Lecture 3.1: Algebra of matrix

Basic definitions of matrices are given in Lecture 1.

3.1.1 Properties of a matrix

1. **Transpose of a Matrix:** A transpose of a matrix is obtained by interchanging rows and corresponding columns of the given matrix. The transpose of the matrix A is denoted A^t .

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad A^t = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Properties of the Transpose of a matrix

1. $(A^t)^t = A$
2. $(AB)^t = B^t A^t$
3. $(kA)^t = kA^t$, where k is a scalar.
4. $(A+B)^t = A^t + B^t$

2. Symmetric Matrix:
A square matrix is symmetric if $A^t = A$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}, \quad A^t = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}, \quad A^t = A$$

3. Skew – symmetric Matrix :
A square matrix is skew symmetric if $A^t = - A$.

$$A = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & -4 \\ 3 & 4 & 0 \end{bmatrix}, \quad A^t = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}, \quad A^t = -A.$$

4. Equality of matrix:
Two matrices are equal, if these of same size and corresponding entries are equal.

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

A and B are equal matrices when these of the same size and corresponding entries are equal.

Example:1. Write down the system of equation, if matrices A and B are equal

$$A = \begin{bmatrix} x-2 & y-3 \\ x+y & z+3 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 3+z \\ z & y \end{bmatrix}$$

Solution: A and B are of the same size, hence

$$A = B \Rightarrow$$

$$\begin{aligned} x-2 &= 1 \\ y-3 &= 3+z \\ x+y &= z \\ z+3 &= y \end{aligned}$$

System of equations are

$$\begin{aligned} x \quad\quad &= 3 \\ y-z &= 6 \\ x+y-z &= 0 \\ -y+z &= -3 \end{aligned}$$

3.1.2 Addition of matrices:

Matrices of the equal size can be added entry wise.

Example:2. **Add the following matrices:**

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & 5 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 2 & 8 \\ 2 & 4 & 1 \end{bmatrix}$$

Solution. We need to add the pairs of entries, and then simplify for the final answer:

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & 5 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 2 & 8 \\ 2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1+4 & 0+2 & 2+8 \\ 3+2 & 5+4 & 4+1 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 10 \\ 5 & 9 & 5 \end{bmatrix}$$

So the answer is:

$$\begin{bmatrix} 5 & 2 & 10 \\ 5 & 9 & 5 \end{bmatrix}$$

Example:3. Find the value of x and y in the following matrix equation

$$\begin{bmatrix} 5 & x \\ 3y & 2 \end{bmatrix} + \begin{bmatrix} -3 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 5 & 7 \end{bmatrix}$$

Solution. Using concept of addition of matrices, we simplify left hand side

$$\begin{bmatrix} 5-3 & x+2 \\ 3y-1 & 2+5 \end{bmatrix} = \begin{bmatrix} 2 & x+2 \\ 3y-1 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 5 & 7 \end{bmatrix}$$

Two matrices are equal when their correspoding entries are equal

x + 2 = 4

2y – 1 = 5

Solving these equations

x = 4 – 2 =2

3y = 5+1

3y = 6,   y = 2

Solution of matrix equation is x = 2, y = 2.

3.1.3 Scalar Multiplication:

If a matrix is multiplied by a scalar α, then each entry is multiplied by scalar α.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}, \quad 2A = 2\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}, \quad 2A = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 2 & 0 \\ 2 & 2 & 4 \end{bmatrix}$$

$$3A = \begin{bmatrix} 3 & 6 & 9 \\ 6 & 3 & 0 \\ 3 & 3 & 6 \end{bmatrix}$$

3.1.4 Matrix Multiplication:

The product of two matrices A and B is possible if the number of columns of A is equal to number of rows in B, the method is being explained by following example:

$$A=\begin{bmatrix}1 & 2 & 4\\ 2 & 6 & 0\end{bmatrix}_{2x3}, \quad B=\begin{bmatrix}4 & 1 & 4 & 3\\ 0 & -1 & 3 & 1\\ 2 & 7 & 5 & 2\end{bmatrix}_{3X4}$$

A x B = C

2x3 3x4 2x4

$$AB=\begin{bmatrix}c_{11} & c_{12} & c_{13} & c_{14}\\ c_{21} & c_{22} & c_{23} & c_{24}\end{bmatrix}$$

c_{11} = 1x4 + 2x0 + 4x2 = 4 + 0 + 8 = 12
c_{12} = 1x1 + 2x(-1) + 4x7 = 1 - 2 + 28 = 27
c_{13} = 1x4 + 2x3 + 4x5 = 4 + 6 + 20 = 30
c_{14} = 1x3 + 2x1 + 4x2 = 3 + 2 + 8 = 13
c_{21} = 2x4 + 6x0 + 0x2 = 8 + 0 + 0 = 8
c_{22} = 2x1 + 6x(-1) + 0x7 = 2 -6 + 0 = -4
c_{23} = 2x4 + 6x3 + 0x5 = 8 + 18 + 0 = 26
c_{24} = 2x3 + 6x1 + 0x2 = 6 + 6 + 0 = 12

$$AB=\begin{bmatrix}12 & 27 & 30 & 13\\ 8 & -4 & 26 & 12\end{bmatrix}$$

NOTE: AB $\neq$ BA

Lecture 3.2 : Inverse of matrix and power of matrix

3.2.1 Inverse of a 2x2 matrix

Consider a 2x2 matrix A = $\begin{bmatrix}a & b\\ c & d\end{bmatrix}$

If ad – bc $\neq$ 0, then $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$

Note: Multiple $(ad-bc)$ is called the **determinant** of matrix A = $\begin{bmatrix}a & b\\ c & d\end{bmatrix}$

Example: Find inverse of matrix A = $\begin{bmatrix}3 & 2\\ 4 & 5\end{bmatrix}$

ad - bc = 3x5 –2x4 =15 – 8 = 7

$A^{-1} = \frac{1}{7}\begin{bmatrix}5 & -2\\ -4 & 3\end{bmatrix}$.

Properties of Inverse

1. $A^{-1}A = A A^{-1} = I$
2. If A and B are invertible matrices of the same size , then AB is also invertible and
   $(AB)^{-1} = B^{-1}A^{-1}$

3.2.2 <u>Power of a matrix</u>

1. $A^0 = I$
2. $A^n = A.A.A\ldots A$ (n-factors), where n>0.
3. $A^{-n} = (A^{-1})^n = A^{-1}.A^{-1}.A^{-1}....A^{-1}$ (n- factors) , where n>0.
4. $A^rA^s = A^{r+s}$
5. $(A^r)^s = A^{rs}$
6. $(A^{-1})^{-1} = A$
7. $(A^n)^{-1} = (A^{-1})^n, \quad n = 0,1,2,...$
8. $(kA)^{-1} = \frac{1}{k}A^{-1}$, where k is a scalar.

Example:4. Let A be an invertible matrix and suppose that inverse of 7A is $\begin{bmatrix} -2 & 7 \\ 1 & -3 \end{bmatrix}$, find matrix A

Solution: $(7A)^{-1} = \frac{1}{7}A^{-1} = \begin{bmatrix} -2 & 7 \\ 1 & -3 \end{bmatrix}$

$$A^{-1} = 7\begin{bmatrix} -2 & 7 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} -14 & 49 \\ 7 & -21 \end{bmatrix}$$

$$A = (A^{-1})^{-1} = -\frac{1}{49}\begin{bmatrix} -21 & -49 \\ -7 & -14 \end{bmatrix} = \frac{7}{49}\begin{bmatrix} 3 & 7 \\ 1 & 2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3 & 7 \\ 1 & 2 \end{bmatrix}$$

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Example:5. Let A be a matrix $\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$ compute A^3, A^{-3}, $A^2 - 2A + I$.

Solution:

$$A^2 = AA = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 28 & 1 \end{bmatrix}$$

$$A^{-3} = (A^3)^{-1} = \frac{1}{8}\begin{bmatrix} 1 & 0 \\ -28 & 8 \end{bmatrix}$$

$$A^2 - 2A + I = \begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 8 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix}$$

Example:6. Find inverse of the matrix

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Solution:

ad – bc = $\cos^2\theta + \sin^2\theta = 1$,

$$A^{-1} = \frac{1}{1}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Lecture 3.3 Inverse by Elementary Matrix

3.3.1 Elementary Matrix

An nxn matrix is called *elementary matrix*, if it can be obtained from nxn identity matrix by performing a single elementary row operation.

Examples: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a 3x3 identity matrix.

Elementary matrices E_1,E_2 and E_3 can be obtained by single row operation.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \text{-3}R_3$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -2 \end{bmatrix} \text{-2}R_3\text{+}R_2$$

$$E_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

NOTE:

When a matrix A is multiplied from the left by an elementary matrices E, the effect is same as to perform an elementary row operation on A.

Example: 1.

Let A be a 3x4 matrix, $A=\begin{bmatrix}1 & 0 & 2 & 3\\ 2 & -1 & 3 & 6\\ 1 & 4 & 4 & 0\end{bmatrix}$ and

E be 3x3 elementary matrix obtained by row operation $3R_1+R_3$ from an Identity matrix

$$E = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 3 & 0 & 1\end{bmatrix}$$

$$EA= \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 3 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 2 & 3\\ 2 & -1 & 3 & 6\\ 1 & 4 & 4 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 2 & 3\\ 2 & -1 & 3 & 6\\ 4 & 4 & 10 & 9\end{bmatrix}, 3R_1+R_3 .$$

<u>3.3.2 Method for finding Inverse of a matrix</u>

To find the inverse of an invertible matrix, we must find a sequence of elementary row operations that reduces A to the identity and then perform this same sequence of operations on I_n to obtain A^{-1}.

$$\left[A \mid I\right] \text{ to } \left[I \mid A^{-1}\right]$$

Example:2. Find inverse of a matrix $A = \begin{bmatrix}1 & 4\\ 2 & 7\end{bmatrix}$ by using Elementary matrix method.

Solution:

$$[A|I] = \left[\begin{array}{cc|cc}1 & 4 & 1 & 0\\ 2 & 7 & 0 & 1\end{array}\right]$$

$$\approx \left[\begin{array}{cc|cc}1 & 4 & 1 & 0\\ 0 & -1 & -2 & 1\end{array}\right] \quad -2R_1 + R_2$$

$$\approx \left[\begin{array}{cc|cc}1 & 4 & 1 & 0\\ 0 & 1 & 2 & -1\end{array}\right] \quad -R_2$$

$$\approx \left[\begin{array}{cc|cc}1 & 0 & -7 & 4\\ 0 & 1 & 2 & -1\end{array}\right] \quad -4R_2 + R_1$$

$$= \left[I \middle| A^{-1}\right]$$

$$A^{-1} = \begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix}$$

Example:3. Use Elementary matrix method to find inverses of

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix} \qquad \text{if A is invertible.}$$

Solution:

$$[A|I] = \left[\begin{array}{ccc|ccc} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right]$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 3 & 4 & -1 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right] R_1 \leftrightarrow R_2$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right] -3R_1 + R_2, -2R_1 + R_3$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 1 & 0 & -1 & -2 & 1 \end{array}\right] -R_2 + R_3$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & \frac{1}{2} & \frac{-7}{10} & \frac{-2}{5} \end{array}\right] \quad R_2 \leftrightarrow R_3, \frac{(-4R_3 + R_2)}{10}$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & \frac{-11}{10} & \frac{-6}{5} \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & \frac{-1}{2} & \frac{7}{10} & \frac{2}{5} \end{array}\right] \quad -3R_3 + R_1, -R_3$$

$$\approx [I|A^{-1}]$$

$$A^{-1} = \begin{bmatrix} \frac{3}{2} & \frac{-11}{10} & \frac{-6}{5} \\ -1 & 1 & 1 \\ \frac{-1}{2} & \frac{7}{10} & \frac{2}{5} \end{bmatrix}.$$