Solutions to Complex Analysis Prelims

Ben Strasser

In preparation for the complex analysis prelim, I typed up solutions to some old exams. This document includes complete solutions to both exams in 2013, as well as select solutions from some older exams. The problems are organized in reverse chronological order, so the most recent exams appear first. Some of these solutions are my own, others are adapted from online sources or fellow graduate students. These solutions have **not** been checked by any prelim graders.

Fall, 2013

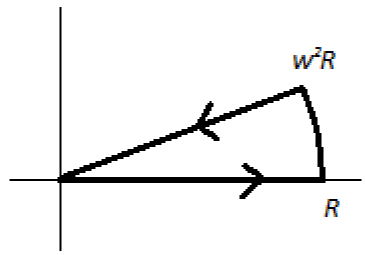
1. *Find the number of zeros of the polynomial* $z^4 + 3z^2 + z + 1$ *in the unit disk.*

   This is the classic setup of a "Rouché's theorem" problem. In its simplest form, Rouché's theorem states that if f and g are analytic functions inside a simply connected region Ω and satisfy
   $$|f(z) - g(z)| < |f(z)| + |g(z)|$$
   on $\partial\Omega$, then f and g have the same number of zeros (counting multiplicity) inside Ω.

   Let $f(z) = z^4 + 3z^2 + z + 1$ and let $g(z) = 3z^2 + 1$. Then on the boundary of the unit disk,
   $$|f(z) - g(z)| = |z^4 + z| \le 2 \le |3z^2 + 1| = |g(z)|.$$
   We are done if we can show that we cannot have both of the above inequalities be simultaneously equal. Consider the case where $|g(z)| = 2$ and $|z| = 1$. This happens exactly when $z^2 = -1$, so $z = \pm i$. Plugging these in, we see that
   $$|f(i) - g(i)| = |1 + i| = \sqrt{2} = |1 - i| = |f(-i) - g(-i)|,$$
   so if $|g(z)| = 2, |f(z) - g(z)| < 2$ on the boundary of the disk. By Rouché's theorem, we conclude that f and g have the same number of zeros inside the disk. Since g has exactly two zeros ($i/\sqrt{3}$ and $-i/\sqrt{3}$) inside the disk, we conclude that f has exactly two zeros in the disk. $\square$

2. *Let* $n \ge 2$ *be an integer. Evaluate the following integral*
   $$\int_0^\infty \frac{1}{1 + x^n} dx$$
   Carefully justify all your steps.

   Let $f(z) = \frac{1}{1+z^n}$. Let $\omega = e^{\pi i/n}$ be a primitive n^{th} root of -1. Consider the contour γ_R illustrated (poorly) below:



This contour consists of three parts, I_R along the line $[0, R] \subseteq \mathbb{R}$, C_R on the circle of radius R, and $I_{\omega^2,R}$ along the line $\omega^2[0, R]$. We orient these contours so that $\gamma_R = I_R + C_R + I_{\omega^2,R}$ is a positive contour around the wedge. f has exactly one simple pole inside γ_R at ω. Therefore, by the residue theorem, we have

$$\begin{aligned}\int_{\gamma_R} f(z)dz &= 2\pi i \mathrm{Res}_{z=\omega} f(z) \\ &= 2\pi i \frac{1}{n\omega^{n-1}} = -\frac{2\pi i \omega}{n}.\end{aligned}$$

We now show that the integral of f along C_R vanishes as $R \to \infty$. By the triangle inequality

$$\left|\int_{C_R} f(z)dz\right| \leq \int_{C_R} \frac{1}{|1 - R^n|}|dz| \leq \frac{2\pi R}{|1 - R^n|} \to 0 \text{ as } R \to \infty.$$

Therefore,

$$\begin{aligned}-\frac{2\pi i\omega}{n} = \lim_{R\to\infty} \int_{\gamma_R} f(z)dz &= \lim_{R\to\infty} \int_{I_R} f(z)dz + \int_{C_R} f(z)dz + \int_{I_{\omega^2,R}} f(z)dz \\ &= \lim_{R\to\infty} \int_0^R \frac{1}{1+x^n}dx + \int_R^0 \frac{1}{1+(\omega^2 x)^n}\omega^2 dx \\ \Rightarrow -\frac{2\pi i\omega}{n} &= \int_0^\infty \frac{1}{1+x^n}dx - \omega^2 \int_0^\infty \frac{1}{1+x^n}dx.\end{aligned}$$

Rearranging the above identity yields

$$\begin{aligned}\int_0^\infty \frac{1}{1+x^n}dx &= -\frac{2\pi i\omega}{n(1-\omega^2)} \\ &= -\frac{2\pi i e^{\pi i/n}}{n(1-e^{2\pi i/n})} \\ &= \frac{\pi}{n}\frac{2i}{e^{\pi i/n} - e^{-\pi i/n}} = \frac{\pi}{n\sin(\pi/n)}.\end{aligned}$$

phew. □

3. *Suppose f is analytic on $\{z : 0 < |z| < 1\}$ and $|f(z)| \leq \log(1/|z|)$. Prove that f is identically 0.*

First, we show f has a removable singularity at 0.

$$\lim_{z \to 0} |zf(z)| \leq \lim_{z \to 0} |z| \log(1/|z|) \leq \lim_{z \to 0} |z| \sqrt{1/|z|} = 0,$$

as desired. Therefore, there is a function g analytic on $D = \{|z| < 1\}$ which satisfies $g|_{D \setminus \{0\}} = f$. The condition $|g(z)| < \log(1/|z|)$ means that $|g(z)| \to 0$ as $|z| \to 1$, so by the maximum modulus principle, $g \equiv 0$ on D. Thus, $f \equiv 0$ on $D \setminus \{0\}$. $\square$

4. *Show the equating $\sin z = z$ has infinitely many solutions in the complex plane.*

Let $f(z) = \sin z - z$. By Picard's great theorem, there is at most one complex number which f does not take as a value infinitely many times. That is, there is at most one $w_0 \in \mathbb{C}$ such that

$$\sin z - z = w_0$$

does not have infinitely many solutions. It must be true that $f(z) = \sin z - z = w_0 + 2\pi$ has infinitely many solutions in $\mathbb{C}$, which means

$$f(z + 2\pi) = \sin(z + 2\pi) - (z + 2\pi) = w_0$$

has infinitely many solutions. This contradicts the assumption that $\sin z - z = w_0$ has at most finitely many solutions, so we conclude that f takes each value in $\mathbb{C}$ infinitely many times. This implies, in particular, there are infinitely many solutions to

$$f(z) = \sin z - z = 0.$$

$\square$

5. (a) *State Schwartz' Lemma.*

Schwartz' Lemma: Let D be the open unit disk. If $f : D \to D$ is an analytic function fixing 0, then

(i) for all $z \in D$, $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$.

(ii) Furthermore, if $|f'(0)| = 1$ or $|f(z)| = |z|$ for any $z \in D \setminus \{0\}$, then $f(z) = \alpha z$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$.

(b) *Let $f : D \to D$ be a holomorphic map from the unit disk to itself. Prove that for all $z \in D$, $\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2}$.*

This result is sometimes called the Schwartz-Pick Theorem.
Throughout this problem, we use the fact that for any $w_0 \in D$, the Möbius transformation

$$m(z) = \frac{w_0 - z}{1 - \overline{w_0} z}$$

maps D to D and swaps w_0 and 0. For a given $z_0 \in z$, let

$$g(z) = \frac{z_0 - z}{1 - \overline{z_0} z} \quad \text{and} \quad h(z) = \frac{f(z_0) - z}{1 - \overline{f(z_0)} z}.$$

Then $h \circ f \circ g$ is an analytic map from D to D which fixes 0, so we may apply Schwartz' lemma to obtain

$$\left| \frac{f(z_0) - f(g(z_0))}{1 - f(g(z_0))\overline{f(z_0)}} \right| \leq |z|.$$

Letting $w = g^{-1}(z)$, we get

$$\begin{aligned} \left| \frac{f(z_0) - f(w)}{1 - f(w)\overline{f(z_0)}} \right| &\leq \left| \frac{z_0 - w}{1 - \overline{z_0} w} \right| \\ \Rightarrow \left| \frac{f(z_0) - f(w)}{z_0 - w} \right| &\leq \left| \frac{1 - f(w)\overline{f(z_0)}}{1 - \overline{z_0} w} \right|. \end{aligned}$$

Taking the limit as $w \to z_0$ (which we can do since g is bijective on D), we get

$$|f'(z_0)| \leq \frac{1 - |f(z_0)|^2}{1 - |z_0|^2} \quad \Rightarrow \quad \frac{|f'(z_0)|}{1 - |f(z_0)|^2} \leq \frac{1}{1 - |z_0|^2}$$

for all $z_0 \in D$, as desired. $\square$

6. *Determine all continuous functions on* $\{z : 0 < |z| \leq 1\}$ *which are harmonic on* $\{z : 0 < |z| < 1\}$ *and which are identically 0 on* $\{z : |z| = 1\}$.

I claim that any such function is of the form $c \log |z|$ for some $c \in \mathbb{R}$ (which could be 0). Define $f(z) = \log |z|$, let g be any function satisfying the given conditions, and let h be the Möbius transformation given by

$$h(z) = \frac{z - i}{z + i},$$

which maps the upper half plane to the unit disk analytically (and continuously on the boundary). Observe that the functions harmonic $f \circ h$ and $g \circ h$ on $\mathbb{H} \setminus \{i\}$ are continuous (and identically 0) on $\mathbb{R}$. By the Schwartz reflection principle for harmonic functions, we can define harmonic functions F and G on $\mathbb{C} \setminus \{-i, i\}$ such that

$$F|_{\overline{\mathbb{H}} \setminus \{i\}} = f \circ h \quad \text{and} \quad G|_{\overline{\mathbb{H}} \setminus \{i\}} = g \circ h.$$

By construction, $F(z) \equiv 0 \equiv G(z)$ for all $z \in \mathbb{R}$. On the real line, we also have $0 = \partial^2 F / \partial x^2 = \partial^2 F / \partial y^2 = \partial^2 G / \partial x^2 = \partial^2 G / \partial y^2$, so for some constants $\alpha, \beta \in \mathbb{R}$, $\partial F / \partial y = \alpha$ and $\partial G / \partial y = \beta$. Note that $\alpha \neq 0$ since F is not constant by assumption. It follows that the analytic functions

$$\beta \left(\frac{\partial F}{\partial x} - i \frac{\partial F}{\partial y} \right) \quad \text{and} \quad \alpha \left(\frac{\partial G}{\partial x} - i \frac{\partial G}{\partial y} \right)$$

are equal on $\mathbb{R}$. By the identity principle, we have that the above functions are equal on $\mathbb{C}\setminus\{-i,i\}$. But then integrating the real part of the above equation with respect to x we have

$$\frac{\beta}{\alpha}F = G$$

since F and G both fix the origin. It follows that for all $z \in \{z \ : \ 0 < |z| \le 1\}$,

$$\frac{\beta}{\alpha}F\circ h^{-1}(z) = \frac{\beta}{\alpha}\log|z| = g(z) = G\circ h^{-1}(z),$$

as desired. □

7. (a) *Prove the series* $\sum_{-\infty}^{\infty}\frac{1}{(z-n)^2}$ *converges to a meromorphic function on* $\mathbb{C}$.

By Weierstrass' theorem, it is sufficient to show that $f(z) = \sum_{-\infty}^{\infty}\frac{1}{(z-n)^2}$ converges uniformly on compact sets (which avoid $\mathbb{Z}$). Our strategy is to use the Weierstrass M test, which states that it is sufficient to produce positive real numbers M_n such that $\left|\frac{1}{(z-n)^2}\right| \le M_n$ and for which

$$\sum_{n=-\infty}^{\infty} M_n < \infty.$$

With this in mind, let $K \subseteq \mathbb{C}\setminus\mathbb{Z}$ be a compact set. Let N_1 and N_2 be integers such that

$$N_1 < \text{Re } z < N_2 \qquad \text{for all } z \in K,$$

and let $\epsilon = d(K,\mathbb{Z}) = \min\{|z-n| \ : \ z\in K, n\in\mathbb{Z}\}$. Now for any $z \in K$ and $n_1 \le N_1$, we have $|z-n_1| > |N_1-n_1|$, and for any $n_2 \ge N_2$, we have $|z-n_2| > |N_2-n_2|$. Finally, for any $N_1 < n_3 < N_2$, $|z-n_3| > \epsilon$. Since

$$\frac{(N_2-N_1)}{\epsilon^2} + \sum_{n=-\infty}^{N_1}\frac{1}{(N_1-n)^2} + \sum_{n=N_2}^{\infty}\frac{1}{(N_2-n)^2} < \infty,$$

the series for f converges uniformly on K by the Weierstrass M test. By Weierstrass' theorem, the limit function f is meromorphic. □

(b) *Prove that there is an entire function* $h(z)$ *so that* $\frac{\pi^2}{\sin^2(\pi z)} = \sum_{-\infty}^{\infty}\frac{1}{(z-n)^2} + h(z)$.

Consider the function

$$g(z) = \frac{\pi^2}{\sin^2(\pi z)} - \sum_{-\infty}^{\infty}\frac{1}{(z-n)^2}.$$

g is a meromorphic function with singularities at each integer. Therefore, it is sufficient to show that each of these singularities is removable. First, consider the singularity at 0. $\frac{\pi^2}{\sin^2(\pi z)}$ has a pole of order 2 at 0 since $\sin z$ has a simple 0 at 0. Furthermore,

$$\lim_{z\to 0}\frac{z^2\pi^2}{\sin^2(\pi z)} = \lim_{z\to 0}\frac{z^2\pi^2}{(\pi z)^2(1+\sum_{n=3}^{\infty}(-1)^n(\pi z)^{2n-3}/(2n-1)!} = 1,$$

so the singular part of $\frac{\pi^2}{\sin^2(\pi z)}$ at 0 is $1/z^2$ ($\frac{\pi^2}{\sin^2(\pi z)}$ is even, so we do not need to consider a $1/z$ Laurent coefficient). Since this is also the singular part of f at 0, we conclude that g has a removable singularity at 0. g is periodic with period 1, so g only has removable singularities. Therefore, there is an entire function h such that $h|_{\mathbb{C}\setminus\mathbb{Z}} = g$, and further

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{-\infty}^{\infty} \frac{1}{(z-n)^2} + h(z).$$

It turns out that the function h is identically 0, but we do not need to show this. $\square$

8. *Show that the total number of poles of an elliptic function f in its fundamental parallelogram is ≥ 2.*

Observe that this problem is false is we do not assume that

(i) all elliptic functions are non-constant, and

(ii) the number of poles of an elliptic function is counted with multiplicity.

Assuming the above, let f be an elliptic function with period lattice $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ and fundamental parallelogram P, chosen so that f has no poles on ∂P. First, we observe that $\overline{P}$ is a compact set, so if f has no poles, f is constant by Liouville's theorem. Assume for the sake of contradiction that f has exactly one simple pole inside P. By Cauchy's residue theorem,

$$\int_{\partial P} f(z)dz \neq 0.$$

For some $a \in \mathbb{C}$, P is the convex hull of $\{a, a+\omega_1, a+\omega_2, a+\omega_1+\omega_2\}$. Therefore,

$$\begin{aligned}\int_{\partial P} f(z)dz &= \int_a^{a+\omega_1} f(z)dz + \int_{a+\omega_1}^{a+\omega_1+\omega_2} f(z)dz + \int_{a+\omega_1+\omega_2}^{a+\omega_2} f(z)dz + \int_{a+\omega_2}^{a} f(z)dz \\ &= \int_a^{a+\omega_1} f(z)dz - \int_{a+\omega_2}^{a+\omega_1+\omega_2} f(z)dz + \int_{a+\omega_1}^{a+\omega_1+\omega_2} f(z)dz - \int_a^{a+\omega_2} f(z)dz.\end{aligned}$$

By the periodicity of f, we have

$$\int_a^{a+\omega_1} f(z)dz = \int_{a+\omega_2}^{a+\omega_1+\omega_2} f(z)dz \text{ and } \int_{a+\omega_1}^{a+\omega_1+\omega_2} f(z)dz = \int_a^{a+\omega_2} f(z)dz,$$

so

$$\int_{\partial P} f(z)dz = \int_a^{a+\omega_1} f(z)dz - \int_a^{a+\omega_1} f(z)dz + \int_{a+\omega_1}^{a+\omega_1+\omega_2} f(z)dz - \int_{a+\omega_1}^{a+\omega_1+\omega_2} f(z)dz = 0,$$

a contradiction. Therefore, f must have at least two poles in P (counting multiplicity). $\square$

1. *Prove that for any $a \in \mathbb{C}$ and $n \geq 2$, the polynomial $az^n + z + 1$ has at least one root in the disk $|z| \leq 2$.*

   Let $f(z) = az^n + z + 1$ and let $D(0,2) = \{|z| \leq 2\}$. There are two cases.

   Case 1: $|a| < 1/2^n$.
   Let $g(z) = z + 1$. Then
   $$|f(z) - g(z)| = |az|^n < 1 \leq |g(z)|$$
   on the boundary of $D(0,2)$. Therefore, by Rouché's theorem, f has exactly one root in the disk $D(0,2)$.

   Case 2: $|a| \geq 1/2^n$.
   By the fundamental theorem of algebra, we can factor f as
   $$f(z) = a\prod_{k=1}^{n}(z - \alpha_k) = az^n + z + 1$$
   for some complex numbers $\{\alpha_k\}$. Then
   $$(-1)^n a \prod_{k=1}^{n}(\alpha_k) = 1 \Rightarrow \prod_{k=1}^{n}|\alpha_k| \leq 2^n.$$
   In particular, there must exist at least one α_k with $|a_k| \leq 2$. □

2. *Suppose $f(z)$ is analytic on the unit disk $D(0,1)$ and continuous on the closed unit disk $\overline{D(0,1)}$. Assume that $f(z) = 0$ on an arc of the circle $z = 1$. Show that $f(z) \equiv 0$.*

   Let $D = D(0,1)$ be the (open) unit disk. We recall that the map $h : \mathbb{H} \to D$ given by $h(z) = \frac{z-i}{z+i}$ analytically and bijectively maps the upper half plane to the unit disk (continuously on the boundary). Then the function $g : \mathbb{H} \to \mathbb{C}$
   $$g(z) = f \circ h(z)$$
   is an analytic map which is continuous on $\mathbb{R}$. In particular, g is identically 0 on some interval $I \subseteq \mathbb{R}$. Let $U \subseteq \mathbb{H}$ be some connected open set such that $\overline{U} \cap \mathbb{R} = I$. By the Schwartz reflection principle, there is an analytic continuation of g to some function G on $U \cup I \cup U'$, where $U' = \{\overline{u} \ : \ u \in U\}$. Since $G \equiv 0$ on I, by the identity principle $G \equiv 0$ on $U \cup I \cup U'$. As $G|_U = g$, we get that $g \equiv 0$ on U. Again by the identity principle we have that $g(z) \equiv 0$ on $\mathbb{H}$. Finally, h is invertible and
   $$g \circ h^{-1}(z) = f(z),$$
   so we conclude that $f(z) \equiv 0$. □

3. *Prove that meromorphic functions on the extended complex plane are rational functions.*

Let $\mathbb{C}_\infty$ be the extended complex plane, and let $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a non-constant meromorphic function. Since $\mathbb{C}_\infty$ is compact, by the identity principle for compact Riemann surfaces, f has at most finitely many zeros and finitely many poles (each necessarily of finite order since f is meromorphic).

If the reader worries about citing a result about Riemann surfaces, one can instead prove that f has finitely many zeros and finitely many poles (counting multiplicity) like so: since the closed unit disk is compact, $f(z)$ and $f(1/z)$ each can only have finitely many zeros and finitely many poles inside the closed unit disk, $\overline{D}$, by the usual identity principle. Furthermore, the map $z \mapsto 1/z$ is a bijective map between zeros and poles of $f(z)$ inside $\overline{D}$ and the zeros and poles of $f(1/z)$ inside $\overline{D}$.

Let $a_1, \cdots, a_k$ be the zeros of f with corresponding orders $n_1, \cdots, n_k$, and let $b_1, \cdots, b_\ell$ be the poles of f with corresponding orders $m_1, \cdots, m_\ell$. Consider

$$g(z) = \frac{\prod_{j=1}^{k}(z-a_j)^{n_j}}{\prod_{d=1}^{\ell}(z-b_d)^{m_d}}.$$

g has exactly the same zeros and poles of f with exactly the same multiplicities, so $h : \mathbb{C}_\infty \to \mathbb{C}_\infty$ given by

$$h(z) = \frac{f(z)}{g(z)}$$

is a meromorphic function which has no zeros or poles in $\mathbb{C}_\infty$. Thus, h extends to a nonzero bounded entire function, so by Liouville, $h(z) \equiv c$ for some nonzero constant c. Then

$$f(z) = cg(z) = c\frac{\prod_{j=1}^{k}(z-a_j)^{n_j}}{\prod_{d=1}^{\ell}(z-b_d)^{m_d}},$$

so f is a rational function. $\square$

4. *Suppose u is harmonic and bounded in $\{z \in \mathbb{C} \;:\; 0 < |z| < 1\}$. Show that $\{z = 0\}$ is a removable singularity of u. That is, show that $u(0)$ can be defined so that u becomes harmonic in the full disk $\{|z| < 1\}$.*

For a circle C_r of radius r centered at 0 and any (piecewise) continuous function u_r on C_r, the Poisson integral

$$P_{u_r}(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - |z|^2}{|z - re^{i\theta}|^2} u_r(e^{i\theta})d\theta$$

defines the unique harmonic function inside C_R which approaches u_r on the boundary. This result is sometimes called Schwartz' theorem.

Let D be the unit disk. We wish to show that the function u is given by the Poisson integral on $D \setminus \{0\}$. For some $r \in (0, 1)$, define $g : \{0 < |z| \leq r\}$ as

$$g(z) = u(z) - P_{u_r}(z),$$

where $u_r(z) = u|_{C_r}$. For any $\epsilon > 0$, define

$$g_\epsilon(z) = g(z) + \epsilon \log(|z|/r).$$

P_{u_r} is a continuous function on the closed disk of radius r, so it is bounded, and u is bounded by assumption, so g is bounded. Therefore,

$$\limsup_{z \to 0} g_\epsilon(z) < 0,$$

so for some $\delta > 0$ and for all $|z| \leq \delta$, $g_\epsilon(z) \leq 0$. By construction, we also have that $g_\epsilon(z) = 0$ for all $|z| = r$, since $u(z) = P_{u_r}(z)$ for all such z. By the maximum principle for harmonic functions, we get that $g_\epsilon(z) \leq 0$ for all $\delta < z < r$. Sending ϵ to 0, we get that $g(z) \leq 0$ for all $\delta < z < r$. Since this argument works for any $\delta' \in (0, \delta)$, we can say

$$g(z) \leq 0.$$

Applying the analogous argument to $-g$, we can conclude that $g \equiv 0$ on its domain. It follows that we can define $u(0) = P_{u_r}(0)$ so that u becomes harmonic at 0. $\square$

5. *Evaluate the following integral*

$$\int_0^\infty \frac{1}{1 + x^6}$$

Carefully justify all your steps.

This is problem 2 on the Fall 2013 exam if we set $n = 6$.

6. (a) *State Schwartz' Lemma.*

(b) *Let $f : D \to D$ be a holomorphic map from the unit disk to itself. Prove that for all $z \in D$, $\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2}$.*

This is problem 6 on the Fall 2013 exam.

7. *Find a one to one holomorphic map that sends the unit disk $D(0, 1)$ onto the slit disk $D(0, 1) \setminus \{[0, 1)\}$.*

Let $D = D(0, 1)$ be the unit disk. The Möbius transformation

$$h(z) = \frac{z - i}{z + i} = \frac{x^2 + y^2 - 1}{x^2 + (y + 1)^2} - i\frac{2x}{x^2 + (y + 1)^2} = h(x + iy)$$

sends the upper half plane, $\mathbb{H}$, to D analytically and bijectively. By writing it in real and imaginary parts, it is clear that h sends the first quadrant to $\{z \in D \ : \ \text{Im } z < 0\}$. Let

$g(z) = \sqrt{z}$ be the map which sends the upper half plane to the first quadrant (analytically and bijectively). Then

$$h \circ g \circ h^{-1}$$

maps D analytically (and bijectively) to $\{z \in D \ : \ \text{Im } z < 0\}$. Since the map $z \mapsto z^2$ sends $\{z \in D \ : \ \text{Im } z < 0\} \to D \setminus \{[0,1)\}$ analytically, we have that

$$(h \circ g \circ h^{-1}(z))^2 = \left(\frac{\sqrt{\frac{z+1}{iz-i}} - i}{\sqrt{\frac{z+1}{iz-i}} + i} \right)^2$$

maps D analytically and bijectively to $D \setminus \{[0,1)\}$, as desired. □

8. *Show that the total number of poles of an elliptic function f in its fundamental parallelogram is ≥ 2.*

This is problem 8 on the Fall 2013 exam.

1. *Show that each Möbius transformation maps a straight line or circle onto a straight line or circle.*

   The group of Möbius transformations is generated by translations, dilations, and a single inversion $z \mapsto 1/z$. Since every translation and every dilation clearly maps circles to circles and lines to lines, we need only show that the inversion map always sends circles and lines to circles and lines. To do this, we need two basic facts. First, if $u, v, x, y \in \mathbb{R}$ and $u + iv = w = 1/z = 1/(x + iy)$, then a straightforward calculation yields

   $$u = \frac{x}{x^2 + y^2}, v = \frac{-y}{x^2 + y^2}, x = \frac{u}{u^2 + v^2}, \text{ and } y = \frac{-v}{u^2 + v^2}.$$

   Secondly, it is not hard to show that every circle and line in $\mathbb{C}$ is given by the set of solutions to a non-degenerate ($B^2 + C^2 > 4AD$) quadratic equation of the form

   $$A(x^2 + y^2) + Bx + Cy + D = 0,$$

   and any every solution to such a non-degenerate quadratic equation is a circle ($A \neq 0$) or a line ($A = 0$). Given $S = \{(x, y) \ : \ A(x^2 + y^2) + Bx + Cy + D = 0\}$ for $B^2 + C^2 > 4AD$, then by an earlier observation, the image of S under $z \mapsto 1/z$ is exactly the set of all $u + iv \in \mathbb{C}$ satisfying

   $$A\left(\frac{u^2}{(u^2 + v^2)^2} + \frac{v^2}{(u^2 + v^2)^2}\right) + B\frac{u}{u^2 + v^2} - C\frac{v}{u^2 + v^2} + D = 0.$$

   Clearing the denominators gives

   $$A + Bu - Cv + D(u^2 + v^2) = 0,$$

   which is also a non-degenerate quadratic equation. We conclude that circles and lines get mapped to circles and lines under inversion, so all Möbius transformations send circles and lines to circles and lines. $\square$

2. *Let f be a complex valued function in the unit disk $D(0, 1)$ such that $g = f^2$ and $h = f^3$ are both analytic. Prove that f is analytic in $D(0, 1)$.*

   Since h and g are analytic,

   $$f(z) = \frac{h(z)}{g(z)}$$

   is meromorphic in $D(0, 1)$. If f is not analytic, then f has a pole at $a \in D(0, 1)$ of order $n > 0$. Then we can write

   $$f(z) = \frac{r(z)}{(z - a)^n}$$

   where r is analytic and nonzero around a. Then r^2 is also analytic and nonzero around a. This implies

   $$g(z) = \frac{r(z)^2}{(z - a)^{2n}}$$

   has a pole of order $2n$ at a, contradicting the assumption that g is analytic. Thus f is a meromorphic function on $D(0, 1)$ without poles, so f is analytic. $\square$

4. *Exhibit a function f such that at each positive integer n, f has a pole of order n, and f is analytic and nonzero at each other complex number.*

Let

$$g(z) = \prod_{n=1}^{\infty} \left(\left(1 - \frac{z}{n}\right) e^{\frac{z}{n} + \frac{z^2}{2n^2}} \right)^n .$$

If the product defining g converges, then it defines an analytic function with zeros of order n at each positive integer n. For a given n, consider

$$\begin{aligned}\left(1 - \frac{z}{n}\right) e^{\frac{z}{n} + \frac{z^2}{n^2}} &= \left(1 - \frac{z}{n}\right) \sum_{k=0}^{\infty} \tfrac{1}{k!} \left(\tfrac{z}{n} + \tfrac{z^2}{2n^2}\right)^k \\ &= 1 - \frac{1}{3}(\tfrac{z}{n})^3 - \frac{1}{4}(\tfrac{z}{n})^4 + O((\tfrac{z}{n})^5)\end{aligned}$$

Since the power series for e^w converges absolutely on all of $\mathbb{C}$, each term of the product is of the form $(1 - z^3 O(1/n^3))^n$, and further, there is some universal constant C (independent of n) such that the coefficient of z^3 is bounded by C/n^3. It follows that our series converges if

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} z^3 \frac{C}{n^3}$$

converges uniformly on compact sets. However, the above series can be rewritten as

$$C z^3 \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which converges uniformly on compact sets by the Weierstrass M test. Therefore, g defines an entire function with zeros of order n at each positive integer n. If we let $f = 1/g$, then f is a meromorphic function with poles of order n at each positive integer n, and f is analytic everywhere else by the entirety of g. Furthermore, f is never 0 since g is entire and therefore has no poles in $\mathbb{C}$. $\square$

5. *Suppose that an entire holomorphic function $g(z)$ satisfies the equation $g(1-z) = 1 - g(z)$ for all $z \in \mathbb{C}$. Show that $g(\mathbb{C}) = \mathbb{C}$.*

   We need to assume g is non-constant, or $g(z) \equiv 1/2$ is a counter example.

   If g is non-constant, by Picard's theorem we have that there is at most one point $a \in \mathbb{C}$ which is not in the image of g. Since $g(1/2) = 1 - g(1/2)$, we have that $g(1/2) = 1/2$. If $a \neq 1/2$, $1 - a \neq a$ and there is some $z \in \mathbb{C}$ with $g(z) = 1 - a$, which implies $g(1-z) = a$. Thus, g must take every value in $\mathbb{C}$. $\square$

8. *State and prove a version of the Schwartz reflection principle.*

   Let $u(x, y)$ be a continuous function on $\overline{\mathbb{H}}$ which is harmonic on $\mathbb{H}$ and identically 0 on $\mathbb{R}$. Then there is a harmonic function U on $\mathbb{C}$ such that $U|_{\overline{\mathbb{H}}} = u$.

   Proof. Define
$$U(x, y) = \begin{cases} u(x, y) & \text{if } x + iy \in \overline{\mathbb{H}} \\ -u(x, -y) & \text{else.} \end{cases}$$
   U is certainly harmonic on $\mathbb{H}$ since u is. Furthermore, on the lower half plane,
$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0,$$
   so U is harmonic away from $\mathbb{R}$.

   Let $x \in \mathbb{R}$. To show U is harmonic at x, we need only show that U satisfies the mean value property at x. That is, we need to show that for any $r > 0$,
$$U(x + 0i) = 0 = \frac{1}{2\pi} \int_0^{2\pi} U(x + re^{i\theta}) d\theta.$$
   This integral must vanish since
$$\begin{aligned} \int_0^{2\pi} U(x + re^{i\theta}) d\theta &= \int_0^{\pi} U(x + re^{i\theta}) d\theta + \int_{\pi}^{2\pi} U(x + re^{i\theta}) d\theta \\ &= \int_0^{\pi} u(x + re^{i\theta}) d\theta + \int_{\pi}^{2\pi} -u(x + re^{-i\theta}) d\theta \\ &= \int_0^{\pi} u(x + re^{i\theta}) d\theta - \int_0^{\pi} u(x - re^{-i\theta}) d\theta \\ &= \int_0^{\pi} u(x + re^{i\theta}) d\theta - \int_0^{\pi} u(x + re^{i\theta}) d\theta = 0. \end{aligned}$$
   Thus, U is harmonic on $\mathbb{R}$, so U is harmonic on $\mathbb{C}$. $\square$

Note that the above argument can be used to show the Schwartz reflection principle for analytic functions on $\mathbb{H}$ which take real values on the real axis by fixing u as the imaginary part of any such function.

5. *Let $f(z)$ be an entire holomorphic function. Suppose that $f(z) = f(z+1)$ and $|f(z)| \leq e^{|z|}$ for all $z \in \mathbb{C}$. Prove that f is constant.*

Observe that the function

$$g(z) = \frac{1}{\sin(\pi z)}$$

also satisfies $g(z+1) = g(z)$, and g has a simple pole at each integer. Therefore

$$h(z) = \frac{f(z) - f(0)}{\sin(\pi z)}$$

is an entire, periodic function with period 1. We claim that h is identically 0. To show this, we will first show that h is bounded on $S = \{x + iy \in \mathbb{C} \ : \ 0 \leq x \leq 1\}$. For any $x + iy \in S$,

$$\begin{aligned} |h(x+iy)| &= \left| \frac{f(x+iy) - f(0)}{\sin(\pi x + iy)} \right| \\ &\leq \frac{2e^{|x+iy|} + |f(0)|}{|e^{-\pi y}e^{i\pi x} - e^{\pi y}e^{-i\pi x}|} \ll \frac{e^{|y|}}{e^{|\pi y|}}. \end{aligned}$$

It follows that $|h(z)| \to 0$ uniformly as $|y| \to \infty$, so h is bounded on S. Since $h(z+1) = h(z)$, we have that h is bounded on $\mathbb{C}$, and so by Liouville, h is constant. Furthermore, since $|h(x+iy)| \to 0$ as $y \to \infty$, we conclude that $h \equiv 0$ in $\mathbb{C}$. Then $f(z) - f(0) \equiv 0$ on $\mathbb{C}$, so f is constant. $\square$