12.4 First-order partial derivatives

In this section, we generalize the notion of derivative to functions of more than one variable.

First, recall that for a function f of a single variable, we define the derivative function as $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, for any values of x for which the limit exists.

At any particular value $x=a$, we interpret $f'(a)$ as the instantaneous rate of change of the function with respect to x at that point.

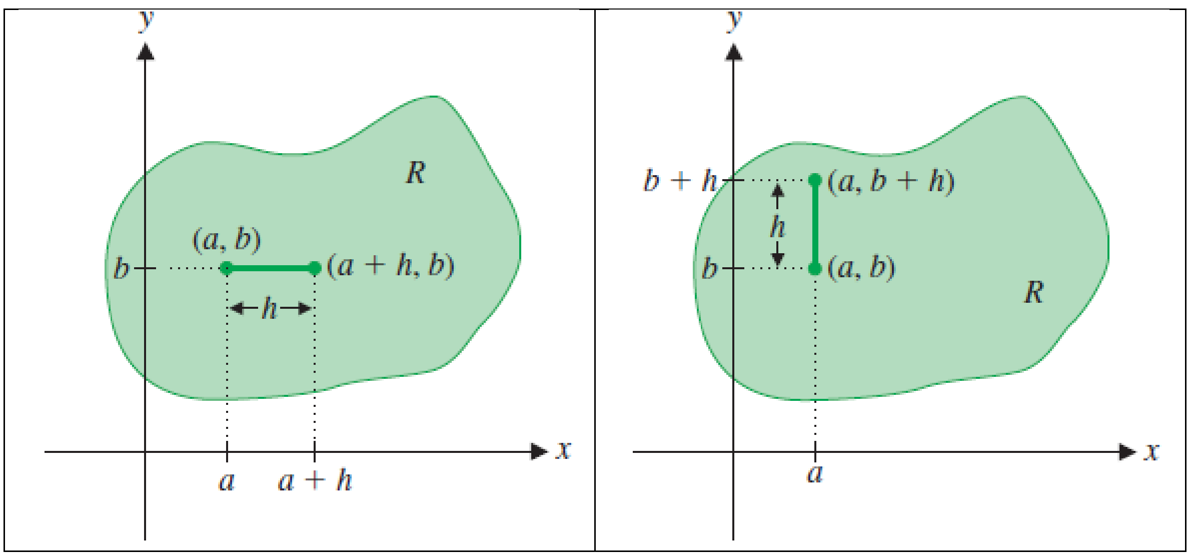
Definition 1

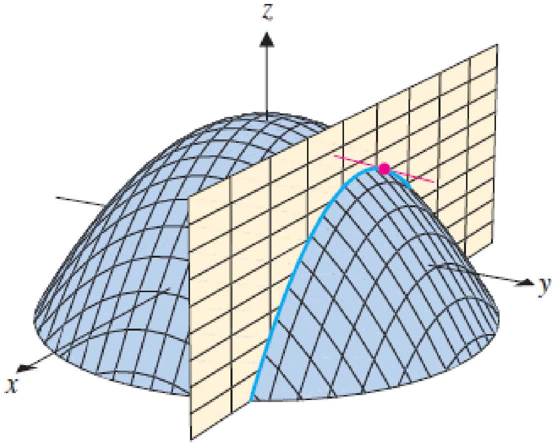
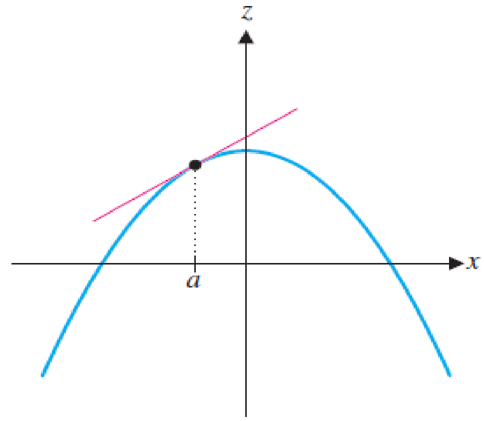
The ***partial derivative of*** $f(x,y)$ ***with respect to*** x *, written* $\frac{\partial f}{\partial x}$ *, is defined by*

$\frac{\partial f}{\partial x}(x,y)=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$*, for any values of* x *and* y *for which the limit exists.*

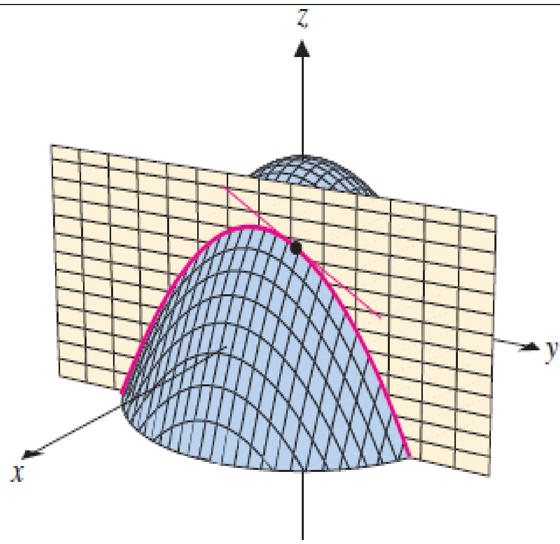
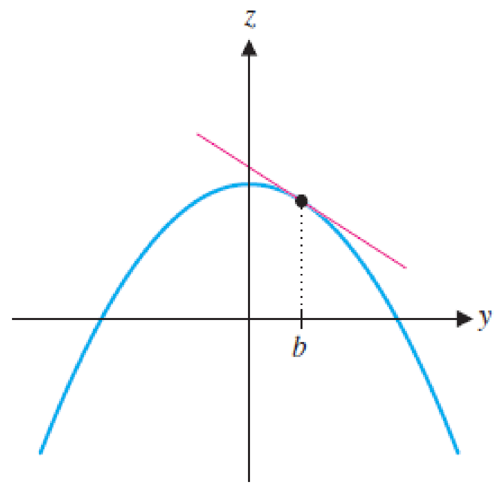
The ***partial derivative of*** $f(x,y)$ ***with respect to*** y *, written* $\frac{\partial f}{\partial y}$*, is defined by,*

$\frac{\partial f}{\partial y}(x,y)=\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$*, for any values of x and y for which the limit exists.*



 Intersection of the surface $z = f(x, y)$ with the plane $y = b$.	 The curve $z = f(x, b)$.

- $\dfrac{\partial f}{\partial x}(a,b)$ gives the slope of the tangent line to the curve at $x = a$.

 Intersection of the surface $z = f(x, y)$ with the plane $x = a$.	 The curve $z = f(a, y)$.

- $\dfrac{\partial f}{\partial y}(a,b)$ gives the slope of the tangent line to the curve at $y = b$.

Remark 1

- To compute the partial derivative $\frac{\partial f}{\partial x}$, you simply take an ordinary derivative with respect to x, while treating y as a constant. Similarly, you compute $\frac{\partial f}{\partial y}$ by taking an ordinary derivative with respect to y, while treating x as a constant.
- For $z = f(x,y)$, we write $\frac{\partial f}{\partial x}(x,y) = f_x(x,y) = \frac{\partial z}{\partial x}(x,y) = \frac{\partial}{\partial x}\left[f(x,y)\right]$.
- The expression $\frac{\partial}{\partial x}$ is a **partial differential operator.** It tells you to take the partial derivative (with respect to x) of whatever expression follows it. Similarly, we have

$$\frac{\partial f}{\partial y}(x,y) = f_y(x,y) = \frac{\partial z}{\partial y}(x,y) = \frac{\partial}{\partial y}\left[f(x,y)\right].$$

Example 1 (Computing Partial Derivatives)

For $f(x,y) = 3x^2 + x^3y + 4y^2$, compute $\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y), f_x(1,0)$ and $f_y(2,-1)$.

Solution

Compute $\frac{\partial f}{\partial x}$ by treating y as a constant. We have

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x}\left[3x^2 + x^3y + 4y^2\right] = 6x + 3x^2y.$$

The partial derivative of $4y^2$ with respect to x is 0, since $4y^2$ is treated as if it were a constant when differentiating with respect to x. Next, we compute $\frac{\partial f}{\partial y}$ by treating x as a constant. We have

$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial y}\left[3x^2 + x^3y + 4y^2\right] = x^3 + 8y.$$

Substituting values for x and y, we get $f_x(1,0) = \frac{\partial f}{\partial x}(1,0) = 6$ and $f_y(2,-1) = \frac{\partial f}{\partial y}(2,-1) = 0.$

Remark 2

Since we are holding one of the variables fixed when we compute a partial derivative, we have the product rules: $\frac{\partial}{\partial x}(uv) = \frac{\partial u}{\partial x}v + u\frac{\partial v}{\partial x}$ and $\frac{\partial}{\partial y}(uv) = \frac{\partial u}{\partial y}v + u\frac{\partial v}{\partial y}$

and the quotient rule: $\frac{\partial}{\partial x}\left(\frac{u}{v}\right)=\frac{\frac{\partial u}{\partial x}v-u\frac{\partial v}{\partial x}}{v^2}$,

with a corresponding quotient rule holding for $\frac{\partial}{\partial y}\left(\frac{u}{v}\right)=\frac{\frac{\partial u}{\partial y}v-u\frac{\partial v}{\partial y}}{v^2}$.

Example 2 (Computing Partial Derivatives)

For $f(x,y)=e^{xy}+\frac{x}{y}$, compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Solution

For $y\neq 0$, we have $\frac{\partial f}{\partial x}(x,y)=\frac{\partial}{\partial x}\left[e^{xy}+\frac{x}{y}\right]=y\,e^{xy}+\frac{1}{y}$. Also,

$$\frac{\partial f}{\partial y}(x,y)=\frac{\partial}{\partial y}\left[e^{xy}+\frac{x}{y}\right]=x\,e^{xy}-\frac{x}{y^2}.$$

Example 3 (Computing Partial Derivatives)

For $f(x,y,z)=\sin(x^2y^3z)+xy\ln z$, compute $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$.

Solution

For $z>0$, we have

$$\frac{\partial f}{\partial x}(x,y,z)=\frac{\partial}{\partial x}\left[\sin\left(x^2y^3z\right)+xy\ln z\right]=2xy^3z\cos(x^2y^3z)+y\ln z.$$

Also,

$$\frac{\partial f}{\partial y}(x,y,z)=\frac{\partial}{\partial y}\left[\sin\left(x^2y^3z\right)+xy\ln z\right]=3x^2y^2z\cos(x^2y^3z)+x\ln z.$$

And, $\frac{\partial f}{\partial z}(x,y,z)=\frac{\partial}{\partial z}\left[\sin\left(x^2y^3z\right)+xy\ln z\right]=x^2y^3\cos(x^2y^3z)+\frac{xy}{z}$.

12.5 Higher-order partial derivatives

Notice that the partial derivatives found in the preceding examples are themselves functions of two variables. We have seen that second- and higher-order derivatives of functions of a single variable provide much significant information. Not surprisingly, **higher-order partial derivatives** are also very important in applications.

For functions of two variables, there are four different second-order partial derivatives. The partial derivative with respect to x of $\frac{\partial f}{\partial x}$ is $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$, usually abbreviated as $\frac{\partial^2 f}{\partial x^2}$ or f_{xx} . Similarly, taking two successive partial derivatives with respect to y gives us $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}=f_{yy}$.

For **mixed second-order partial derivatives,** one derivative is taken with respect to each variable. If the first partial derivative is taken with respect to x, we have $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$, abbreviated as $\frac{\partial^2 f}{\partial y\,\partial x}$, or $(f_x)_y = f_{xy}$. If the first partial derivative is taken with respect to y, we have $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$, abbreviated as $\frac{\partial^2 f}{\partial x\,\partial y}$, or $(f_y)_x = f_{yx}$.

Example 1 (Computing Second-Order Partial Derivatives)

Find all second-order partial derivatives of $f(x,y) = x^2 y - y^3 + \ln x$.

Solution

We start by computing the first-order partial derivatives: For $x > 0$,

$\frac{\partial f}{\partial x}(x,y) = 2xy + \frac{1}{x}$ and $\frac{\partial f}{\partial y}(x,y) = x^2 - 3y^2$. We then have

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(2xy + \frac{1}{x}\right) = 2y - \frac{1}{x^2},$$

$$\frac{\partial^2 f}{\partial y\,\partial x}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(2xy + \frac{1}{x}\right) = 2x,$$

$$\frac{\partial^2 f}{\partial x\,\partial y}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(x^2 - 3y^2\right) = 2x,$$

and finally, $\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(x^2 - 3y^2\right) = -6y$.

Remark 1

Notice in example 1 that $\frac{\partial^2 f}{\partial y\,\partial x}(x,y) = \frac{\partial^2 f}{\partial x\,\partial y}(x,y)$. It turns out that this is true for most, but *not all*, of the functions that you will encounter.

Theorem 1

If $f_{xy}(x,y)$ and $f_{yx}(x,y)$ are continuous on an open set containing (a,b), then $f_{xy}(a,b) = f_{yx}(a,b)$.

Example 2 (Computing Higher-Order Partial Derivatives)

For $f(x,y) = \cos(xy) - x^3 + y^4$, compute f_{xyy} and f_{xyyy}.

Solution

We have $f_x = \frac{\partial}{\partial x}\left(\cos(xy) - x^3 + y^4\right) = -y\sin(xy) - 3x^2$.

Differentiating f_x with respect to y gives us

$$f_{xy} = \frac{\partial}{\partial y}\left(-y\sin(xy) - 3x^2\right) = -\sin(xy) - xy\cos(xy) \text{ and}$$

$$f_{xyy} = \frac{\partial}{\partial y}\left(-\sin(xy) - xy\cos(xy)\right)$$
$$= -x\cos(xy) - x\cos(xy) + x^2 y\sin(xy)$$
$$= -2x\cos(xy) + x^2 y\sin(xy).$$

Finally, we have

$$f_{xyyy} = \frac{\partial}{\partial y}\left(-2x\cos(xy) + x^2 y\sin(xy)\right)$$
$$= 2x^2\sin(xy) + x^2\sin(xy) + x^3 y\cos(xy)$$
$$= 3x^2\sin(xy) + x^3 y\cos(xy).$$

Example 3 (Partial Derivatives of Functions of Three Variables)

For $f(x,y,z) = \sqrt{xy^3 z} + 4x^2 y$, defined for $x,y,z \geq 0$, compute f_x , f_{xy} and f_{xyz} .

Solution

To keep x , y and z as separate as possible, we first rewrite f as

$$f(x,y,z) = x^{1/2} y^{3/2} z^{1/2} + 4x^2 y .$$

To compute the partial derivative with respect to x , we treat y and z as constants and obtain $f_x = \frac{\partial}{\partial x}\left[x^{1/2} y^{3/2} z^{1/2} + 4x^2 y\right] = \left(\frac{1}{2}x^{-1/2}\right) y^{3/2} z^{1/2} + 8xy .$

Next, treating x and z as constants, we get

$$f_{xy} = \frac{\partial}{\partial y}\left[\frac{1}{2}x^{-1/2} y^{3/2} z^{1/2} + 8xy\right] = \left(\frac{1}{2}x^{-1/2}\right)\left(\frac{3}{2}y^{1/2}\right) z^{1/2} + 8x .$$

Finally, treating x and y as constants, we get

$$f_{xyz} = \frac{\partial}{\partial z}\left[\frac{3}{4}x^{-1/2} y^{1/2} z^{1/2} + 8x\right] = \left(\frac{1}{2}x^{-1/2}\right)\left(\frac{3}{2}y^{1/2}\right)\left(\frac{1}{2}z^{-1/2}\right)$$
$$= \frac{3}{8}x^{-1/2} y^{1/2} z^{-1/2}.$$

Notice that this derivative is defined for $x,z > 0$ and $y \geq 0$. Further, you can show that all first-, second- and third-order partial derivatives are continuous for $x,y,z > 0$, so that the order in which we take the partial derivatives is irrelevant in this case.

12.6 Tangent planes and Linear approximations

Recall that the tangent line to the curve $y = f(x)$ at $x = a$ stays close to the curve near the point of tangency. This enables us to use the tangent line to approximate values of the function close to the point of tangency (see Figure 1).

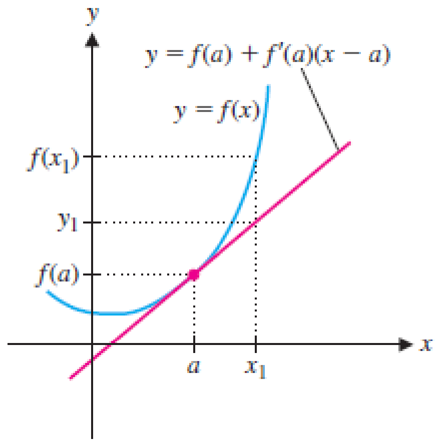


Figure1: Linear approximation.

The equation of the tangent line is given by: $y = f(a) + f'(a)(x - a)$. We called this the *linear approximation* to $f(x)$ at $x = a$.
In much the same way, we can approximate the value of a function of two variables near a given point using the tangent *plane* to the surface at that point. For instance, the graph of $z = 6 - x^2 - y^2$ and its tangent plane at the point $(1,2,1)$ are shown in Figure 2.

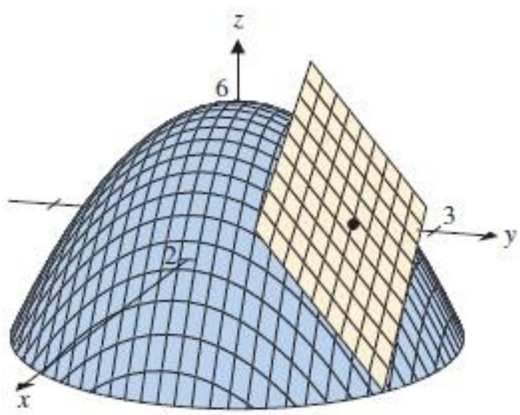


Figure2: $z = 6 - x^2 - y^2$ and the tangent plane at $(1,2,1)$.

Notice that near the point $(1,2,1)$, the surface and the tangent plane are very close together.

Theorem1

Suppose that $f(x,y)$ has continuous first partial derivatives at (a,b). A normal vector to the tangent plane to $z = f(x,y)$ at (a,b) is then $\left(f_x(a,b), f_y(a,b), -1\right)$.
Further, an equation of the tangent plane is given by
$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$ ***or***
$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$ **.**

Remark1

- A **vector normal** to the plane is then given by the cross product:
  $$(0,\ 1, f_y(a,b)) \quad \times \quad (0, f_x(a,b), 1) = (f_x(a,b), f_y(a,b), -1) \ .$$
- The line orthogonal to the tangent plane and passing through the point $(a, b, f(a,b))$ is given by $\begin{cases} x = a + t\, f_x(a,b) \\ y = b + t\, f_y(a,b) \\ z = f(a,b) - t \end{cases}$.

This line is called the **normal line** to the surface at the point $(a, b, f(a,b))$.

Example1 (Finding Equations of the Tangent Plane and the Normal Line)

Find equations of the tangent plane and the normal line to $z = 6 - x^2 - y^2$ at the point $(1,2,1)$.

Solution

For $f(x,y) = 6 - x^2 - y^2$, we have $f_x = -2x$ and $f_y = -2y$. This gives us $f_x(1,2) = -2$ and $f_y(1,2) = -4$. So a normal vector is then $(-2,-4,-1)$.

An equation of the tangent plane is: $z = 1 - 2(x-1) - 4(y-2)$.

Equations of the normal line are $\begin{cases} x = 1 + 2t \\ y = 2 - 4t\ , \quad t \in \mathbb{R}. \\ z = 1 - t \end{cases}$

A sketch of the surface, the tangent plane and the normal line is shown in Figure 3.

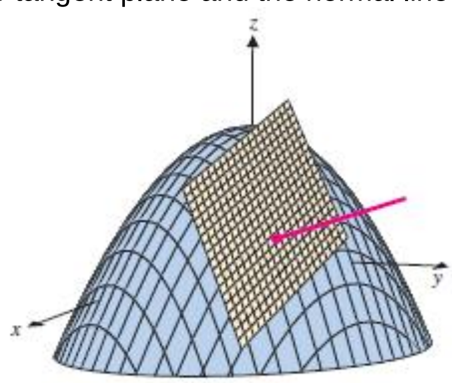


Figure3: Surface, tangent plane and normal line at the point $(1,2,1)$.

Example2 (Finding Equations of the Tangent Plane and the Normal Line)

Find equations of the tangent plane and the normal line to $z = x^3 + y^3 + \dfrac{x^2}{y}$ at the point $(2,1,13)$.

Solution

Here, $f_x = 3x^2 + \frac{2x}{y}$ and $f_y = 3y^2 - \frac{x^2}{y^2}$, so that $f_x(2,1) = 12 + 4 = 16$ and $f_y(2,1) = 3 - 4 = -1$. So a normal vector is then $(16,-1,-1)$.

An equation of the tangent plane is: $z = 13 + 16(x-2) - (y-1)$.

Equations of the normal line are $\begin{cases} x = 2 + 16t \\ y = 1 - t \quad , \quad t \in \mathbb{R}. \\ z = 13 - t \end{cases}$

A sketch of the surface, the tangent plane and the normal line is shown in Figure 4.

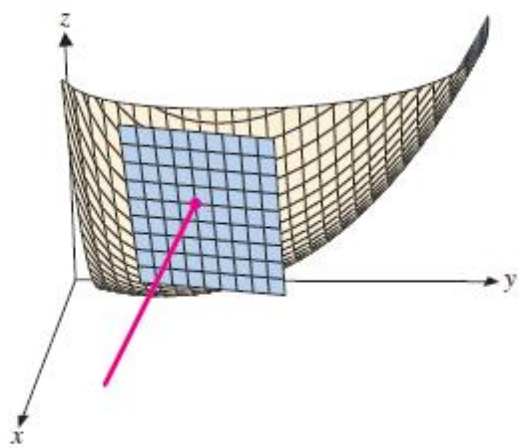


Figure4: Surface, tangent plane and normal line at the point $(2,1,13)$.