Proof $f'(z) = 0$ implies $f(z)$ locally constant.

I am currently reading "Theory of Complex Functions" by Remmert and I encountered this theorem:

> Let $f : D \to \mathbb{C}$ be a complex holomorphic function. If $f'(z) = 0, \forall z \in D$, then f is locally constant in D.

Here is part of the proof:

> Consider any open ball $B = B_r(b) \subset D$ and any $z \in B$. Let L denote the line segment from b

> $\delta = \delta(c) > 0$, such that $|f(w) - f(c)| < \epsilon|w - c|, \forall w \in B_\delta(c)$.
>
> Because finitely many discs $B_\delta(c)$ suffice to cover the line L, there is a succession of points $z_0 = b, z_1, \cdots, z_n = z$ on L such that $|f(z_i) - f(z_{i-1})| < \epsilon|z_i - z_{i-1}|, 1 \leq i \leq n$." $\cdots$

I understand why finitely many discs cover L since it is compact. My question is how do you prove that there is a succession of points $z_0, \cdots, z_n$ such that $|f(z_i) - f(z_{i-1})| < \epsilon|z_i - z_{i-1}|$? The statements seems intuitively obvious but I can't provide the rigorous argument.

complex-analysis

edited May 2 '17 at 14:37
Nosrati
23.3k 6 19 52

asked May 2 '17 at 1:57
KnobbyWan
167 1 10

That is kind of a silly proof. The FTC of one variable calculus will give the result almost immediately. – zhw. May 3 '17 at 21:57

In the context of a textbook, I can well imagine that the proof of this theorem would precede a proof of the FTC. In the context of overall simplicity of concepts, applying the FTC of

Lee Mosher May 5 '17 at 13:04

@LeeMosher This is complex analysis. Why do you say applying the fundamental theorem of calculus, a very basic freshman level result, is like using a sledge hammer? – zhw. May 5 '17 at 15:32

Perhaps we are talking about different levels of simplicity? Although you are correct that most students see the fundamental theorem of calculus long before they see complex holomorphic functions, my point is that in a textbook which develops complex analysis from first principles, e.g. the textbook in the question above, it is simpler to introduce covering properties as in this question before one gives the proof of the fundamental theorem of calculus. – Lee Mosher May 5 '17 at 15:39

@LeeMosher i've never seen a textbook that does this. This would be a textbook, for example, that would need to define what a partial derivative is in order to proceed to the CR equations, that would need to redo the product rule, the chain rule, etc., and would need to restart the very definition of the integral for continuous functions. – zhw. May 6 '17 at 1:42

2 Answers

Lebesgue number lemma, a standard tool for these kinds of arguments.

By continuity it follows that for each $c \in L$ there exists an open ball $B(c)$ centered on c with the property that for any $w, z \in B(c)$ we have $|f(w) - f(z)| < \epsilon|w - z|$. (My choice of $B(c)$ is more restrictive than in the text you quote).

Since L is a compact metric space, by the Lebesgue number lemma there exists $\lambda > 0$ such that for every subset $A \subset L$, if the diameter of A is less than λ then there exists c such that $A \subset B(c)$.

Choose $n > \frac{\text{Length}(L)}{\lambda}$. Let $b = z_0, z_1, \ldots, z_n = z$ be the sequence of points on L that subdivides it into subintervals of length equal to $\frac{\text{Length}(L)}{n} < \lambda$.

For each $i = 1, \ldots, n$ the set $\{z_{i-1}, z_i\}$ has diameter $< \lambda$ and so $|f(z_{i-1}) - f(z_i)| < \epsilon|z_{i-1}$ -
.

answered May 2 '17 at 14:58

Lee Mosher
46.3k 3 35 80

I understand your argument in the third, fourth, and last paragraph. However, how do you show that there is an open ball $B(c)$ such that for any

Hopefully this is good enough.

Let $B_{\delta(c_1)}(c_1), B_{\delta(c_2)}(c_2), \ldots B_\delta$ be the finite cover of L by discs. Then for each c_i, $|f(w) - f(c_i)| < \varepsilon|w - c_i|$ when $w \in B_{\delta(c_i)}(c_i)$. (This is terrible notation) for each pair c_i, c_{i+1}, we can find a $c_{i+1/2}$ which belongs to $B_{\delta(c_i)}(c_i) \cap B_{\delta(c_{i+1})}(c_{i+1})$. Set $z_1 = c_1$, $z_2 = c_{1+1/2}$ $z_3 = c_2$, etc.

Now given any z_i and z_{i-1}, one of these points is the center of a disc and the other is contained in it. WLOG suppose it is z_i that belongs to the center of a disc. Then

$$|f(z_i) - f(z_{i-1})| < \varepsilon|z_i -$$

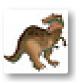
and this set $z_1, \ldots z_m$ fits the bill.

I'm sure you've drawn a picture of this situation already. Just take the centers of the discs along with a point of intersection for each pair of disc which intersect.

It seems like a "connectedness argument" from topology would work for this as well, and could be easier depending on how much you know. For this argument you would fix a point z in D and find a proof you must just show that the the set $\{w \in B_r(z) \ : \ f(w) = p\}$ is both open in closed in the subspace topology of $B_r(z)$: $B_r(z)$ is obviously connected, and the only open and closed subsets of a connected space are the empty set and $B_r(z)$ itself, in which case it would have to be $B_r(z)$ (it is nonempty since z is a member). The set $\{w \in B_r(z) \ : \ f(w) = c\}$ is the inverse image of a closed set under a continuous function, so it is closed, and so the only hard part is to show that this set is open. Depending on how much you know this might not even be hard.

edited May 2 '17 at 10:45

answered May 2 '17 at 10:39



Blake
1,106 5 16

It seems that your argument requires the intersection of $B_{\delta(c_i)}(c_i) \cap B_{\delta(c_{i+1})}(c_{i+1})$ to be non empty $\forall i$. What happens if the intersection is empty for some i? – KnobbyWan May 2 '17 at 11:47

I think it's not a big deal. Since if such i exist, then choose the least integer, say j, such that $j > i + 1$ and $B(c_i) \cap B(c_j$ is not empty. This is possible because a line is connected. Then choose provide... – KnobbyWan May 2 '17 at 12:15

@Khoria Yes. You could probably even dig into a parametrization $\gamma : [0,1] \to B$ of L and produce the sequence of points inductively. This is one of those things that you could go about very rigorously, but it would almost seem silly since it's so intuitive (not recommending against it thought, it just might drive you a little nuts.) – Blake May 2 '17 at 12:44