Lecture1.1. Basic definitions of matrices

Basic definitions of matrices

1.1.1 Matrix

A n x m matrix is a rectangular array arranged in n-rows and m-columns

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & . & . & a_{1m} \\ a_{21} & a_{22} & a_{23} & . & . & a_{2m} \\ a_{31} & a_{32} & a_{33} & . & . & a_{3m} \\ . & . & . & . & . & . \\ a_{n1} & a_{n2} & a_{n3} & . & . & a_{nm} \end{bmatrix}$$

a_{ij} is an element in ith row anf jth column.

Dimension of matrix.

The numbers of rows and ciolumns ofa matrix are called dimensions.

Examples.

A 2 x 3 *matrix* is rectangular array of objects, written in 2-rows and 3- columns. These objects can be numbers or functions.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -5 & \sqrt{2} & x \end{bmatrix}$$ is 2 x 3 matrix,

with $a_{11} = 2, a_{12} = 1, a_{13} = 0, a_{21} = -5, a_{22} = \sqrt{2}$ and $a_{32} = x$

$$B = \begin{bmatrix} 3 & 2 & 4 \\ 5 & 9 & 8 \\ -1 & 0 & 4 \end{bmatrix}$$ is 3 x 3 matrix

$C = [3]$ is 1 x 1 matrix

In a matrix information can be shown more clearly and in compact form.

Example.1.

Distances between major four cities of Kingdom of Saudi Arabia Riyadh, Makkah, Medina and Jeddah can be shown in following 4 x 4 matrix

	Riyadh	*Makkah*	*Madina*	*Jeddah*
Riyadh	0	794	718	847
Makkah	794	0	337	67
Madina	718	337	0	327
Jeddah	847	67	327	0

Example.2. Production of the plant in First week is shown in 3 x 4 matrix

	Plant 1	*Plant* 2	*Plant* 3	*Plant* 4
Pepsi	5000	5400	3000	1000
7Up	4000	3200	2000	500
Marinda	1000	3400	1000	600

1. **Size of a Matrix:** If a matrix A has n rows and m columns, then we say A is " n by m matrix" and we write it as " n x m"

Examples:

(i) $\begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}$ is 2 x 2 matrix (ii) $\begin{bmatrix} 0 & 1 & 2 \\ 9 & 7 & 4 \\ 3 & 5 & 1 \end{bmatrix}$ is 3x3 matrix

(iii) $\begin{bmatrix} 1 & x & x^2 & e^x \\ x+1 & 2 & 0 & 2x \\ 0 & 0 & 5 & x \end{bmatrix}$ is 3x4 matrix (3 rows x 4 columns)

2. **Square Matrix:** If $n = m$ that is number rows and columns are equal, then the matrix is square matrix.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},$$ 2x2 is a **square matrix**

If number of rows and columns are not equal ($n \neq m$) then matrix is called **Rectangular matrix**.

$$B = \begin{bmatrix} 2 & 3 & 1 & 0 \\ 3 & 1 & 0 & 7 \\ 1 & 1 & -1 & 5 \end{bmatrix}$$ is 3x 4 matrix

3. **Row Matrix:** Matrix with only one row and can contain any number of columns

$$B = \begin{bmatrix} 1 & 2 & 4 & 3 \end{bmatrix},$$ 1 x 4 is a **row matrix**

4. **Column Matrix:** Matrix with only one column and can contain any number of rows

$$C = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix},$$ 4x1 is a **column matrix**

5. **Zero Matrix:** A zero matrix is a matrix of nay order whose all entries are zero.

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$ is a **zero matrix**.

6. **Diagonal Matrix:** A square matrix with all its non- diagonal entries are zero.

Examples.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$$

7. Unit Matrix: A diagonal matrix with all diagonal entries are one '1'

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Lecture 1.2: Linear Equation and Different Forms

1.2.1 Linear Equation

An equation of straight line passing through origin is described as

$$y = m\,x \qquad .1$$

in which variable y is expressed in terms of x and the constant m, is called Linear Equation.

Note: In Linear Equation exponents (Power) of the variable is always ' one'.

Example:3

$2x + 3y = 5,\quad x - y = 2$ are linear equations in two variables x and y and are known as equations of line.

Example:4

$$2x + 3y + 4z = 5,\quad x - y + 2z = 2$$

are linear equations in three variables x, y and z and are known as equations of plane.

1.2.2 Linear Equation in *n* variables:

$$a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n = b \qquad .2$$

where $x_1, x_2, x_3, \dots, x_n$ are variables and

$a_1, a_2, a_3, \dots, a_n$ and b are constants are also called coefficients.

1.2.3 Linear System:

A linear system of m linear equations and n unknowns can be written as

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n = b_1\\
&a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ... + a_{2n}x_n = b_2\\
&a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + ... + a_{3n}x_n = b_3\\
&\dots\dots\dots\dots\dots\dots\dots\dots \qquad .3\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + ... + a_{mn}x_n = b_m
\end{aligned}
$$

where $x_1, x_2, x_3, ..., x_n$ are variables or unknowns and *a*'s and *b*'s are constants.

Example.1. A linear system of 4 linear equations and 3 unknowns is

$$
\begin{aligned}
&2x + 3y + 4z = 5\\
&x - 2y + z = 2\\
&3x + 7y + 3z = 3\\
&x + 3y + 4z = -2
\end{aligned}
$$

Example.2. A linear system of 3 linear equations and 3 unknowns is

$$
\begin{aligned}
&x + \sqrt{3}y + 4z = 0\\
&x - \frac{2}{3}y + z = -2\\
&3x + y + 3z = 3
\end{aligned}
$$

Different Ways of writing System of Linear Equations

1.2.4 Equation Form

System of linear equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_{31} = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_{31} = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_{31} = b_3$$

1.2.5 **Matrix Form**

can be written in the form of matrices product

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

1.2.6 **Matrix Equation Form**

or we may write it in the form AX=b,

where A= $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, X = $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, b = $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

1.2.7 **Augmented matrix Form**

Augmented matrix is $[A:b] = \left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$

Example:3. Write the matrix and augmented form of the system of linear equations

3x – y + 6z = 6
x + y + z = 2
2x + y +4z = 3

Solution:

1. Matrix form of the system is

$$\begin{bmatrix} 3 & -1 & 6 \\ 1 & 1 & 1 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix} \quad \Rightarrow \quad AX = B$$

2. Augmented form is $[A:b]=\begin{bmatrix} 3 & -1 & 6 & 6 \\ 1 & 1 & 1 & 2 \\ 2 & 1 & 4 & 3 \end{bmatrix}$.

Lecture 1.3 Solution of System of Linear Equations

1.3.1 Solution:

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ... + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + ... + a_{3n}x_n = b_3 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + ... + a_{mn}x_n = b_m \end{array} \qquad .3$$

Solution of the linear system (3) is a sequence of n numbers $s_1, s_2, s_3, ..., s_n$, which satisfies system (3) when we substitute $x_1 = s_1, x_2 = s_2, x_3 = s_3, ..., x_n = s_n$.

1.3.2 Solutions of the system of Linear Equations.

1. The system has exactly one solution, unique solution. **System is consistent.**
2. The system has an infinite many solutions. **System is consistent**
3. The system has no solution. **System is inconsistent**

Example.1. Only one solution

Solve the system of equations

$$x - 3y = -3 \qquad \mathbf{E_1}$$
$$2x + y = 8 \qquad \mathbf{E_2}$$

Solution: Eliminating x from both equations, we subtract twice E_1 from E_2

$-2E_1 + E_2 \Rightarrow$

$$\begin{array}{r} -2x + 6y = 6 \\ 2x + y = 8 \\ \hline +7y = 14 \end{array} \quad \Rightarrow \quad y = 2$$

From Eq. 1

$x = -3 + 3y$

$x = -3 + 6 = 3$

Solution is $x = 3$ and $y = 2$. System is consistent

Check Substitute the solution in Equations 1 and 2

Equation 1 $\Rightarrow$ 3 – 3(2) = 3 – 6 = -3

Equation 2 $\Rightarrow$ 2(3) +2 = 6 + 2 = 8 .

Example.2. No solution

Solve the system of equations

$$x - 3y = -7 \qquad \rightarrow 1$$
$$2x - 6y = 7 \qquad \rightarrow 2$$

Solution:

$2E_1 - E_2 \Rightarrow$

$$\begin{aligned} 2x - 6y &= -7 \\ -2x + 6y &= -14 \\ \hline 0 + 0 &= -21 \end{aligned}$$

This makes no sense as $0 \neq -21$, hence there is no solution. System is inconsistent

NOTE: **Consistent,** the system of equations is consistent if the system has at least one solution.

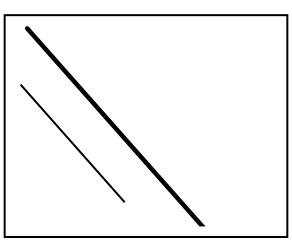
Inconsistent, the system of equations is inconsistent, if the system has no solution.

Example: ***Inconsistent and consistent system of equations***

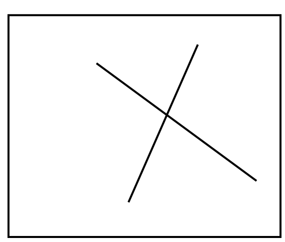
For the system of linear equations which is represented by straight lines:

$$a_1x - b_1y = c_1 \qquad \rightarrow l_1$$
$$a_2x - b_2y = c_2 \qquad \rightarrow l_2$$

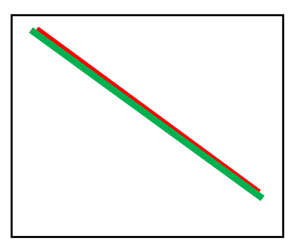
There are three possibilities:



No solution
[inconsistent]



one solution
[consistent]



infinite many
[consistent]

Note:1. A system will have unique solution (only one solution) when number of unknowns is equal to number of equations.

Note:2. A system is over determined, if there are more equations then unknowns and it will be mostly inconsistent.

Note:3. A system is under determined if there are less equations then unknowns and it may turn inconsistent.

Example.3. Consider the system of linear equations

$$x + 2y = 2 \qquad \mathbf{E_1}$$
$$2x + 4y = 4 \qquad \mathbf{E_2}$$

Solution. Adding multiple E_1 by –2 to E_2

$$-2x - 4y = -4 \qquad E_1$$
$$2x + 4y = 4 \qquad E_2$$

$$0 = 0 \quad \Rightarrow \quad x + 2y = 2$$
$$x = 2 - 2y$$

Let $y = 2 \Rightarrow x = 2 - 4 = -2$, solution is $x = -2$ and $y = 2$

There is solution for each value of y,

hence system has **infinite** many solutions. Such system is known as consistent.

Example.4. **Infinite many solutions**

Find the solution of $4x - 2y = 1$

Solution.

we can assign an arbitrary value to x and solve for y , or choose an arbitrary value for y and solve for x .If we follow the first approach and assign x an arbitrary value ,we obtain

$$x = t_1, \; y = 2t_1 - \frac{1}{2} \qquad \textbf{or} \qquad x = \frac{1}{2}t_2 + \frac{1}{4}, \; y = t_2$$

arbitrary numbers $t_1, \; t_2$ are called parameter.

for example $t_1 = 2$ yields the solution $x = 2, \; y = \frac{7}{2}$