'transport' equations have general solutions of the form $z(x,y) = g(x-t)$ and $w(x,y) = f(x+t)$ for any differentiable functions $f, g : \mathbb{R} \to \mathbb{R}$. Hence one class of solutions to the wave equation is

$$u(x,t) = f(x+t) + g(x-t). \tag{2.9}$$

This is the most general solution to (2.5), for, making the change of coordinates

$$\xi = x + t, \qquad \eta = x - t, \tag{2.10}$$

the Chain Rule in two variables in (ξ, η)-coordinates the wave operator (**??**) becomes

$$L = -4\,\partial_\xi \partial_\eta \tag{2.11}$$

For, the Chain Rule in two variables says that

$$\partial_x = \partial_\xi + \partial_\eta, \qquad \partial_t = \partial_\xi - \partial_\eta,$$

and a simple rearrangement of this gives (2.11). So to solve the wave equation we have to solve

$$\partial_\xi \partial_\eta \, u(\xi, \eta) = 0.$$

And we already know that

$$u_{\xi\eta} = 0 \quad \Longrightarrow \quad u(\xi,\eta) = f(\xi) + g(\eta) \;\text{ with }\; f, g : \mathbb{R}^1 \to \mathbb{R}^1. \tag{2.12}$$

Hence substituting back (2.10) into (2.12) we obtain (2.9) resolving it into "left-moving" and "right-moving" waves.

Exercise: check directly that (2.9) **really does solve the wave equation**

The solution (2.9) is general, but also rather vague. To get a more precise solution we have to specify some *initial (or 'boundary') conditions.* To this end, we augment the PDE to the initial value problem

$$\begin{cases} u_{tt} - u_{xx} = 0, & \\ u(x,0) = \phi(x) & \text{(initial shape of the wave)} \\ u_t(x,0) = \psi(x) & \text{(initial speed of the wave).} \end{cases} \tag{2.13}$$

A direct computation shows that the solution (2.9) is refined in the case of (2.13) to

$$u(x,t) \;=\; \frac{1}{2}\left(\phi(x+t) + \phi(x-t)\right) \;+\; \frac{1}{2}\int_{x-t}^{x+t} \psi(s)\, ds. \tag{2.14}$$

This says that if we specify what the wave in the (x,t)-plane must look like along the x-axis, and we specify its speed along that line, then the wave form (= graph of

$(x,t) \mapsto u(x,t)$ is determined throughout $\mathbb{R}^2$. This is much like what we discovered with the transport equation –that PDE is first-order so it was enough just to specify $u(x,t)$ along a curve, while because here we are dealing with a 2nd order PDE we also have to specify its first derivative.

For example, using (2.14), the solution to the PDE initial value problem

$$\begin{cases} u_{tt} - u_{xx} = 0, \\ u(x,0) = e^{-x}, \\ u_t(x,0) = \cos x, \end{cases} \tag{2.15}$$

is

$$u(x,t) = e^{-x}\,\cosh(t) + \cos(x)\sin(t).$$

2.0.3 Example: Laplace equation

The Laplace equation is the 2nd order linear homogeneous PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{2.16}$$

or, in alternative notation,

$$u_{xx} + u_{yy} = 0. \tag{2.17}$$

Easy solutions to spot just by observation are

$$u(x,y) = ax + by + c, \quad u(x,y) = x^2 - y^2, \quad u(x,y) = xy. \tag{2.18}$$

By the *superposition principle* (=" if u and v solve the PDE then so does $u+v$") it therefore follows that $u(x,y) = \lambda x^2 + \mu xy - \lambda y^2 + ax + by + c$ is also a solution for any constants λ, μ, a, b, c. A less obvious polynomial solution is

$$u(x,y) = x^3 - xy^2 + 2xy^2 \tag{2.19}$$

and, in fact, there are particular polynomial solutions in each degree — one thing we will shortly explain is how to find those polynomials. All of these solutions exist on all of $\mathbb{R}^2$.

Another not so obvious solution is

$$u(x,y) = \log(x^2 + y^2) \quad \text{valid on } \mathbb{R}^2 \backslash \{(0,0)\}. \tag{2.20}$$

On the other hand, if we look for solutions to the Laplace equation PDE bvp: in the punctured unit disc $D = \{(x,y) \mid 0 < x^2 + y^2 < 1\}$ with the boundary condition

$$u(x,y) = 0 \quad \text{for} \quad (x,y) \in S^1 = \{(x,y) \mid x^2 + y^2 = 1\}, \tag{2.21}$$

then none of the polynomials above are solutions, but (2.20) is a solution .

A rather different type of solution is

$$u_m(x,y) = \sin(mx)\,\sinh(my) \quad \text{any} \quad m \in \mathbb{R}.$$

Likewise, $u_m(x,y) = \cos(mx)\,\sinh(my), u_m(x,y) = \sin(mx)\,\cosh(my), u_m(x,y) = \cos(mx)\,\cosh(my)$, are all solutions, and so by linearity so is any linear sum of these solutions. This is useful for 'fitting' a solution to a given boundary problem; for example, with X the solid rectangle $X = [0,\pi] \times [0,\pi]$ the Laplace equation PDE bvp:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{for } (x,y) \in [0,\pi] \times [0,\pi] \tag{2.22}$$

subject to the boundary conditions

$$u(0,y) = 0, \quad u(\pi,y) = 0, \quad u(x,0) = 0, \quad \text{and} \quad u(x,\pi) = \sin^3 x, \tag{2.23}$$

has solution

$$u(x,y) = \left(\frac{3}{4\sinh\pi}\right)\sin(x)\,\sinh(y) + \left(\frac{1}{4\sinh(3\pi)}\right)\sin(3x)\,\sinh(3y). \tag{2.24}$$

<u>Excercise</u>: ***Check directly that*** (2.24) ***is a solution to the PDE bvp*** (2.22)***,*** (2.23) – ***draw the region*** X ***and where the boundary conditions are being imposed.***

2.0.4 Example: heat equation

The heat equation is the 2nd order linear homogeneous PDE

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial u} = 0 \quad \text{for} \quad (x,t) \in \mathbb{R}^1 \times (0,\infty), \tag{2.25}$$

or, in alternative notation,

$$u_t = u_{xx} \quad \text{for} \quad (x,t) \in \mathbb{R}^1 \times (0,\infty). \tag{2.26}$$

Note that this is only specified in the half-plane $t > 0$ (this is needed for non-trivial solutions to exist).

An easy 'trivial' solution to spot is

$$u_0(x,t) = \alpha x^2 + \beta x + \gamma + 2\alpha t \tag{2.27}$$

for any constants α, β, γ.

An important less obvious solution is

$$u_0(x,t) = \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{x^2}{4t}}, \tag{2.28}$$

we will derive this later on.

On the other hand, in the half-infinite strip $X = [0,\pi]\times(0,\infty)$ this is not a solution to the heat equation PDE boundary value problem:

$$u_t = u_{xx} \quad \text{for } (x,t) \in [0,\pi] \times (0,\infty) \tag{2.29}$$

subject to

$$u(x,t) \to 0 \text{ as } t \mapsto \infty, \quad u(\pi,t) = 0, \quad u(0,t) = 0, \quad \text{and} \quad u(x,0) = \sin^3 x, \tag{2.30}$$

(note that the first of these is a 'boundary' condition 'at infinity'). But

$$u(x,t) = \frac{3}{4}\, e^{-t} \sin x - \frac{1}{4}\, e^{-9t} \sin(3x) \tag{2.31}$$

is a solution to this PDE bvp!

Notice how the specific geometry of bvps radically changes the form of the solutions in all the above example solutions — look again and try to identify the differences!

3 Fourier Transforms

The **Fourier transform (FT)** transforms an integrable function $f : \mathbb{R} \to \mathbb{R}$ of one variable x into another function $\widehat{f} : \mathbb{R} \to \mathbb{R}$ of one variable ξ, while the **inverse Fourier transform (IFT)** reverses the transformation, provided $\widehat{f}$ is integrable. This is an important process for studying differential equations (and partial differential equations) because the FT turns differentiation of f with respect to x into multiplication of $\widehat{f}$ by ξ, exchanging (hard) questions of differentiability of f for (easier) questions of growth rates of $\widehat{f}$ at infinity (i.e. as $|\xi| \to \infty$), which turns (hard to solve) differential equations into (relatively easy to solve) algebraic (polynomial) equations.

Let us denote the FT by F and the IFT by F^{-1}. Then F is defined by

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto f(x), \qquad \overset{\mathsf{F}}{\rightsquigarrow} \qquad \widehat{f} := \mathsf{F}(f) : \mathbb{R} \to \mathbb{R}, \quad \xi \mapsto \widehat{f}(\xi),$$

with

$$\widehat{f}(\xi) := \int_{-\infty}^{\infty} e^{i\xi t} f(t)\; dt. \tag{3.1}$$

The inverse Fourier transform is defined by

$$\breve{g}(x) := \mathsf{F}^{-1}(g)(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x} g(\xi)\ d\xi. \tag{3.2}$$

3.1 F and F^{-1} are inverse to each other (when defined!)

The notation and terminology suggest F^{-1} reverses what F does to f. This is true provided all the integrals in question exist (we will assume they do!). That is,

$$I = \mathsf{F}^{-1} \circ \mathsf{F} \quad \text{where} \quad I(f) = f \qquad (\text{i.e. } I(f)(x) = f(x)\ \forall x \in \mathbb{R}). \tag{3.3}$$

That is, $f = \mathsf{F}^{-1}(\mathsf{F}(f))$ or, equivalently, $f = \mathsf{F}^{-1}\left(\widehat{f}\right)$. From (5.5), this is the equality

$$f(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x}\,\widehat{f}(\xi)\ d\xi := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x}\left(\int_{-\infty}^{\infty} e^{i\xi t} f(t)\ dt\right)\ d\xi. \tag{3.4}$$

3.1.1 Example: compute $\mathsf{F}(e^{-|x|})$

We have

$$\begin{aligned}
\widehat{f}(\xi) &:= \int_{-\infty}^{\infty} e^{i\xi t} e^{-|t|}\ dt = \int_{0}^{\infty} e^{i\xi t} e^{-t}\ dt + \int_{-\infty}^{0} e^{i\xi t} e^{t}\ dt \\
&= \left[\frac{1}{i\xi - 1}\, e^{(i\xi-1)t}\right]_0^{\infty} + \left[\frac{1}{i\xi + 1}\, e^{(i\xi+1)t}\right]_{-\infty}^{0} \\
&= \lim_{a\to\infty}\left[\frac{1}{i\xi - 1}\, e^{(i\xi-1)t}\right]_0^{a} + \lim_{a\to\infty}\left[\frac{1}{i\xi + 1}\, e^{(i\xi+1)t}\right]_{-a}^{0} \\
&= -\frac{1}{i\xi - 1} + \frac{1}{i\xi + 1},
\end{aligned}$$

since $\lim_{a\to\infty} e^{(i\xi-1)a} = 0$ (for, here $|e^{(i\xi-1)t}| = e^{-t}$), and similarly $\lim_{a\to\infty} e^{(i\xi+1)(-a)} = 0$. Thus

$$\widehat{f}(\xi) = \frac{2}{1+\xi^2}. \tag{3.5}$$

Perhaps what is more interesting is that applying the inverse FT now gives,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x}\ \underbrace{\frac{2}{1+\xi^2}}_{=\ \widehat{f}(\xi)}\ d\xi = e^{-|x|} \tag{3.6}$$

This is the identity (3.4) applied to $f(x) = e^{-|x|}$. Thus,

$$\int_{-\infty}^{\infty} \frac{e^{-i\xi x}}{1+\xi^2}\, d\xi = \pi\, e^{-|x|}. \tag{3.7}$$

3.1.2 Example:

Let $\beta > 0$, and set

$$f(x) = \begin{cases} 1, & |x| \leq \beta, \\ 0, & |x| > \beta. \end{cases} \tag{3.8}$$

Then

$$\widehat{f}(\xi) = \int_{-\beta}^{\beta} e^{i\xi t}\, dt = \left[\frac{e^{i\xi t}}{i\xi}\right]_{-\beta}^{\beta} = \frac{e^{i\beta t}}{i\xi} - \frac{e^{-i\beta t}}{i\xi} = \frac{2\sin(\beta\xi)}{\xi}.$$

3.1.3 Example: compute $\mathsf{F}(e^{-\alpha x^2})$, where $\alpha \in (0,\infty)$.

In the following, for notational brevity, write $\int = \int_{-\infty}^{\infty}$. We have

$$\widehat{f}(\xi) := \int e^{i\xi t} e^{-\alpha t^2}\, dt = \int e^{-\alpha t^2 + i\xi t}\, dt \overset{w=\sqrt{\alpha}t}{=} \frac{1}{\sqrt{\alpha}} \int e^{-w^2 + \frac{i\xi}{\sqrt{\alpha}}w}\, dw.$$

We are aiming now to reduce this to 'something' times the integral $\int e^{-r^2}\, dr = \sqrt{\pi}$. So we want to try and write the exponent as an exact square, well we have

$$w^2 - \frac{i\xi}{\sqrt{\alpha}} w = \left(w - \frac{i\xi}{2\sqrt{\alpha}}\right)^2 + \frac{\xi^2}{4\alpha},$$

and so $\widehat{f}(\xi) = \frac{e^{-\frac{\xi^2}{4\alpha}}}{\sqrt{\alpha}} \int e^{-\left(w - \frac{i\xi}{2\sqrt{\alpha}}\right)^2}\, dw$. But $\int e^{-\left(w - \frac{i\xi}{2\sqrt{\alpha}}\right)^2}\, dw = \int e^{r^2}\, dr$, and hence

$$\widehat{f}(\xi) = \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{\xi^2}{4\alpha}}. \tag{3.9}$$

4 ODEs and the FT

[1] Linearity: for (integrable) $f, g : \mathbb{R}^1 \to \mathbb{R}^1$ and any $\lambda, \mu \in \mathbb{C}$, one has

$$\mathsf{F}(\lambda f + \mu g)(\xi) \;=\; \lambda \mathsf{F}(f)(\xi) + \mu \mathsf{F}(g)(\xi) \tag{4.1}$$

[2] The FT turns differentiation into multiplication:

$$\mathsf{F}(f')(\xi) \;=\; -i\xi\, \mathsf{F}(f)(\xi). \tag{4.2}$$

Here, $f'(x) = df/dx$ is the derivative of f, and, crucially, we assume that

$$f(x) \to 0 \quad \text{as} \quad |x| \to \infty. \tag{4.3}$$

Iterating the latter property we get for $m \in \mathbb{N}$

$$\mathsf{F}(f^{(m)})(\xi) \;=\; (-i\xi)^m\, \mathsf{F}(f)(\xi) \tag{4.4}$$

where $f^{(m)} = d^m f/dx^m$. That iteration therefore requires from (5.9) that

$$f^{(k)}(x) \to 0 \quad \text{as} \quad |x| \to \infty \quad \text{for } k = 0, 1, \ldots, m-1. \tag{4.5}$$

4.1 Exact solutions to constant coefficient ODEs

If we combine the above two properties with the invertibility of the FT (i.e. that $F^{-1}(F(f)) = f$) then we can solve 'any' constant coefficient ODE

$$a_n\, f^{(n)}(x) + a_{n-1}\, f^{(n-1)}(x) + \cdots + a_1\, f'(x) + a_0 f(x) = g(x), \tag{4.6}$$

where $a_n, \ldots, a_0 \in \mathbb{C}$ are constants and where $g : \mathbb{R}^1 \to \mathbb{R}^1$ is given, and the objective is to determine the function $f : \mathbb{R}^1 \to \mathbb{R}^1$.

Specifically, applying the FT to both sides of (4.6), and using Fourier inversion gives

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\xi} \widehat{f}(\xi)\, d\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\xi} \frac{\widehat{g}(\xi)}{p_n(\xi)}\, d\xi. \tag{4.7}$$

Substituting

$$\widehat{g}(\xi) \;=\; \int_{-\infty}^{\infty} e^{i\xi y} g(y)\, dy$$

into (4.7) and rearranging (in particular - changing the order of integration) we get our solution

$$f(x) = \int_{-\infty}^{\infty} k(x,y)\, g(y)\, dy, \qquad \text{where } k(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i(y-x)\xi}}{p_n(\xi)}\, d\xi. \tag{4.8}$$

4.2 An example

For

$$-\frac{d^2 f}{dx^2} + f(x) = g(x) \tag{4.9}$$

subject to $f(x) \to 0$ and $f'(x) \to 0$ as $|x| \to \infty$, our polynomial is $p_2(\xi) = \xi^2 + 1$. Hence

$$k(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i(y-x)\xi}}{1+\xi^2}\, d\xi.$$

From equation (3.9) we have

$$k(x,y) = \frac{1}{2}\, e^{-|x-y|} = \begin{cases} \frac{1}{2}\, e^{-x} e^{y}, & \text{for } x \geq y, \\ \frac{1}{2}\, e^{x} e^{-y}, & \text{for } x \leq y. \end{cases}.$$

That is, the general solution to (4.9) is $f(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|}\, g(y)\, dy$, which we can rewrite using (4.2) as

$$f(x) = \frac{e^{-x}}{2}\int_{-\infty}^{x} e^{y}\, g(y)\, dy + \frac{e^{x}}{2}\int_{x}^{\infty} e^{-y}\, g(y)\, dy. \tag{4.10}$$

5 PDEs and the FT

A general constant coefficient homogeneous linear PDE in $\mathbb{R}^2$ up to order 2 has the form (recall)

$$a_1\frac{\partial^2 u}{\partial x^2} + a_2\frac{\partial^2 u}{\partial x \partial y} + a_3\frac{\partial^2 u}{\partial y^2} + a_4\frac{\partial u}{\partial x} + a_5\frac{\partial u}{\partial y} + a_6 u(x,y) = v(x,y), \tag{5.1}$$

for some constants $a_k \in \mathbb{C}$; if we took $a_1 = a_2 = a_3 = 0$ this reduces to a first order PDE. In alternative (rather lighter) notation,

$$a_1 u_{xx} + a_2 u_{xy} + a_3 u_{yy} + a_4 u_x + a_5 u_y + a_6 u = v. \tag{5.2}$$

Here, u and v are functions of two variables on $\mathbb{R}^2$

$$u : \mathbb{R}^2 \longrightarrow \mathbb{R}^1, \quad (x,y) \mapsto u(x,y), \qquad v : \mathbb{R}^2 \longrightarrow \mathbb{R}^1, \quad (x,y) \mapsto v(x,y),$$

(you studied such functions in Calculus II cm112a; for example, $v(x,y) = x^2 + y^2$).

We will restrict our attention is specific examples to the homogeneous case: $v = 0$.

5.0.1 Definition of $\widehat{u}$

We are going to Fourier transform just one of the variables: let us FT the x-variable,

$$\mathsf{F} = \mathsf{F}_{x\to\xi} \;:\; \text{functions on } \underbrace{(x,y) - \text{space}}_{=\,\mathbb{R}^2} \quad\longrightarrow\quad \text{functions on } \underbrace{(\xi,y) - \text{space}}_{=\,\mathbb{R}^2}.$$

Specifically,

$$u : \mathbb{R}^2 \to \mathbb{R}^1, \quad (x,y) \mapsto u(x,y), \quad \overset{\mathsf{F}}{\rightsquigarrow} \quad \widehat{u} := \mathsf{F}_{x\to\xi}(u) : \mathbb{R}^2 \to \mathbb{R}^1, \quad (\xi,y) \mapsto \widehat{u}(\xi,y),$$

where

$$\widehat{u}(\xi,y) := \int_{-\infty}^{\infty} e^{i\xi t} u(t,y)\, dt. \tag{5.3}$$

For notational convenience, we may use any of

$$\widehat{u}(\xi, y) \;=\; \mathsf{F}(u)(\xi, y) \;=\; \mathsf{F}_{x\to\xi}(u)(\xi, y) \;=\; \mathsf{F}_{x\to\xi, y\to y}(u)(\xi, y) \tag{5.4}$$

to denote the x-FT of u. The last two can be useful in reminding us 'what variable is being changed into what'. The inverse Fourier transform

$$\mathsf{F}^{-1}_{\xi\to x} := \mathsf{F}^{-1}_{\xi\to x, y\to y} \;:\; \text{functions on } \underbrace{(\xi, y) - \text{space}}_{=\,\mathbb{R}^2} \quad\longrightarrow\quad \text{functions on } \underbrace{(x, y) - \text{space}}_{=\,\mathbb{R}^2},$$

is defined on $w : \mathbb{R}^2 \to \mathbb{R}^1,\ (\xi, y) \mapsto w(\xi, y)$, by

$$\breve{w}\,(x, y) := \frac{1}{2\pi}\int_{\xi=-\infty}^{\infty} e^{-i\xi x} w(\xi, y)\; d\xi. \tag{5.5}$$

Again, for notational convenience, we may use any of

$$\breve{w}\,(x, y) \;=\; \mathsf{F}^{-1}(w)(x, y) \;=\; \mathsf{F}^{-1}_{\xi\to x}(w)(x, y) \;=\; \mathsf{F}^{-1}_{\xi\to x, y\to y}(w)(x, y)). \tag{5.6}$$

5.0.2 Basic properties of $\mathsf{F}_{x\to\xi, y\to y}$:

As in the case of ODEs, the two principal properties needed to use the FT to solve PDEs are:

[1] **Linearity**: for (integrable) $u, v : \mathbb{R}^2 \to \mathbb{R}^1$ and any $\lambda, \mu \in \mathbb{C}$, one has

$$\mathsf{F}_{x\to\xi, y\to y}(\lambda u + \mu v)(\xi, y) \;=\; \lambda\widehat{u}(\xi, y) + \mu\widehat{v}(\xi, y). \tag{5.7}$$

[2] $\mathsf{F}_{x\to\xi, y\to y}$ turns partial differentiation in x into multiplication by ξ:

$$\widehat{u_x}\,(\xi, y) \;=\; -i\xi\,\widehat{u}(\xi, y). \tag{5.8}$$

Here, $u_x(x, y) = \partial u/\partial x$and we must assume that

$$u(x, y) \to 0 \quad \text{as} \quad |x| \to \infty. \tag{5.9}$$

Iterating the latter property we get for $m \in \mathbb{N}$

$$\mathsf{F}\left(\frac{\partial^m u}{\partial x^m}\right)(\xi, y) \;=\; (-i\xi)^m\,\widehat{u}(\xi, y). \tag{5.10}$$

provided that

$$\frac{\partial^k u}{\partial x^k}(x, y) \to 0 \quad \text{as} \quad |x| \to \infty \quad \text{for } k = 0, 1, \ldots, m-1. \tag{5.11}$$

One of the differences in the $\mathbb{R}^2$ case is that there are now more types of derivative to consider: mixed partial derivatives and partial derivatives in y alone. Plugging $u_y = \partial u/\partial y$ instead of u into (5.3) we easily see that

$$\widehat{u_y}\,(\xi, y) \;=\; \frac{\partial}{\partial y}\,\widehat{u}(\xi, y). \tag{5.12}$$

And hence, combining (5.8) and (5.12),

$$\widehat{u_{xy}}\,(\xi, y) \;=\; (-i\xi)\,\frac{\partial}{\partial y}\,\widehat{u}(\xi, y), \tag{5.13}$$

(which is the same as $\widehat{u_{yx}}\,(\xi, y)$ of course), whilst iterating (5.12) we have

$$\widehat{u_{yy}}\,(\xi, y) \;=\; \frac{\partial^2}{\partial y^2}\,\widehat{u}(\xi, y), \tag{5.14}$$

and similarly for higher partial derivatives.

The other fact we need to know is how the FT affects initial conditions: if

$$u(x, 0) = f(x) \tag{5.15}$$

for some specific one variable $f : \mathbb{R}^1 \to \mathbb{R}^1$ (for example, $u(x, 0) = \sin(x)$), then

$$\widehat{u}(\xi, 0) = \widehat{f}(\xi). \tag{5.16}$$

5.0.3 Example: the transport equation

Recall from Lecture 2 that this is the first-order PDE

$$\begin{cases} u_x + c\,u_y = 0, \\ u(x, 0) = f(x), \end{cases} \tag{5.17}$$

where c is a constant, and that it has the unique solution

$$u(x, y) = f\left(x - \frac{y}{c}\right). \tag{5.18}$$

We deduced (5.18) by the good fortune of spotting that the Chain Rule could be used. We will now derive (5.18) using FT methods; the point being that the FT is a general method which applies to any PDE (we do not in general want to rely on good fortune alone!). Applying the FT $\mathsf{F}_{x\to\xi, y\to y}$ in the x-variable to $u_x + c\,u_y = 0$ we have

$$\widehat{u_x}\,(\xi, y) + c\,\widehat{u_y}\,(\xi, y) = 0. \tag{5.19}$$

From (7.10) (with $m = 1$) and (7.12) this is the same as $(-i\xi)\,\widehat{u}(\xi, y) + c\,\frac{\partial}{\partial y}\,\widehat{u}(\xi, y) = 0$, provided (for the first summand) that $u(x, y) \to 0$ as $|x| \to \infty$.

By writing (**??**) as $\frac{\partial}{\partial y}\widehat{u}(\xi,y) = \frac{i\xi}{c}\widehat{u}(\xi,y)$ we see it has solution

$$\widehat{u}(\xi,y) = \widehat{u}(\xi,0)\, e^{\frac{i\xi}{c}y}. \tag{5.20}$$

From (7.16), this is the same thing as

$$\widehat{u}(\xi,y) = \widehat{f}(\xi)\, e^{\frac{i\xi}{c}y}. \tag{5.21}$$

So (5.21) is the solution in (ξ, y) space. We now have to apply the inverse FT to the ξ variable to transform this back to the solution in (x, y) space

$$u(x,y) = \mathsf{F}^{-1}_{\xi\to x, y\to y}\left(\widehat{u}\right)(x,y) = \frac{1}{2\pi}\int_{\xi=-\infty}^{\infty} e^{-i\xi\left(x-\frac{y}{c}\right)}\,\widehat{f}(\xi)\ d\xi = f\left(x-\frac{y}{c}\right),$$

where the final equality uses equation (3.4).

5.1 Example

Solve the following PDE using Fourier transforms:

$$\frac{\partial^6 u}{\partial x^6} - \frac{\partial^2 u}{\partial y^2} = 0, \qquad u = u(x,y),$$

subject to the conditions

$$\frac{\partial^m u}{\partial x^m} \to 0 \quad \text{as} \quad |x| \to \infty \quad \text{for} \ \ m = 0,1,2,3,4,5$$

and

$$\frac{\partial u}{\partial y}(x,0) = 0 \quad \text{and} \quad u(x,0) = \frac{1}{1+x^2}. \tag{5.22}$$

Applying the Fourier transform $\mathsf{F}_{x\to\xi,y\to y}$ in the x-variable to $\frac{\partial^6 u}{\partial x^6} - \frac{\partial^2 u}{\partial y^2} = 0$ gives the ODE in y

$$\left(\frac{\partial^2}{\partial y^2} + \xi^6\right)\widehat{u}(\xi,y) = 0. \tag{5.23}$$

This has solution

$$\widehat{u}(\xi,y) = A(\xi)\cos(\xi^3 y) + B(\xi)\sin(\xi^3 y), \tag{5.24}$$

with $A(\xi)$, $B(\xi)$ constants in y, but which may depend on ξ. In fact, we see $A(\xi) = \widehat{u}(\xi,0)$ and so

$$A(\xi) = \mathsf{F}_{x\to\xi}(u(x,0))(\xi), \quad \text{where, recall,} \ \ u(x,0) = \frac{1}{1+x^2}. \tag{5.25}$$

From Example 3.1.1

$$\mathsf{F}_{x\to\xi}\left(\frac{1}{1+x^2}\right)(\xi) = \pi e^{-|\xi|}. \tag{5.26}$$

On the other hand, we easily see $B(\xi)=0$. So the solution in (ξ, y) space is

$$\widehat{u}(\xi, t) = \pi e^{-|\xi|}\,\cos(\xi^3 y). \tag{5.27}$$

Applying the inverse FT transforms this back to the solution in (x, y) space

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x}\,\pi e^{-|\xi|}\,\cos(\xi^3 y)\; d\xi, \tag{5.28}$$

or, more neatly,

$$u(x,y) = \int_{0}^{\infty} e^{-|\xi|}\,\cos(\xi x)\,\cos(\xi^3 y)\; d\xi. \tag{5.29}$$

6 The method of separation of variables and Fourier series

The second analytical method for solving PDEs we are going to consider computes solutions to constant coefficient PDEs on certain symmetric regions of $X \subset \mathbb{R}^2$ which have **bounded geometry**, or **semi-bounded geometry**. The geometry of X has a determining influence on the form of the solution to a PDE, and, conversely, if we can solve a given PDE on X then the solutions will often tell us about the geometry of X; this is an important idea used to study spaces which are twisted in complicated ways which, particularly in higher dimensions, we cannot easily understand by intuitive ideas alone.

The way in which the geometry of a flat region X of $\mathbb{R}^2$ is encoded into a PDE is by specifying **boundary conditions** on the solution functions, meaning that the solution is require to assume specified values along the edge of X. The idea, then, is to look for solutions to a given PDE

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + p\frac{\partial u}{\partial x} + q\frac{\partial u}{\partial y} + ru = 0$$

which have the form

$$u(x,y) \;=\; f(x)\,g(y), \quad \leftarrow \quad \text{this is what is called } \textbf{`separation of variables'} \tag{6.1}$$

where

$$f:\mathbb{R}^1 \longrightarrow \mathbb{R}^1,\ \ x \mapsto f(x), \quad \text{and} \quad g:\mathbb{R}^1 \longrightarrow \mathbb{R}^1,\ \ y \mapsto g(y),$$

are functions of 1 variable. For a well-posed problem, the boundary conditions for the PDE will separate into boundary conditions for two ODEs — one ODE for f and one ODE for g. The whole process each time repeats the following steps:

1. Separate variables and split the PDE into two ODEs each with boundary (or initial) conditions.
2. Solve one of the ODEs using its homogeneous boundary conditions (i.e. the ones equal to zero) and as far as is possible determine the constants β_n in any trigonometric factors $\sin(\beta_n x)$ and $\cos(\beta_n x)$ which occur.
3. Use this to then also solve the second ODE, again using its homogeneous boundary conditions. Hence write down for each β_n a solution to the PDE — the constants β_n are at this point still undetermined.
4. By summing such solutions over n use *Fourier Series* to compute the undetermined coefficients β_n, and hence write down an exact solution to the PDE as an infinite sum of functions of the form $\beta_n f_n(x)\, g_n(y)$, where one (at least) of $f_n(x)$ or $g_n(y)$ is a trigonometric function.

6.1 Example: separation of variables solution of the Laplace equation on on $[0, L] \times [0, L]$

Consider the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \text{or} \quad u_{xx} + u_{yy} = 0 \tag{6.2}$$

on the solid square $X = \{(x, y) \mid 0 \le x \le L,\ 0 \le y \le L\}$ subject to the boundary conditions

$$\begin{aligned} u(0, y) &= 0, && (6.3) \\ u(L, y) &= 0, && (6.4) \\ u(x, 0) &= 0, && (6.5) \\ u(x, L) &= \phi(x) \ \text{ for some given } \phi : \mathbb{R}^1 \to \mathbb{R}^1 && (6.6) \end{aligned}$$

for $(x, y) \in (0, L) \times (0, L)$.

We try to construct a solution by adding together solutions having the form

$$u(x, y) = f(x)g(y)$$

for some $f : \mathbb{R}^1 \to \mathbb{R}^1$ and $g : \mathbb{R}^1 \to \mathbb{R}^1$. Substituting in (6.2) gives the two ODEs

$$f''(x) + \mu^2 f(x) = 0 \quad \text{and} \quad g''(y) - \mu^2 g(y) = 0 \tag{6.7}$$

with solutions if $\mu \neq 0$

$$f_\mu(x) = A_\mu \cos(\mu x) + B_\mu \sin(\mu x) \quad \text{and} \quad g_\mu(y) = C_\mu \cosh(\mu x) + D_\mu \sinh(\mu x) \quad (6.8)$$

for some undetermined constants $A_\mu, B_\mu, C_\mu, D_\mu$. If $\mu = 0$ then

$$f_0(x) = ax + b, \quad g_0(y) = cy + d, \tag{6.9}$$

for some constants a, b, c, d.

Since $u_\mu(x, y) = f_\mu(x)g_\mu(y)$ is a solution so is any function

$$u(x, y) = \Sigma_{\mu \in S} f_\mu(x)g_\mu(y)$$

for S a set of labels, such as the integers or the positive integers.

We can use the boundary conditions to determine the constants, as follows. (6.3) implies from (6.8) that $A_\mu = 0$. Likewise, (6.5) implies $C_\mu = 0$. The boundary condition (6.4) requires $f_\mu(L) = 0$ and hence that $\mu = n\pi/L$ with $n \in \mathbb{Z}$ an integer, and also that $a = 0$.

Thus, any solution to the Laplace equation with the first three boundary conditions (6.3), (6.4), (6.5) is a sum of the form

$$u(x, y) = \Sigma_{n=1}^\infty E_n \sin(\frac{n\pi x}{L}) \sinh(\frac{n\pi y}{L}) \tag{6.10}$$

for some constants E_n.

To compute the constants E_n one uses the boundary condition (6.6) , which requires

$$\phi(x) = \Sigma_{n=1}^\infty E_n \sinh(n\pi) \cdot \sin(\frac{n\pi x}{L}) \;\; = \;\; \Sigma_{n=1}^\infty F_n \cdot \sin(\frac{n\pi x}{L}) \tag{6.11}$$

some constants F_n (since we can then compute the $E_n = F_n/\sinh(n\pi)$). For this we need Fourier series (below).

6.2 Example

By the procedure outlined above, we likewise obtain that the Laplace boundary problem on $[0, L] \times [0, M]$

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \\ u_x(0, y) &= 0, \\ u(x, 0) &= 0, \\ u(L, y) &= 0, \end{aligned}$$

has the class of solutions

$$u(x,y) = \Sigma_{n=0}^{\infty}\ K_n \cos\left(\frac{(2n-1)\pi}{2L}x\right)\sinh\left(\frac{(2n-1)\pi}{2L}y\right). \tag{6.12}$$

Thus, a conclusion of all this is that the particular boundary conditions imposed produce different products of trigonometric and hyperbolic functions. Likewise, the possible values for the constants μ are determined by the boundary conditions. As an exercise, you might like to try and manufacture boundary conditions such that the solution involves only sums of $\cos(\mu x)\cosh(\mu y)$ or $\sin(\mu x)\cosh(\mu y)$.

6.3 Fourier Series

Fourier series allow us to write a general function

$$\phi : [0, L] \to \mathbb{R}, \quad x \mapsto \phi(x), \tag{6.13}$$

no matter how complicated it is (provided it is, say, at least piecewise continuous) as a sum of (relatively simple) sines and cosines. Precisely, there exist real values constants $A_n, B_n \in \mathbb{R}$ such that

$$\phi(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{for } 0 < x < L, \tag{6.14}$$

and also

$$\phi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \quad \text{for } 0 \le x \le L. \tag{6.15}$$

Those constants are given by the specific formulae

$$B_n = \frac{2}{L}\int_0^L \phi(t)\ \sin\left(\frac{n\pi t}{L}\right)\ dt \tag{6.16}$$

and

$$A_n = \frac{2}{L}\int_0^L \phi(t)\ \cos\left(\frac{n\pi t}{L}\right)\ dt. \tag{6.17}$$

For example, for

$$\phi(x) = x, \tag{6.18}$$

from (6.16) you can compute that as a sine series,

$$x = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\ \sin\left(\frac{n\pi x}{L}\right) \quad \text{for } 0 < x < L, \tag{6.19}$$

while as a cosine series

$$x = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\ \cos\left(\frac{(2m-1)\pi x}{L}\right) \quad \text{for } 0 \le x \le L. \tag{6.20}$$

6.4 Using this to find the constants E_n in (6.10)

To determine the constants B_n we specify a third boundary condition

$$u(x, L) = \phi(x) \qquad \forall\ 0 < x < L, \tag{6.21}$$

for some $\phi : \mathbb{R}^1 \to \mathbb{R}^1$. From (6.12) this is the requirement that

$$\phi(x) = \sum_{n=1}^{\infty} E_n \sinh(n\pi)\, \sin\left(\frac{n\pi x}{L}\right) \tag{6.22}$$

That is, that $B_n \sinh(n\pi)$ is the nth Fourier sine coefficient of the function $\phi(x)$ on $(0, L)$, so therefore

$$E_n \sinh(n\pi) = \frac{2}{L} \int_0^L \phi(t)\, \sin\left(\frac{n\pi t}{L}\right)\, dt.$$

So this gives an exact solution to the original PDE; that is, in terms of the given boundary conditions which allow us to say exactly what the value is of the (initially) undetermined constants that turn up in solving the PDE without boundary conditions.

For example, if we set

$$\phi(x) = x$$

then, as computed earlier,

$$E_n \sinh(n\pi) = \frac{2}{L} \int_0^L t\, \sin\left(\frac{n\pi t}{L}\right)\, dt = \frac{2L}{\pi} \frac{(-1)^{n+1}}{n}, \tag{6.23}$$

and hence the solution of the PDE subject the four given boundary conditions is

$$u(x, y) = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \sinh(n\pi)}\, \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi y}{L}\right). \tag{6.24}$$

Notice that if we had specified fewer boundary conditions then we would only have been able to compute a solution which contained arbitrary constants (a ‘family of solutions’), but not a unique solution, while if we had specified more boundary conditions then there would not exist any solution at all. Thus, in setting up the PDE a delicate balance has to be achieved in ensuring that the boundary problem is ‘well posed’.

7 The Cauchy Residue Theorem and Fourier Transforms

For the remainder of this course we will be thinking hard about how the following theorem allows one to explicitly evaluate a large class of Fourier transforms. This will enable us to write down explicit solutions to a large class of ODEs and PDEs.

The Cauchy Residue Theorem:

Let $C \subset \mathbb{C}$ be a simple closed contour. Let $f : \mathbb{C} \to \mathbb{C}$ be a complex function which is holomorphic along C and inside C except possibly at a finite number of points $a_1, \ldots, a_m$ at which f has a pole. Then, with C oriented in an anti-clockwise sense,

$$\int_C f(z)\,dz \;=\; 2\pi i \sum_{k=1}^{m} \mathrm{res}(f, a_k), \tag{7.1}$$

where $\mathrm{res}(f, a_k)$ is the residue of the function f at the pole $a_k \in \mathbb{C}$.

You are probably not yet familiar with the meaning of the various components in the statement of this theorem, in particular the underlined terms and what is meant by the contour integral $\int_C f(z)\,dz$, and so our first task will be to explain the terminology. The Cauchy Residue theorem has wide application in many areas of pure and applied mathematics, it is a basic tool both in engineering mathematics and also in the purest parts of geometric analysis. The idea is that the right-side of (7.47), which is just a finite sum of complex numbers, gives a simple method for evaluating the contour integral; on the other hand, sometimes one can play the reverse game and use an 'easy' contour integral and (7.47) to evaluate a difficult infinite sum (allowing $m \to \infty$). More broadly, the theory of functions of a complex variable provides a considerably more powerful calculus than the calculus of functions of two real variables ('Calculus II').

As listed on the course webpage, a good text for this part of the course is:
H A Priestley, Introduction to Complex Analysis (2nd Edition) (OUP)

We start by considering complex functions and the sub class of holomorphic functions.

7.1 Holomorphic functions

Within the space of all functions $f : \mathbb{C} \to \mathbb{C}$ there is a distinguished subspace of *holomorphic functions*, often also called *analytic functions*. Being holomorphic is just a *local property*, meaning that whether a function is *holomorphic at a point* $a \in \mathbb{C}$ depends only on the value of f at a and, for some small real number $\varepsilon > 0$, and its behaviour in a small disc

$$B_\varepsilon(a) : \{z \in \mathbb{C} \mid |z - a| < \varepsilon\}$$

around a. (By definition, $B_\varepsilon(a)$ consists of those complex numbers whose distance from a is less than ε.)

Working definition: *A function is holomorphic at* $a \in \mathbb{C}$ *if it is independent of* $\overline{z}$ *near* a *and has no singularity at* $z = a$ *(meaning it is well defined at all points near* a *and is differentiable (smooth) in* z*) there.*

In practise, those are the properties we look for in order to identify whether function is holomorphic at a given point: it must be a function of z alone and must be differentiable, the latter meaning (in practise) that if you replace z by a real variable x then you recognize the resulting function as differentiable in the usual (real variable) sense.

In particular, for any given $b \in \mathbb{C}$ the exponential function

$$f(z) = e^{bz}$$

is holomorphic at all $z \in \mathbb{C}$. That immediately implies that all the trigonometric and hyperbolic functions

$$f(z) = \sin z, \quad f(z) = \cos z, \quad f(z) = \sinh z, \quad f(z) = \cosh z, \tag{7.2}$$

are all holomorphic at all $z \in \mathbb{C}$. The implication is immediate because of the following properties which tell us we can build many holomorphic functions just by knowing a few simple ones:

$$f,\ g\ ,: \mathbb{C} \to \mathbb{C}\ \textbf{holomorphic}\ \text{at}\ a \in \mathbb{C}\ \Rightarrow\ f.g,\ f{+}g,\ f{\circ}g\ \textbf{holomorphic}\ \text{at}\ a \in \mathbb{C}. \tag{7.3}$$

and

$$\frac{f}{g}\ \textbf{holomorphic}\ \text{at}\ a \in \mathbb{C}\ \textbf{provided}\ g(a) \neq 0. \tag{7.4}$$

Thus, as another example, because $f(z) = z$ is holomorphic (everywhere) then so is any polynomial

$$f(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0, \quad a_j \in \mathbb{C}. \tag{7.5}$$

On the other hand,

$$f(z) = \frac{e^z}{z-b} \tag{7.6}$$

is holomorphic at all points *except* at $z = b$.

When we have a function which is holomorphic at $a \in \mathbb{C}$ then its derivative

$$f'(a) := \left.\frac{\partial f}{\partial z}\right|_{z=a} \in \mathbb{C}$$

at a is defined and we may compute it in the usual way — as a partial derivative with respect to z. All the usual identities hold:

$$\frac{\partial}{\partial z} z^n = n z^{n-1} \qquad \text{(if } n < 0 \text{ then for } z \neq 0\text{).}$$

$$\text{For any fixed } \lambda \in \mathbb{C}, \quad \frac{\partial}{\partial z} e^{\lambda z} = \lambda\, e^z$$

and hence

$$\frac{\partial}{\partial z}\cos z = -\sin z, \qquad \frac{\partial}{\partial z}\sin z = \cos z,$$
$$\frac{\partial}{\partial z}\cosh z = \sinh z, \qquad \frac{\partial}{\partial z}\sinh z = \cosh z.$$

'Hence' because all the usual properties of (partial) differentiation hold: if f, g : $\mathbb{C} \to \mathbb{C}$ are holomorphic at $z \in \mathbb{C}$ then

$$(f+g)'(z) = f'(z) + g'(z), \quad (\lambda f)'(z) = \lambda f'(z),$$
$$(fg)'(z) = f(z)g'(z) + f'(z)g(z), \quad (f \circ g)'(z) = g'(z)\, f'(g(z)),$$

and also $(f(z)/g(z))' = (g(z)f'(z) - f(z)g'(z))/(g^2(z))$ provided $g(z) \neq 0$.

A more mathematically rigorous definition of holomorphic: *Let* $a \in \mathbb{C}$ *and let* $\varepsilon > 0$ *be a positive real number. A function is holomorphic at* $a \in \mathbb{C}$ *if its partial derivatives are continuous and there exists an* $\varepsilon > 0$ *such that there is a power series expansion*

$$f(z) = \sum_{n=0}^{\infty} c_n\,(z-a)^n \quad \text{valied for all } z \text{ with } |z-a| < \varepsilon. \tag{7.7}$$

Thus the expansion must hold for all z *in an 'open disc' of radius* ε *centred at* a, *that is, for the set of points which have distance less than* ε *from* a.

In fact, when this holds it is just the complex 'Taylor series' expansion: the coefficients are given by

$$c_n = \frac{1}{n!}\, f^{(n)}(a), \qquad \text{where } f^{(n)}(a) := \left.\frac{\partial^n f}{\partial z^z}\right|_{z=a}. \tag{7.8}$$

For example, the exponential is can be expanded around $a = 0$ into the Taylor power series

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \tag{7.9}$$

This particular expansion, in fact, holds for all z, i.e. we can take ε arbitrarily large. We can likewise compute its expansion (7.7) around any other $a \in \mathbb{C}$ to see

$$e^z \;=\; e^a e^{z-a} \;\overset{(7.9)}{=}\; e^a \cdot \sum_{n=0}^{\infty} \frac{1}{n!} (z-a)^n \;=\; \sum_{n=0}^{\infty} \underbrace{\frac{e^a}{n!}}_{=\,c_n} (z-a)^n. \tag{7.10}$$

As another example,

$$f(z) := \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \text{valied for all } z \text{ with } |z| < 1. \tag{7.11}$$

That is, (7.7) holds for $f(z) = 1/(1-z)$ at $a = 0$ with $\varepsilon = 1$; that is, the expansion is valid for all z which distance less than 1 from the origin.

We can immediately deduce from (7.11) that there is a power series expansion around any $a \in \mathbb{C}\backslash\{1\}$ — indeed, as **you** can see (7.11) implies the power series expansion

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{1}{(1-a)^{n+1}} (z-a)^n \quad \text{valied for all } |z-a| < |1-a|, \tag{7.12}$$

that is, valid provided z is closer to a that a is to 1.

Of course, it is 'obvious' from (7.4) that $f(z) = \frac{1}{1-z}$ is holomorphic everywhere except at $z = 1$ because it is the quotient of two functions (1 and $1-z$) which really are obviously holomorphic everywhere, so the only points where f will fail to be holomorphic are where the denominator has zeroes, i.e. at $z = 1$.

7.2 The Cauchy-Riemann equations.

Recall that $\phi : \mathbb{C} \to \mathbb{C}$ is said to be *holomorphic* at $a \in \mathbb{C}$ means that ϕ has an expansion valid for z sufficiently near to a

$$\phi(z) \;=\; \beta_0 \;+\; \beta_1 (z-a) \;+\; \beta_2 (z-a)^2 \;+\; \beta_3 (z-a)^3 \;+\; \cdots \tag{7.13}$$

in *positive* powers of $z - a$.

Another characterization of holomorphic is as follows:

Suppose that

$$f : \mathbb{C} \to \mathbb{C}, \quad z = x + iy \mapsto f(z) = u(x, y) + iv(x, y),$$

is differentiable at $a \in \mathbb{C}$ — meaning that the partial derivatives $u_x = \partial u/\partial x$, u_y, v_x v_y exist and are continuous. Then f is holomorphic at $a \in \mathbb{C}$ if and only if in some small enough disc $D_\varepsilon(a)$ centred at $a \in \mathbb{C}$ one has

$$\frac{\partial f}{\partial \overline{z}} = 0. \tag{7.14}$$

An equivalent way to state (7.14) is

$$u_x = v_y \quad \text{and} \quad u_y = -v_x. \tag{7.15}$$

So (7.14) says that f is independent of $\overline{z}$ in D_ε, which, since it is differentiable and hence has no singularities, is what we said holomorphic intuitively means.

Example: *Find, in terms of $z = x + iy$, the most general holomorphic function whose real part is $e^x \sin y$.*

Solution: *Set $f(z) = u + iv$ with $u = e^x \sin y$. The Cauchy-Riemann equations state that*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

which gives

$$e^x \sin y = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -e^x \cos y.$$

The first of these equations gives $v = -e^x \cos y + f_1(x)$ and substituting this in the second gives

$$-e^x \cos y + f_1'(x) = -e^x \cos y, \qquad f_1(x) = C, \quad C \in \mathbb{R}.$$

Hence

$$f(z) = e^x \sin y + i(-e^x \cos y + C) = -ie^z + iC, \quad C \in \mathbb{R}.$$

Example: *Show that if a holomorphic function has constant real part, then the function is constant*

Solution: *With $z = x + iy$ we have $f(z) = c + iv(x, y)$ for some real constant c. The Cauchy-Riemann equations therefore imply that*

$$\frac{\partial v}{\partial y} = 0, \qquad \frac{\partial v}{\partial x} = 0,$$

which says that v is independent of both x and y. Hence $v(x, y) = c'$ is a real constant and thus $f(z) = c + ic'$ is likewise constant.

7.3 Integration along a contour

We begin by noting that if $g : [a, b] \to \mathbb{C}$ is a continuous complex valued function such that $g(t) = g_1(t) + ig_2(t)$, where g_1, g_2 are real valued on $[a, b]$, then

$$\int_a^b g(t)\,dt = \int_a^b g_1(t)\,dt + i\int_a^b g_2(t)\,dt.$$

Note that it is immediate from the definition that

$$\int_a^b g'(t)\,dt = g(b) - g(a) \tag{7.16}$$

since we know this holds for the real valued functions g_1 and g_2.

Now let $\gamma : [a, b] \to \mathbb{C}$ be a smooth curve and suppose that $f : \mathbb{C} \to \mathbb{C}$ is a function which is continuous in a region containing the path traced out by γ. We wish to define the integral of f along the curve γ,

$$\int_\gamma f(z)\,dz$$

A natural way to proceed is to partition $[a, b]$ as above by points $t_0, t_1, \ldots t_{n-1}, t_n$ such that $a = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = b$. The points $z_j = \gamma(t_j)$ define a polygon with vertices at $z_0, z_1, \ldots, z_n$. We may form the sum

$$\sum_{j=1}^{n} f(t_j)(z_j - z_{j-1}) = \sum_{j=1}^{n} f(t_j)\,\underbrace{\frac{\gamma(t_j) - \gamma(t_{j-1})}{t_j - t_{j-1}}}_{\gamma'(t_j)\ +\ o(t)}\,\underbrace{(t_j - t_{j-1})}_{\delta t_j}.$$

(Note, f could be evaluated at any point $s_j \in [t_{j-1}, t_j]$. As we take finer and finer partitions we can take the limit of these sums and as the length of the longest interval tends to zero, it tends to $\int_a^b f(\gamma(t))\gamma'(t)\,dt$.

We will take this integral as our *definition* of $\int_\gamma f(z)\,dz$. To be precise:

Definition 7.1 *Let $\gamma : [a,b] \to \mathbb{C}$ be a smooth curve and suppose that f is a function which is continuous in a region containing the path of γ. Then*

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

It will be useful to note the following basic properties: Let γ be a contour. For constants $\alpha, \beta \in \mathbb{C}$

- $$\int_\gamma (\alpha f(z) + \beta g(z))\,dz = \alpha \int_\gamma f(z)\,dz + \beta \int_\gamma g(z)\,dz, \tag{7.17}$$

- $$\int_{\widetilde{\gamma}} f(z)\,dz = -\int_\gamma f(z)\,dz, \tag{7.18}$$

  where $\widetilde{\gamma}$ is the reverse curve to γ

- Let
  $$\gamma : [a,b] \longrightarrow \mathbb{C},\quad t \mapsto \gamma(t), \qquad \mu : [c,d] \longrightarrow \mathbb{C},\quad s \mapsto \mu(s),$$
  be two parametrizations of a path $C \subset \mathbb{C}$. Then if they both have the same direction
  $$\int_\gamma f(z)\,dz = \int_\mu f(z)\;dz. \tag{7.19}$$
  That is, $\int_C f(z)\,dz$ is independent of the choice of parametrization (up to sign).

7.3.1 Example

Evaluate $\int_\gamma \overline{z}\,dz$, where $\gamma = \gamma_1 + \gamma_2$, γ_1 being the line segment from 1 to 0 and γ_2 being the line segment from 0 to $2 + 2i$.

In this case the contour consists of two lines and we need to parametrize them separately by, for example,

$$\gamma_1 : [0,1] \to \mathbb{C}, \qquad \gamma_1(t) = 1 - t,$$

and

$$\gamma_2 : [0,1] \to \mathbb{C}, \qquad \gamma_2(t) = t(2 + 2i)$$

So

$$\int_C \overline{z}\ dz = \int_{\gamma_1} \overline{z}\ dz\ +\ \int_{\gamma_2} \overline{z}\ dz.$$

We have in each case

$$\int_{\gamma_j} \overline{z}\ dz = \int_0^1 \overline{\gamma_j(t)}\, \gamma_j'(t)\ dt$$

from which we find $\int_{\gamma_1} \overline{z}\ dz = -1/2$ and $\int_{\gamma_2} \overline{z}\ dz = 4$, so that

$$\int_\gamma \overline{z}\ dz = \frac{7}{2}.$$

7.3.2 Example

Evaluate $\int_C z^2\ dz$, where $C = C_1 + C_2$ with C_1 the line segment from -2 to 2 on the real axis, and C_2 the semi-circle of radius 2 and centre 0 in the upper-half plane from 2 to -2.

The contour consists of two pieces which we need to parametrize separately. Parametrize C_1 by, for example, $\gamma_1 : [-2, 2] \to \mathbb{C},\ \gamma_1(t) = t$, and C_2 by, for example, $\gamma_2 : [0, \pi] \to \mathbb{C},\ \gamma_2(t) = 2e^{it}$. So $\int_C z^2\ dz = 0$. Make sure you can see why.

7.3.3 A particularly important Example

This example is one of the key elements that goes into the Cauchy Residue theorem.

Let $C(a, r)$ be the circle of radius $r > 0$ with centre at $a \in \mathbb{C}$. Then

$$\int_{C(a,r)} \frac{1}{(z-a)^n}\ dz \quad = \quad \begin{cases} 0 & n \neq 1 \\ 2\pi i & n = 1. \end{cases} \tag{7.20}$$

Let us emphasize

$$\frac{1}{2\pi i} \int_{C(a,r)} \frac{1}{(z-a)}\ dz \quad = \quad 1 \tag{7.21}$$

for *any* $r > 0$.

7.4 Some fundamental theorems

In practise, we evaluate real integrals analytically by 'reverse differentiation', using the FTC for real integrals. A similar result is true for certain types of contour integration:

Theorem 7.2 *Let $f : \mathbb{C} \to \mathbb{C}$ be a complex function, and let C be a contour beginning at $p \in \mathbb{C}$ and ending at $q \in \mathbb{C}$. If $f = F'$ is the derivative of a function F which is holomorphic at each point of C (recall here $F'(z) := \partial F/\partial z$) then*

$$\int_\gamma f(z)\, dz = F\big(q\big) - F\big(p\big). \tag{7.22}$$

In particular, if C is a closed contour and $f = F'$ then

$$\int_C f(z)\, dz = 0. \tag{7.23}$$

(Recall, a 'closed contour' is a contour which looks something like a circle, or a loop of string, it has no end points.)

The FTC can greatly simplify the evaluation and analysis of contour integrals.

Example: As a simple example, let $f(z) = (z - a)^m$, where m is an integer. If C is a smooth curve which does not pass through $a \in C$ and which starts at p and ends at q then

$$\int_C (z-a)^m\, dz = \frac{1}{m+1}\big[q^{m+1} - p^{m+1}\big] \quad \text{provided } m \neq -1 \tag{7.24}$$

On the other hand, when $m = -1$ and C_{closed} is a simple (no self-crossings) closed contour then

$$\int_{C_{\text{closed}}} (z-a)^{-1}\, dz = \begin{cases} 2\pi i, & \text{if a is inside the contour} \\ 0, & \text{if a is outside the contour.} \end{cases} \tag{7.25}$$

We can write (7.25) for the case of a circle $C(a, r)$ centred at a of radius $r > 0$ as

$$\int_{C(a,r)} (z-a)^{-1}\, dz = 2\pi i. \tag{7.26}$$

Exercise: How about $\int_{C(b,r)} (z - a)^{-1}\, dz$ where $b \neq a$? (Your answer will depend on $|a - b|$).

In fact, (7.25) follows from the simpler fact (7.26) because of the following fundamental theorem:

Theorem 7.3 *(Cauchy's Theorem) Let $f : \mathbb{C} \to \mathbb{C}$ be a complex function, and let $C_{\text{closed}} \subset \mathbb{C}$ be a simple closed contour. Then if f is holomorphic along and at all points inside C_{closed}, then*

$$\int_{C_{\text{closed}}} f(z)\, dz = 0. \tag{7.27}$$

The proof of this theorem is given in full detail in the course 6CCM322A Complex Analysis which is a 3rd year mathematics option. You need to know the statement of the above theorems, but not the proofs. Cauchy's Theorem 7.3 combined with (7.26) yields the Cauchy Residue theorem (CRT).

7.5 Poles and residues

If $\phi : \mathbb{C} \to \mathbb{C}$, on the other hand, if $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto f(z)$, has an *isolated singularity* at $a \in \mathbb{C}$ (meaning that f is not holomorphic at a but it is holomorphic at those points $z \in \mathbb{C}$ with $0 < |z-a| < \epsilon$ some $\epsilon > 0$) and the singularity at a looks like

$$f(z) = \frac{\beta_{-m}}{(z-a)^m} \ + \ \ldots \ + \ \frac{\beta_{-1}}{(z-a)} \ + \phi(z), \quad 0 < |z-a| < \epsilon, \tag{7.28}$$

with ϕ holomorphic for $|z-a| < \epsilon$, then f is said to have a **pole of order m at a**. Note that the complex numbers occuring in (7.28)

$$\beta_{-r} = \beta_{-r}(a)$$

will depend on the point $a \in \mathbb{C}$. If (7.28) holds with $m = 1$, i.e. if for some $\varepsilon > 0$

$$f(z) = \frac{\beta_{-1}(a)}{(z-a)} \ + \phi(z), \quad 0 < |z-a| < \epsilon, \tag{7.29}$$

with ϕ holomorphic at $a \in \mathbb{C}$, then f is said to have a **simple pole at a**.

If f has a pole at a (of some order) then the **residue of f at a** is defined by

$$\operatorname{res}(f, a) \ = \ \beta_{-1}(a). \tag{7.30}$$

(7.28) can be equivalently written

$$f(a+h) = \frac{\beta_{-m}(a)}{h^m} \ + \ \ldots \ + \ \frac{\beta_{-1}(a)}{h} \ + \phi(a+h), \quad 0 < |h| < \epsilon, \tag{7.31}$$

which can sometimes be easier for computing (7.30); note also that (**??**) may be similarly written

$$\phi(a+h) = \sum_{n \geq 0} \beta_n(a)\, h^n, \qquad \beta_n(a) = \frac{\phi^{(n)}(a)}{n!}, \qquad \text{for all `}small\text{' } h. \tag{7.32}$$

7.6 Computing residues

In order to use the Cauchy Residue Theorem effectively we need to have some methods for computing residues. For particularly simple functions one can do this directly, but more generally it is usually easier to resort to one of the following formulae for computing residues.

Suppose that

$$f(z) = \frac{\phi(z)}{z-a} \quad \textit{where } \phi \textit{ is holomorphic at } a. \tag{7.33}$$

Then

$$\mathrm{res}(f,a) \;=\; \phi(a). \tag{7.34}$$

Suppose

$$f(z) = \frac{\phi(z)}{(z-a)^m}, \quad \phi \;\; \textit{holomorphic at } a, \tag{7.35}$$

then

$$\mathrm{res}(f,a) \;=\; \frac{\phi^{(m-1)}(a)}{(m-1)!}. \tag{7.36}$$

Suppose that

$$f(z) \;=\; \frac{\phi(z)}{\psi(z)} \tag{7.37}$$

with ϕ and ψ holomorphic at $a \in \mathbb{C}$, and that

$$\psi(a) = 0 \quad \text{and} \quad \psi^{'}(a) \neq 0.$$

Then f has a simple *pole at $z = a$ and*

$$\mathrm{res}(f,a) \;=\; \frac{\phi(a)}{\psi'(a)}. \tag{7.38}$$

Example

$$f(z) = \frac{1}{z^4+1}$$

has poles at

$$e^{\frac{\pi}{4}i}, \quad e^{\frac{3\pi}{4}i}, \quad e^{\frac{5\pi}{4}i}, \quad e^{\frac{7\pi}{4}i}.$$

We have $\psi'(e^{\frac{(2k+1)}{4}\pi i}) = 4e^{\frac{3}{4}(2k+1)\pi i} \neq 0$, so we can apply the formula to get

$$\mathrm{res}\left(\frac{1}{z^4+1}, e^{\frac{(2k+1)}{4}\pi i}\right) \;=\; \frac{1}{4}\, e^{-\frac{3}{4}(2k+1)\pi i}. \tag{7.39}$$

For instance,

$$\mathrm{res}\left(\frac{1}{z^4+1}, e^{\frac{\pi}{4}i}\right) \;=\; \frac{1}{4}\, e^{-\frac{3}{4}\pi i} = -\frac{1}{4\sqrt{2}}(1+i). \tag{7.40}$$

7.7 Evaluating integrals with the CRT

The Cauchy Residue Theorem:

Let $C \subset \mathbb{C}$ be a simple closed contour. Let $f : \mathbb{C} \to \mathbb{C}$ be a complex function which is holomorphic along C and inside C except possibly at a finite number of points $a_1, \ldots, a_m$ at which points f has poles. Then, with C oriented in an anti-clockwise sense,

$$\int_C f(z)\, dz \;=\; 2\pi i \sum_{k=1}^{m} \operatorname{res}(f, a_k), \tag{7.41}$$

where $\operatorname{res}(f, a_k)$ is the residue of the function f at the point $a_k \in \mathbb{C}$. If f has no poles inside C (f is holomorphic inside C) then $\int_C f(z)\, dz = 0$.

7.8 The right-hand side of the Cauchy Residue Formula (7.47): adding-up residues

If $\phi(z)$ is analytic at a, then from (7.47) and formula (15.9) of online lecture 15 we have

$$\begin{aligned} \int_C \frac{\phi(z)}{z-a}\, dz \;&=\; 2\pi i \operatorname{res}\left(\frac{\phi(z)}{z-a}, a\right) \\ &=\; \begin{cases} 2\pi i\, \phi(a), & \text{if } a \text{ is inside } C \\ 0, & \text{if } a \text{ is outside } C. \end{cases} \end{aligned} \tag{7.42}$$

Let

$$C(a, R) \;=\; \text{circle centred at } a \in \mathbb{C} \text{ and with radius } R > 0.$$

7.9 Example

the functions e^z and z are holomorphic everywhere the only pole of $f(z) = \frac{e^z}{z}$ is at $z = 0$ with, by (7.42), residue equal to $e^0 = 1$. Hence

$$\int_{C(0,1)} \frac{e^z}{z}\, dz = 2\pi i. \tag{7.43}$$

7.10 Example

To evaluate

$$\int_\gamma \frac{e^z}{(z-1)(z-3)}\,dz,$$

taken round the circle γ given by $|z| = 2$ in the positive (anti-clockwise) sense, we have $f(z) = \frac{\phi(z)}{z-a}$ with $\phi(z) = \frac{e^z/(z-3)}{z-1}$ holomorphic everywhere *inside* the curve $|z| = 2$ — and, in particular, at $z = 1$. Hence from (7.42) the integral equals $2\pi i\,\phi(1) = -\pi i\,e$. Around $|z| = 1/2$, on the other hand, the CIF says it evaluates to zero

Likewise, from (7.47) we can compute

$$\begin{aligned}\int_C \frac{\phi(z)}{(z-a)^m}\,dz &= 2\pi i \operatorname{res}\left(\frac{\phi(z)}{(z-a)^m}, a\right)\\ &= \begin{cases} 2\pi i\,\frac{\phi^{(m-1)}(a)}{(m-1)!}, & \text{if } a \text{ is inside } C\\ 0, & \text{if } a \text{ is outside } C.\end{cases}\end{aligned}$$

7.11 Example

Applying this, we have

$$\int_C \frac{\sin(3z)}{(z-2i)^2}\,dz = 6\pi i \cosh 6. \tag{7.44}$$

On the other hand, if g analytic at $a \in \mathbb{C}$ while

$$\psi(a) = 0 \quad \text{and} \quad \psi^{'}(a) \neq 0,$$

from (7.47) we infer that if a is *inside* the contour C then

$$\int_C \frac{g(z)}{\psi(z)}\,dz = 2\pi i\,\frac{g(a)}{\psi'(a)}. \tag{7.45}$$

If there are two points $a_1, a_2 \in \mathbb{C}$ which are *inside* the contour C and which are poles of this type, then

$$\int_C \frac{g(z)}{\psi(z)}\,dz = 2\pi i\,\frac{g(a_1)}{\psi'(a_1)} + 2\pi i\,\frac{g(a_2)}{\psi'(a_2)}, \tag{7.46}$$

and so on.

7.12 Example

To evaluate

$$\int_{C(0,5/2)} \cot(\pi z)\, dz.$$

we saw earlier that $\cot(\pi z)$ has a simple pole at each integer $n \in \mathbb{Z}$ with residue $\frac{1}{\pi}$ (independent of n). Hence, as there are five poles inside the contour, we have

$$\int_{C(0,5/2)} \cot(\pi z)\, dz = 10\, i.$$

7.13 Evaluating real integrals using the CRT

The Cauchy Residue Theorem states:

Let $C \subset \mathbb{C}$ be a simple closed contour. Let $f : \mathbb{C} \to \mathbb{C}$ be a complex function which is holomorphic along C and inside C except possibly at a finite number of points $a_1, \ldots, a_m$ at which points f has poles. Then, with C oriented in an anti-clockwise sense,

$$\int_C f(z)\, dz \ = \ 2\pi i \sum_{k=1}^{m} \operatorname{res}(f, a_k), \tag{7.47}$$

where $\operatorname{res}(f, a_k)$ is the residue of the function f at the point $a_k \in \mathbb{C}$.

This provides us with a powerful method of computing real integrals which would be impossible to evaluate using standard real integration techniques on $\mathbb{R}$.

In fact, every time we evaluate a contour integral using (7.47) we evaluate two real integrals. For, if $\gamma : [a, b] \to C \subset \mathbb{C}$ is a parametrisation of C then

$$\int_C f(z)\, dz \ = \ \int_a^b \alpha(t)\ dt \ + \ i \int_a^b \beta(t)\ dt.$$

On the other hand, we can also write the right-hand side $2\pi i \sum_{k=1}^{m} \operatorname{res}(f, a_k)$ of (7.47) as

$$2\pi i \sum_{k=1}^{m} \operatorname{res}(f, a_k) = A + iB \quad \text{for some real numbers } A, B \in \mathbb{R}, \tag{7.48}$$

and hence we obtain the evaluations $\int_a^b \alpha(t)\ dt \ = \ A$ and $\int_a^b \beta(t)\ dt \ = \ B$.

7.13.1 Example:

We computed in Example (16.1) that

$$\int_{C(0,1)} \frac{e^z}{z}\, dz = 2\pi i. \tag{7.49}$$

Using the parametrization $\gamma(\theta) = e^{i\theta},\ 0 \le \theta \le 2\pi$, of $C(0,1)$ we have

$$\int_C \frac{e^z}{z}\ dz = i \int_0^{2\pi} e^{\cos\theta}\ \left(\cos(\sin\theta) + i\sin(\sin\theta)\right)\ d\theta.$$

Equating this with (7.49) gives

$$\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta)\, d\theta = 2\pi, \qquad \int_0^{2\pi} e^{\cos\theta} \sin(\sin\theta)\, d\theta = 0.$$

7.13.2 Exercise:

By considering the contour integral $\int_{C(0,1)} \frac{1}{z^2+4z+1}\, dz$ show that

$$\int_0^{2\pi} \frac{1}{2+\cos(t)}\ dt = \frac{2\pi}{\sqrt{3}}.$$

7.14 Evaluating integrals $\int_{-\infty}^{\infty} f(x)\ dx$ over the whole real line

This will provide us with a method of evaluating Fourier transforms and hence obtaining explicit solutions to a given ODE or PDE.

The idea is to compute $\int_{C_R} f(z)\ dz$ over a simple closed contour, typically, of the form

$$C_R = [-R, R] \cup A_R$$

where f is holomorphic on and inside C_R except possibly at poles $a_1, \ldots, a_{m_R}$ inside C_R, and then allow $R \to \infty$. Since

$$\begin{aligned} \int_{C_R} f(z)\ dz \ &= \ \int_{[-R,R]} f(z)\ dz \ + \ \int_{A_R} f(z)\ dz \\ &= \ \int_{-R}^{R} f(x)\ dx \ + \ \int_{A_R} f(z)\ dz, \end{aligned}$$

the aim is to choose A_R so that

$$\lim_{R\to\infty} \int_{A_R} f(z)\ dz = 0 \tag{7.50}$$

and hence that by the Cauchy Residue Theorem infer that

$$\int_{-\infty}^{\infty} f(x)\ dx = \lim_{R\to\infty} 2\pi i \sum_{k=1}^{m_R} \operatorname{res}(f, a_k). \tag{7.51}$$

Note that the right-hand side must in this case be a real number. This method depends on the convergence of all the limits and the existence of the integrals and so forth — but it works for large classes of functions.

7.14.1 Estimating contour integrals

The following fact is useful in showing properties like (7.50).

We know that for continuous real valued functions $f : [a, b] \to \mathbb{R}$ that

$$\left|\int_a^b f(x)\, dx\right| \leq \int_a^b |f(x)|\, dx.$$

This extends to complex integrals in the following way.

Theorem 7.4 *Let C be a contour in $\mathbb{C}$. Then*

$$\left|\int_C f(z)\, dz\right| \leq \int_a^b |f\big(\gamma(t)\big)| \cdot |\gamma'(t)|\ dt, \tag{7.52}$$

where $\gamma : [a, b] \to C \subset \mathbb{C}$ is a parametrisation of C.

Note that if $|f(z)| \leq M$, $\forall\, z \in C$ (f is bounded by M along C, this implies $|\int_C f(z)\, dz| \leq ML_C$, where L_C is the length of C.

7.14.2 Example:

By considering the contour integral

$$\int_{C_R} \frac{e^{iz}}{z - ia}\, dz,$$

where $C_R = [-R, R] \cup A_R$ and A_R is the semicircle centre at 0 and radius $R > a$ in the upper-half plane, one has

$$\int_{-\infty}^{\infty} \frac{x \sin x + a \cos x}{x^2 + a^2}\, dx = 2\pi e^{-a}$$

and

$$\int_{-\infty}^{\infty} \frac{x \cos x - a \sin x}{x^2 + a^2}\, dx = 0.$$

To see this, we use the CRT as indicated above, and show that $\int_{A_R} f(z)\,dz \to 0$ as $R \to \infty$. For that use (7.52). From the parametrization $\gamma_2(x) = Re^{ix}$ of A_R and Theorem 7.4 we have

$$\left|\int_{A_R} f(z)\,dz\right| = \left|\int_0^\pi \frac{e^{i[R\cos\theta + iR\sin\theta]}Rie^{i\theta}\,d\theta}{Re^{i\theta} - ia}\right| \le \int_0^\pi \frac{Re^{-R\sin\theta}\,d\theta}{|Re^{i\theta} - ia|}$$

$$\le \int_0^\pi \frac{R}{R-a} e^{-R\sin\theta}\,d\theta = \frac{2R}{R-a}\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta.$$

— and since $\sin\theta \ge 2\theta/\pi \ \forall\theta \in [0, \pi/2]$ —

$$\le \frac{2R}{R-a}\int_0^{\pi/2} e^{-2R\theta/\pi}\,d\theta = \frac{\pi}{R-a}(1 - e^{-R}) \to 0 \text{ as } \ R \to \infty.$$

8 An application to ordinary differential equations

We are now going to use a contour integral to help solve an ODE. This principle is, in fact, very general and can be applied to partial differential equations on $\mathbb{R}^m$ (recall that ODEs refer to differential equations on $\mathbb{R}^1$). As a warm up to this you are encouraged to re-read the section "ODEs and the FT".

8.1 The ODE

Here, we are going to see how to find the general solution to the ODE

$$\frac{d^4 f}{dx^4} + f = g \tag{8.1}$$

provided that

$$\frac{d^k f}{dx^k} \to 0 \quad \text{as } |x| \to \infty \text{ for } k = 0, 1, 2, 3. \tag{8.2}$$

The function $g : \mathbb{R}^1 \to \mathbb{R}^1$ is given, and the objective is to determine $f : \mathbb{R}^1 \to \mathbb{R}^1$.

8.2 Applying Fourier transform

The assumption (8.2) means that we can apply the Fourier transform to the ODE (8.1) to obtain

$$f(x) = \int_{-\infty}^{\infty} k(x, y)\, g(y)\ dy$$

where

$$k(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i(y-x)\xi}}{\xi^4+1}\, d\xi. \tag{8.3}$$

It remains, then, to evaluate the integral on the right-side of (8.3).

8.3 Using the Cauchy residue theorem

To evaluate this real integral previous examples suggest that we may get somewhere by evaluating the contour integral

$$I(\alpha) = \int_{C_R} \frac{e^{i\alpha z}}{z^4+1}\, dz$$

around the contour C_R in the z-plane which consists of the portion of the real axis from $-R$ to R, together with the semi-circular arc γ_R parametrized by $\gamma_R(\theta) = Re^{i\theta},\ \ 0 \le \theta \le \pi$.
We assume that

$$\alpha \in \mathbb{R}^1. \tag{8.4}$$

For $R > 1$ $f(z) = \frac{e^{i\alpha z}}{z^4+1}$ has two poles inside C_R at $z_0 = e^{i\pi/4}$ and $z_1 = e^{i\,3\pi/4}$. The Cauchy residue theorem says that

$$\begin{aligned} I(\alpha) &= 2\pi i(\operatorname{res}(f,z_0) + \operatorname{res}(f,z_1)), \qquad (8.5)\\ &= \pi\, e^{-\frac{\alpha}{\sqrt{2}}}\, \sin\left(\frac{\alpha}{\sqrt{2}} + \frac{\pi}{4}\right), \end{aligned}$$

using residue formulae.

On the other hand,

$$I(\alpha) = \int_{-R}^{R} \frac{e^{i\alpha\xi}}{\xi^4+1}\, d\xi + \int_{A_R} \frac{e^{i\alpha z}}{z^4+1}\, dz$$

with A_R the arc component of the contour. But using the estimate for contour integrals, described in the previous section, we find

$$\left|\int_{A_R} \frac{e^{i\alpha z}}{z^4+1}\, dz\right| = \left|\int_0^{\pi} \frac{e^{i\alpha Re^{i\theta}}}{R^4 e^{i4\theta}+1}\, Re^{i\theta}\, d\theta\right| \le \frac{R\pi}{R^4-1} \ \to 0 \quad \text{as } R\to\infty \ \text{ if } \alpha \ge 0.$$

We are then left with

$$\lim_{R\to\infty}\int_{-R}^{R} \frac{e^{i\alpha\xi}}{\xi^4+1}\, d\xi := \int_{-\infty}^{\infty} \frac{e^{i\alpha\xi}}{\xi^4+1}\, d\xi = \pi\, e^{-\frac{\alpha}{\sqrt{2}}}\, \sin\left(\frac{\alpha}{\sqrt{2}} + \frac{\pi}{4}\right).$$

That is,

$$\text{provided } \alpha \geq 0 \qquad \int_{-\infty}^{\infty} \frac{e^{i\alpha\xi}}{\xi^4+1}\, d\xi = \pi\, e^{\frac{-\alpha}{\sqrt{2}}} \sin\left(\frac{\alpha}{\sqrt{2}} + \frac{\pi}{4}\right). \tag{8.6}$$

From (8.3) we want to apply this with $\alpha = y - x$. But $y - x$ can take on any real value, so we need to extend (8.8) to the case $\alpha \leq 0$. To do that we make use of the fact that we have only used two of the four poles of f. So we now evaluate

$$I_{D_R}(\alpha) = \int_{D_R} \frac{e^{i\alpha z}}{z^4+1}\, dz$$

around the contour D_R in the z-plane which consists of the portion of the real axis from R to $-R$, together with the semi-circular arc γ_R parametrized by $\gamma_R(\theta) = Re^{i\theta}, \quad \pi \leq \theta \leq 2\pi$, (note this is positively oriented (anticlockwise) which means the direction along the $[-R, R]]$ is reversed!). We now assume that

$$\alpha \leq 0. \tag{8.7}$$

We now repeat the above process with D_R instead of C_R to find that

$$\text{provided } \alpha \leq 0 \qquad \int_{-\infty}^{\infty} \frac{e^{i\alpha\xi}}{\xi^4+1}\, d\xi = \pi\, e^{\frac{\alpha}{\sqrt{2}}} \sin\left(\frac{-\alpha}{\sqrt{2}} + \frac{\pi}{4}\right). \tag{8.8}$$

Hence:

$$\text{For any } \alpha \in \mathbb{R}^1 \qquad \int_{-\infty}^{\infty} \frac{e^{i\alpha\xi}}{\xi^4+1}\, d\xi = \pi\, e^{\frac{-|\alpha|}{\sqrt{2}}} \sin\left(\frac{|\alpha|}{\sqrt{2}} + \frac{\pi}{4}\right). \tag{8.9}$$

We now obtain from (8.3) and (8.9) that the general solution to (8.1) is given by

$$f(x) = \int_{-\infty}^{\infty} k(x, y)\, g(y)\, dy$$

with

$$k(x, y) = \frac{1}{2}\, e^{-\frac{|x-y|}{\sqrt{2}}} \sin\left(\frac{|x-y|}{\sqrt{2}} + \frac{\pi}{4}\right).$$