12.7 Increments and Differentials

First, we remind you of the notation that we used for functions of a single variable. We defined the **increment** Δy of the function $f(x)$ at $x = a$ to be $\Delta y = f(a+\Delta x) - f(a)$. Referring to Figure 1, notice that for Δx small, $\Delta y \approx dy = f'(a)\Delta x$, where we referred to dy as the **differential** of y.

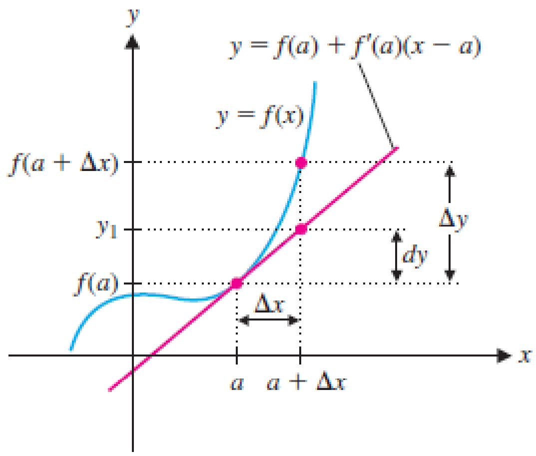


Figure 1: Increments and differentials for a function of one variable.

For $z = f(x, y)$, we define the **increment** of f at (a, b) to be

$$\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b).$$

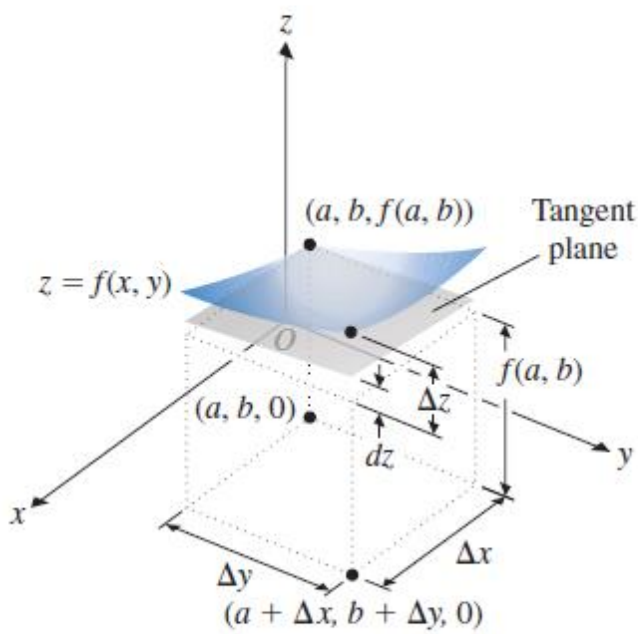


Figure 2: Linear approximation.

Notice that as long as f is continuous in some open region containing (a,b) and f has first partial derivatives on that region, we can write:

$$\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$$

$$= [f(a+\Delta x, b+\Delta y) - f(a, b+\Delta y)] + [f(a, b+\Delta y) - f(a,b)]$$

Adding and subtracting $f(a, b+\Delta y)$.

$$= f_x(u, b+\Delta y)[(a+\Delta x) - a] + f_y(a,v)[(b+\Delta y) - b]$$

Applying the Mean Value Theorem to both terms.

$$= f_x(u, b+\Delta y)\Delta x + f_y(a,v)\Delta y,$$

by the Mean Value Theorem. Here, u is some value between a and $a+\Delta x$, and v is some value between b and $b+\Delta y$ (see Figure 3). This gives us

$$\Delta z = f_x(u, b+\Delta y)\Delta x + f_y(a,v)\Delta y,$$

$$= \{f_x(a,b) + [f_x(u, b+\Delta y) - f_x(a,b)]\}\Delta x + \{f_y(a,b) + [f_y(a,v) - f_y(a,b)]\}\Delta y$$

which we rewrite as $\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$, where $\varepsilon_1 = [f_x(u, b+\Delta y) - f_x(a,b)]$ and $\varepsilon_2 = [f_y(a,v) - f_y(a,b)]$.

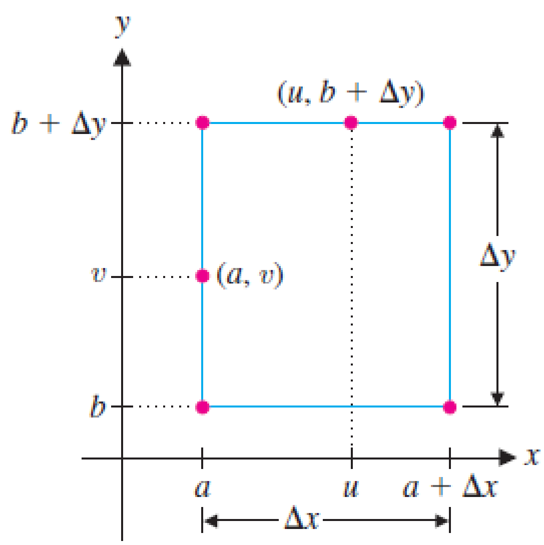


Figure 3: Intermediate points from the Mean Value Theorem.

We have now established the following result.

<u>Theorem1</u>

Suppose that $z = f(x,y)$ ***is defined on the rectangular region*** $R = \{(x,y) \in \mathbb{R}^2 \mid x_0 < x < x_1 \,\&\, y_0 < y < y_1\}$ ***and*** f_x ***and*** f_y ***are defined on*** R ***and are continuous at*** $(a,b) \in R$ ***. Then for*** $(a+\Delta x, b+\Delta y) \in R$ ***,*** $\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$ ***where*** ε_1 ***and*** ε_2 ***are functions of*** Δx ***and*** Δy ***that both tend to zero, as*** $(\Delta x, \Delta y) \to (0,0)$***.***

Example 1 (Computing the Increment Δz)

For $z = f(x, y) = x^2 - 5xy$, find Δz .

Solution

We have

$$\Delta z = f\left(x + \Delta x, y + \Delta y\right) - f\left(x, y\right).$$

$$= \left(x + \Delta x\right)^2 - 5\left(x + \Delta x\right)\left(y + \Delta y\right) - \left[x^2 - 5xy\right].$$

$$= x^2 + 2x\ \Delta x + \left(\Delta x\right)^2 - 5\left(xy + x\ \Delta y + y\ \Delta x + \Delta x\ \Delta y\right) - x^2 + 5xy$$

$$= \left(2x - 5\right)\Delta x + \left(-5x\right)\Delta y + \left(\Delta x\right)\Delta x + \left(-5\Delta x\right)\Delta y.$$

$$= f_x(x, y)\ \Delta x + f_y(x, y)\ \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where $\varepsilon_1 = \Delta x$ and $\varepsilon_2 = -5\Delta x$ both tend to zero, as $\left(\Delta x, \Delta y\right) \to \left(0, 0\right)$.

Example 2

Let $z = f(x, y) = 3x^2 - xy$.

(a) If Δx and Δy are increments of x and y , find Δz .

(b) Use Δz to calculate the change in $f(x, y)$ if $\left(x, y\right)$ changes from $\left(1, 2\right)$ to $\left(1.01, 1.98\right)$.

Solution

(a) We have

$$\Delta z = f\left(x + \Delta x, y + \Delta y\right) - f\left(x, y\right).$$

$$= 3\left(x + \Delta x\right)^2 - \left(x + \Delta x\right)\left(y + \Delta y\right) - \left[3x^2 - xy\right].$$

$$= 3x^2 + 6x\ \Delta x + 3\left(\Delta x\right)^2 - \left(xy + x\ \Delta y + y\ \Delta x + \Delta x\ \Delta y\right) - 3x^2 + xy$$

$$= \left(6x - y\right)\Delta x + \left(-x\right)\Delta y + \left(3\Delta x\right)\Delta x + \left(-\Delta x\right)\Delta y.$$

$$= f_x(x, y)\ \Delta x + f_y(x, y)\ \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where $\varepsilon_1 = 3\Delta x$ and $\varepsilon_2 = -\Delta x$ both tend to zero, as $\left(\Delta x, \Delta y\right) \to \left(0, 0\right)$.

(b) If $\left(x, y\right)$ changes from $\left(1, 2\right)$ to $\left(1.01, 1.98\right)$, substituting $x = 1, y = 2, \Delta x = 0.01,$ and $\Delta y = -0.02$ into the formula for Δz gives us

$$\Delta z = \left[6(1) - 2\right](0.01) - (1)(-0.02) + 3(0.01)^2 - (0.01)(-0.02) = 0.0605.$$

Remark1

If we increment x by the amount $dx = \Delta x$ and increment y by $dy = \Delta y$, then we define the **total differential** of z to be $dz = f_x(x, y)dx + f_y(x, y)dy$.

Definition1

*Let $z = f(x, y)$. We say that f is **differentiable** at $\left(a, b\right)$ if we can write $\Delta z = f_x(a, b)\ \Delta x + f_y(a, b)\ \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$, where ε_1 and ε_2 are both functions of Δx and Δy and $\varepsilon_1, \varepsilon_2 \to 0$, as $\left(\Delta x, \Delta y\right) \to \left(0, 0\right)$. We say that f is differentiable on a region $R \subseteq \mathbb{R}^2$ whenever f is differentiable at every point in R .*

Definition2

*The **linear approximation** to* $f(x,y,z)$ *at the point* (a,b,c) *is given by*

$L(x,y,z)=f(a,b,c)+f_x(a,b,c)(x-a)+f_y(a,b,c)(y-b)+f_z(a,b,c)(z-c).$

Example 3

The dimensions of a closed rectangular box are measured as 3 feet, 4 feet, and 5 feet, with a possible error of $\pm\frac{1}{16}$ inch in each measurement. Use differentials to approximate the maximum error in the calculated value of

(a) The surface area.

(b) The volume.

Solution

(a) The surface area is $S=2(xy+yz+xz)$. So

$$dS=2(y+z)dx+2(x+z)dy+2(x+y)dz.$$

As $dx=dy=dz=\pm\frac{1}{16}$ inch $=\pm\frac{1}{192}$ feet, we get $dS=(18+16+14)\left(\frac{\pm 1}{192}\right)=\pm\frac{1}{4}$ feet^2.

(b) The volume is $V=x\ y\ z$. So

$$dV=yz\ dx+xz\ dy+xy\ dz$$
$$=(20+15+12)\left(\frac{\pm 1}{192}\right)=\pm\frac{47}{192}\text{ feet}^3.$$

12.8 Chain Rule and Implicit Differentiation

The general form of the chain rule says that for differentiable functions f and g ,

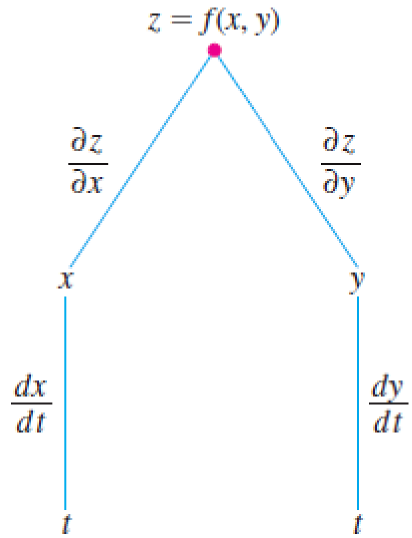
$$\frac{d}{dx}\left[f(g(x))\right]=f'(g(x))g'(x).$$

We now extend the chain rule to functions of several variables.

Theorem1 (Chain Rule)

If $z=f(x(t),y(t))$**, where** $x(t)$ **and** $y(t)$ **are differentiable and** $f(x,y)$ **is a differentiable function of** x **and** y **, then**

$$\frac{dz}{dt}=\frac{d}{dt}\left[f\left(x(t),y(t)\right)\right]=\frac{\partial f}{\partial x}\left(x(t),y(t)\right)\frac{dx}{dt}+\frac{\partial f}{\partial y}\left(x(t),y(t)\right)\frac{dy}{dt} .$$



$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}$$

Example1 (Using the Chain Rule)

For $z=f(x,y)=x^2e^y$, $x(t)=t^2-1$ and $y(t)=\sin t$, find the derivative of $g(t)=f\left(x(t),y(t)\right)$.

Solution

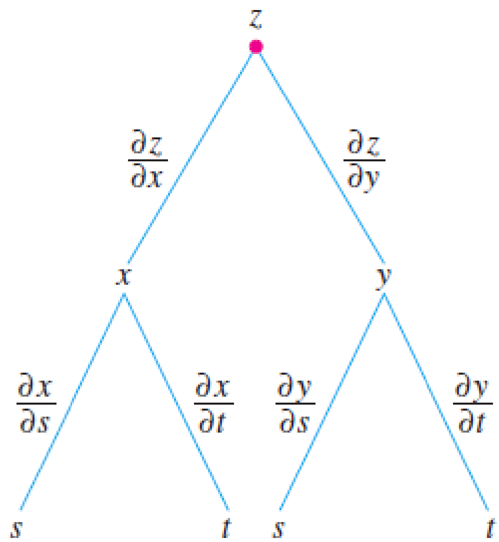
We first compute the derivatives $\frac{\partial z}{\partial x}=2xe^y$, $\frac{\partial z}{\partial y}=x^2e^y$, $x'(t)=2t$ and $y'(t)=\cos t$.

The chain rule (Theorem1) then gives us

$$g'(t)=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}=2xe^y\left(2t\right)+x^2e^y\left(\cos t\right)$$
$$=4t\left(t^2-1\right)e^{\sin t}+\left(\cos t\right)\left(t^2-1\right)^2e^{\sin t}$$
.

<u>Theorem2 (Chain Rule)</u>

Suppose that $z=f(x,y)$, where f is a differentiable function of x and y and where $x=x\left(s,t\right)$ and $y=\left(s,t\right)$ both have first-order partial derivatives. Then we have the chain rules: $\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$ **and** $\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$ **.**



Example 2 (Using the Chain Rule)

Suppose that $f(x,y)=e^{xy}$, $x(u,v)=3u\sin v$ and $y(u,v)=4v^2u$. For $g(u,v)=f\left(x(u,v),y(u,v)\right)$, find the partial derivatives $\frac{\partial g}{\partial u}$ and $\frac{\partial g}{\partial v}$.

Solution

We first compute the partial derivatives $\frac{\partial f}{\partial x}=ye^{xy}$, $\frac{\partial f}{\partial y}=xe^{xy}$, $\frac{\partial x}{\partial u}=3\sin v$ and $\frac{\partial y}{\partial u}=4v^2$. The chain rule (Theorem 2) gives us

$$\frac{\partial g}{\partial u}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}=ye^{xy}\left(3\sin v\right)+xe^{xy}\left(4v^2\right).$$

Substituting for x and y, we get

$$\frac{\partial g}{\partial u}=12uv^2\sin v\; e^{12u^2v^2\sin v}+12uv^2\sin v\; e^{12u^2v^2\sin v}$$

$$=24uv^2\sin v\; e^{12u^2v^2\sin v}.$$

For the partial derivative of g with respect to v, we compute $\frac{\partial x}{\partial v}=3u\cos v$ and $\frac{\partial y}{\partial v}=8uv$. Here, the chain rule gives us :

$$\frac{\partial g}{\partial v}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial v}=ye^{xy}\left(3u\cos v\right)+xe^{xy}\left(8uv\right).$$

Substituting for x and y, we have : $\frac{\partial g}{\partial v}=\left(12u^2v^2\cos v+24u^2v\sin v\right)e^{12u^2v^2\sin v}$.

Example 3 (Converting from Rectangular to Polar Coordinates)

For a differentiable function $f(x,y)$ with $x = r\cos\theta$ and $y = r\sin\theta$, show that $f_r = f_x\cos\theta + f_y\sin\theta$ and $f_{rr} = f_{xx}\cos^2\theta + 2f_{xy}\cos\theta\sin\theta + f_{yy}\sin^2\theta$.

Solution

First, notice that $\frac{\partial x}{\partial r} = \cos\theta$ and $\frac{\partial y}{\partial r} = \sin\theta$. From Theorem 2, we now have

$$f_r = f_x\frac{\partial x}{\partial r} + f_y\frac{\partial y}{\partial r} = f_x\cos\theta + f_y\sin\theta .$$

Be very careful when computing the second partial derivative. Using the expression we have already found for f_r and Theorem2, we have

$$\begin{aligned}
f_{rr} &= \frac{\partial}{\partial r}(f_r) = \frac{\partial}{\partial r}\left(f_x\cos\theta + f_y\sin\theta\right)\\
&= \frac{\partial}{\partial r}\left(f_x\cos\theta\right) + \frac{\partial}{\partial r}\left(f_y\sin\theta\right)\\
&= \left[\frac{\partial}{\partial x}(f_x)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}(f_x)\frac{\partial y}{\partial r}\right]\cos\theta + \left[\frac{\partial}{\partial x}\left(f_y\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(f_y\right)\frac{\partial y}{\partial r}\right]\sin\theta\\
&= \left[f_{xx}\cos\theta + f_{xy}\sin\theta\right]\cos\theta + \left[f_{yx}\cos\theta + f_{yy}\sin\theta\right]\sin\theta\\
&= f_{xx}\cos^2\theta + 2f_{xy}\sin\theta\cos\theta + f_{yy}\sin^2\theta.
\end{aligned}$$

Implicit Differentiation

- Suppose that the equation $F(x,y)=0$ defines y implicitly as a function of x, say $y = f(x)$. We let $z = F(x,y)$, where $x = t$ and $y = f(t)$. From Theorem1, we have $\frac{dz}{dt} = F_x\frac{dx}{dt} + F_y\frac{dy}{dt}$**.** But, since $z = F(x,y) = 0$, we have $\frac{dz}{dt} = 0$. Further, since $x = t$, we have $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = \frac{dy}{dx}$. This gives us $0 = F_x + F_y\frac{dy}{dx}$. Notice that we can solve this for $\frac{dy}{dx}$, provided $F_y \neq 0$. In this case, we have : $\frac{dy}{dx} = -\frac{F_x}{F_y}$.
- Suppose that the equation $F(x,y,z)=0$ implicitly defines a function $z = f(x,y)$, where f is differentiable. Then, we can find the partial derivatives f_x and f_y using the chain rule, as follows. We first let $w = F(x,y,z)$. From the chain rule, we have $\frac{\partial w}{\partial x} = F_x\frac{\partial x}{\partial x} + F_y\frac{\partial y}{\partial x} + F_z\frac{\partial z}{\partial x}$. Notice that since $w = F(x,y,z) = 0$, $\frac{\partial w}{\partial x} = 0$. Also, $\frac{\partial x}{\partial x} = 1$ and $\frac{\partial y}{\partial x} = 0$, since x and y are

independent variables. This gives us $0 = F_x + F_z \frac{\partial z}{\partial x}$. We can solve this for $\frac{\partial z}{\partial x}$, as long as $F_z \neq 0$, to obtain: $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$.

Likewise, differentiating w with respect to y leads us to: $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$, $F_z \neq 0$.

Example 4 (Finding Partial Derivatives Implicitly)

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, given that $F(x, y, z) = xy^2 + z^3 + \sin(xyz) = 0$.

Solution

First, note that using the usual chain rule, we have: $F_x = y^2 + yz\cos(xyz)$,

$F_y = 2xy + xz\cos(xyz)$ and $F_z = 3z^2 + xy\cos(xyz)$.

If $3z^2 + xy\cos(xyz) \neq 0$ then

$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{y^2 + yz\cos(xyz)}{3z^2 + xy\cos(xyz)}$ and $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{2xy + xz\cos(xyz)}{3z^2 + xy\cos(xyz)}$.

12.9 The gradient and Directional derivatives

In this section, we develop the notion of directional derivatives. Suppose that we want to find the instantaneous rate of change of $f(x, y)$ at the point $P(a,b)$ and in the direction given by the *unit* vector $u = <u_1, u_2>$. Let $Q(x, y)$ be any point on the line through $P(a,b)$ in the direction of u . Notice that the vector $\overrightarrow{PQ}$ is then parallel to u . Since two vectors are parallel if and only if one is a scalar multiple of the other, we have that $\overrightarrow{PQ} = h.u$, for some scalar h , so that $\overrightarrow{PQ} = <x-a, y-b> = hu = h<u_1, u_2> = <hu_1, hu_2>$.

It then follows that $x - a = hu_1$ and $y - b = hu_2$, so that $x = a + hu_1$ and $y = b + hu_2$.

The point Q is then described by $(a + hu_1, b + hu_2)$, as indicated in Figure 1. Notice that the average rate of change of $z = f(x, y)$ along the line from P to Q is then

$\frac{f(a + hu_1, b + hu_2) - f(a,b)}{h}$.

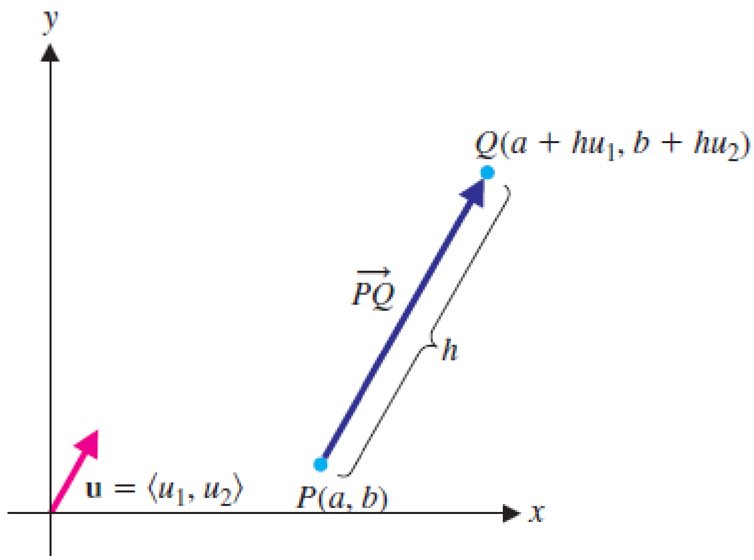


Figure1: The vector $\overrightarrow{PQ}$.

The instantaneous rate of change of $f(x, y)$ at the point $P(a,b)$ and in the direction of the unit vector u is then found by taking the limit as $h \to 0$.

Definition1

*The **directional derivative of** $f(x, y)$ at the point (a,b) and in the direction of the unit vector $u = <u_1, u_2>$ is given by* $D_u f(a,b) = \lim_{h\to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$, *provided the limit exists*.

Remark1:
We can extend the definition of the directional derivative of a function in 3 variables as:
The **directional derivative of** $f(x, y, z)$ at the point (a,b,c) and in the direction of the unit vector $u = <u_1, u_2, u_3>$ is given by

$D_u f(a,b,c) = \lim_{h\to 0} \frac{f(a+hu_1, b+hu_2, c+hu_3) - f(a,b,c)}{h}$, provided the limit exists.

Theorem1

- Suppose that f is differentiable at (a,b) and $u = <u_1, u_2>$ is any unit vector. Then, we can write $D_u f = f_x(a,b)u_1 + f_y(a,b)u_2$.

- Suppose that f is differentiable at (a,b,c) and $u = <u_1, u_2, u_3>$ is any unit vector. Then, we can write $D_u f = f_x(a,b,c)u_1 + f_y(a,b,c)u_2 + f_z(a,b,c)u_3$.

Example 1 (Computing Directional Derivatives)

For $f(x,y)=x^2y-4y^3$, compute $D_u f(2,1)$ for the directions

(a) $u=<\frac{\sqrt{3}}{2},\frac{1}{2}>$

(b) u in the direction from $(2,1)$ to $(4,0)$.

Solution

Regardless of the direction, we first need to compute the first partial derivatives $\frac{\partial f}{\partial x}=2xy$ and $\frac{\partial f}{\partial y}=x^2-12y^2$. Then, $f_x(2,1)=4$ and $f_y(2,1)=-8$.

- For (a), the unit vector is given as $u=<\frac{\sqrt{3}}{2},\frac{1}{2}>$ and so, from Theorem 1 we have $D_u f(2,1)=f_x(2,1)u_1+f_y(2,1)u_2=4\frac{\sqrt{3}}{2}-8\frac{1}{2}=2\sqrt{3}-4<0$. Notice that this says that the function is decreasing in this direction.
- For (b), we must first find the unit vector u in the indicated direction. Observe that the vector from $(2,1)$ to $(4,0)$ corresponds to the position vector $<2,-1>$ and so, the unit vector in that direction is $u=\frac{<2,-1>}{\|<2,-1>\|}=<\frac{2}{\sqrt{5}},\frac{-1}{\sqrt{5}}>$. We then have from Theorem 1 that $D_u f(2,1)=f_x(2,1)u_1+f_y(2,1)u_2=4\frac{2}{\sqrt{5}}+(-8)\frac{(-1)}{\sqrt{5}}=\frac{16}{\sqrt{5}}>0$. So, the function is increasing rapidly in this direction.

For convenience, we define the **gradient** of a function to be the *vector-valued function* whose components are the first-order partial derivatives of f. We denote the gradient of a function f by **grad** f or ∇f.

Definition 2

The ***gradient*** *of* $f(x,y)$ *is the vector-valued function*

$\nabla f(x,y)=<\frac{\partial f}{\partial x}(x,y),\frac{\partial f}{\partial y}(x,y)>=\frac{\partial f}{\partial x}(x,y)\vec{i}+\frac{\partial f}{\partial y}(x,y)\vec{j}$, *provided both partial derivatives exist. Similarly, we define the gradient of* $f(x,y,z)$ *as the vector-valued function*

$\nabla f(x,y,z)=<\frac{\partial f}{\partial x}(x,y,z),\frac{\partial f}{\partial y}(x,y,z),\frac{\partial f}{\partial z}(x,y,z)>=\frac{\partial f}{\partial x}(x,y,z)\vec{i}+\frac{\partial f}{\partial y}(x,y,z)\vec{j}+\frac{\partial f}{\partial z}(x,y,z)\vec{k}$,

provided all the partial derivatives are defined.

Theorem 2

If f is a differentiable function of x and y and u is any unit vector, then

$D_u f(x,y)=\nabla f(x,y).u$

Similarly, if f is a differentiable function of x, y and z and u is any unit vector, then $D_u f(x,y,z) = \nabla f(x,y,z).u$

Example 2 (Finding Directional Derivatives)
For $f(x,y) = x^2 + y^2$, find $D_u f(1,-1)$ for
(a) u in the direction of $v = <-3,4>$.
(b) u in the direction of $v = <3,-4>$.

Solution

First, note that $\nabla f(x,y) = < \frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) > = <2x, 2y>$.

At the point $(1,-1)$, we have $\nabla f(1,-1) = <2,-2>$.

- For (a), a unit vector in the same direction as v is $u = < \frac{-3}{5}, \frac{4}{5} >$. The directional derivative of f in this direction at the point $(1,-1)$ is then

  $D_u f(1,-1) = <2,-2>.< \frac{-3}{5}, \frac{4}{5} > = 2 \times \frac{-3}{5} + (-2) \times \frac{4}{5} = \frac{-14}{5}$.

- For (b), the unit vector is $u = < \frac{3}{5}, \frac{-4}{5} >$ and so, the directional derivative of f in this direction at $(1,-1)$ is $D_u f(1,-1) = <2,-2>.< \frac{3}{5}, \frac{-4}{5} > = 2 \times \frac{3}{5} + (-2) \times \frac{-4}{5} = \frac{14}{5}$.

Theorem 3

Suppose that f is a differentiable function of x and y at the point (a,b). Then

- **the maximum rate of change of f at (a,b) is $\|\nabla f(a,b)\|$, occurring in the direction of the gradient;**
- **the minimum rate of change of f at (a,b) (a, b) is $-\|\nabla f(a,b)\|$, occurring in the direction opposite the gradient;**
- **the rate of change of f at (a,b) is 0 in the directions orthogonal to $\nabla f(a,b)$.**
- **the gradient $\nabla f(a,b)$ is orthogonal to the level curve $f(x,y) = c$ at the point (a,b), where $c = f(a,b)$.**

Example 3 (Finding Maximum and Minimum Rates of Change)

Find the maximum and minimum rates of change of the function $f(x,y) = x^2 + y^2$ at the point $(1,3)$.

Solution

We first compute the gradient $\nabla f = < 2x, 2y >$ and evaluate it at the point $(1,3)$; $\nabla f(1,3) = < 2,6 >$. From Theorem 3, the maximum rate of change of f at $(1,3)$ is $\|\nabla f(1,3)\| = \sqrt{40} = 2\sqrt{10}$ and occurs in the direction of $u = \frac{\nabla f(1,3)}{\|\nabla f(1,3)\|} = < \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} >$.

Similarly, the minimum rate of change of f at $(1,3)$ is $-\|\nabla f(1,3)\| = -\sqrt{40} = -2\sqrt{10}$, which occurs in the direction of $u = -\frac{\nabla f(1,3)}{\|\nabla f(1,3)\|} = < \frac{-1}{\sqrt{10}}, \frac{-3}{\sqrt{10}} >$.

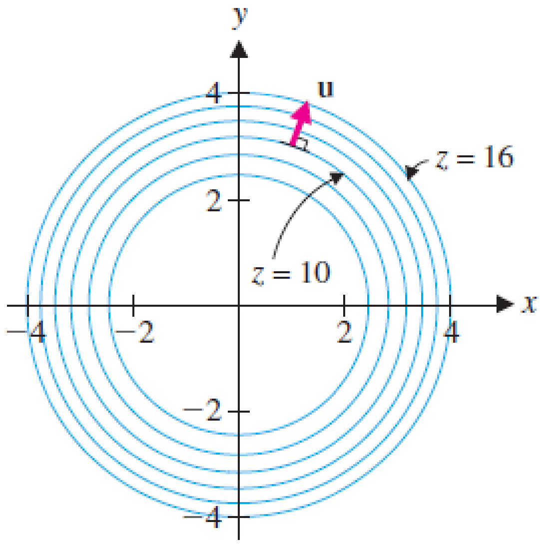


Figure2: Contour Plot of $z = x^2 + y^2$.

Example 4 (Finding the Direction of Maximum Increase)

If the temperature at point (x, y, z) is given by $T(x, y, z) = 85 + \left(1 - \frac{z}{100}\right)e^{-(x^2+y^2)}$, find the direction from the point $(2,0,99)$ in which the temperature increases most rapidly.

Solution

We first compute the gradient

$$\nabla f = < \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} >$$

$$= < -2x\left(1 - \frac{z}{100}\right)e^{-(x^2+y^2)}, -2y\left(1 - \frac{z}{100}\right)e^{-(x^2+y^2)}, \frac{-1}{100}e^{-(x^2+y^2)} >$$

and $\nabla f(2,0,99) = < \frac{-1}{25}e^{-4}, 0, \frac{-1}{100}e^{-4} >$. To find a unit vector in this direction, you can simplify the algebra by canceling the common factor of e^{-4} and multiplying by 100. A unit vector in the direction of $< -4, 0, -1 >$ and also in the direction of $\nabla f\ (2,0,99)$ is then $< \frac{-4}{\sqrt{17}}, 0, \frac{-1}{\sqrt{17}} >$.

<u>Theorem 4</u>

Suppose that $f(x,y,z)$ has continuous partial derivatives at the point (a,b,c) and $\nabla f(a,b,c) \neq 0$. Then, $\nabla f(a,b,c)$ is a normal vector to the tangent plane to the surface $f(x,y,c)=k$, at the point (a,b,c). Further, the equation of the tangent plane is $f_x(a,b,c)(x-a)+f_y(a,b,c)(y-b)+f_z(a,b,c)(z-c)=0$.

Example 5 (Using a Gradient to Find a Tangent Plane and Normal Line to a Surface)

Find equations of the tangent plane and the normal line to $x^3y - y^2 + z^2 = 7$ at the point $(1,2,3)$.

Solution

If we interpret the surface as a level surface of the function $f(x,y,z) = x^3y - y^2 + z^2$, a normal vector to the tangent plane at the point $(1,2,3)$ is given by $\nabla f\ (1,2,3)$. We have $\nabla f = < 3x^2y, x^3 - 2y, 2z >$ and $\nabla f\ (1,2,3) = < 6, -3, 6 >$. Given the normal vector $< 6, -3, 6 >$ and point $(1,2,3)$, an equation of the tangent plane is

$$6(x-1) - 3(y-2) + 6(z-3) = 0 \ .$$

The normal line has parametric equations $\begin{cases} x = 1+6t \\ y = 2-3t \\ z = 3+6t \end{cases}, t \in \mathbb{R}.$