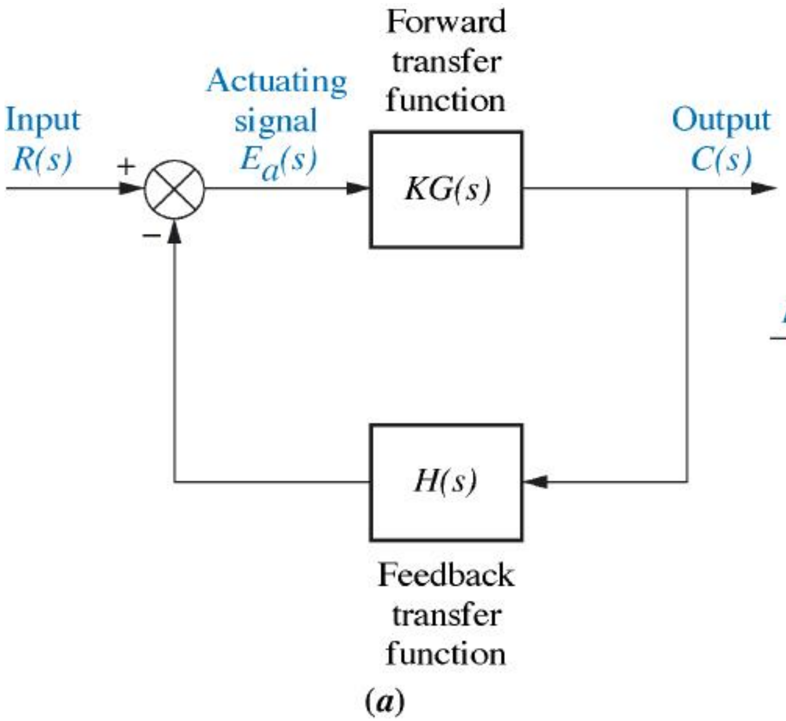
Design via Root Locus

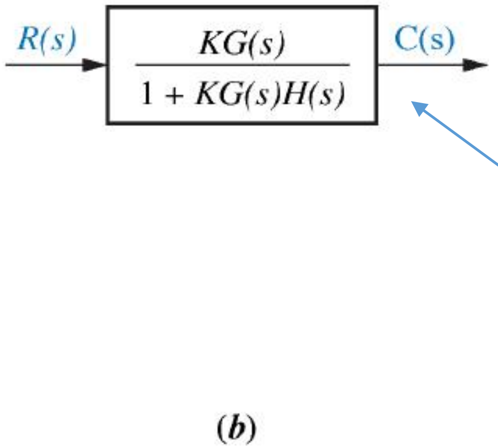
1. Root Locus.
2. Compensator design via Root Locus.
3. Physical Realization of Compensation.

Root Locus Techniques

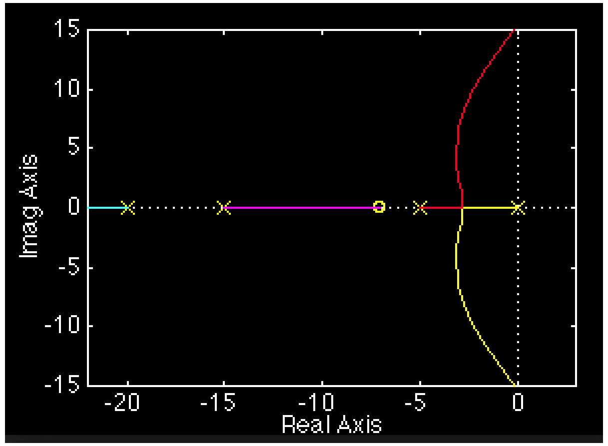
- Root locus is a graphical presentation of the closed-loop poles as a system parameter k is varied.
- The graph of all possible roots of this equation (K is the variable parameter) is called the root locus.
- The root locus gives information about the stability and transient response of feedback control systems.



Closed-loop system



Equivalent Transfer Function



Rout-locus (poles motion graph)

Zeros of T(s) are zeros of G(s) and poles of H(s).

Poles of T(s) depends on gain K

CLCF is a function of K.
Root Locus graphically shows poles of T(s) as *K* varies

Evaluation of a Complex Function via Vectors

Any *complex number*, $\sigma + j\omega$, described in Cartesian coordinates can be graphically represented by a vector,

Magnitude and phase Of F(s) at S

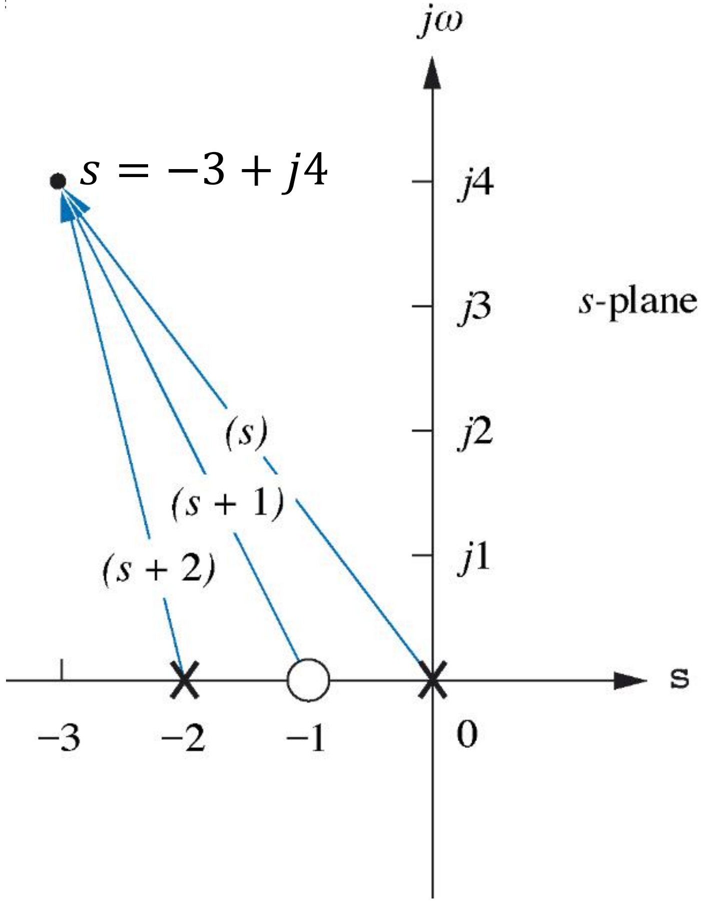
If $F(s) = \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}$ $\Rightarrow$

$$M = \frac{\prod zero\ lengths}{\prod pole\ lengths} = \frac{\prod_{i=1}^{m}|(s+z_i)|}{\prod_{j=1}^{n}|(s+p_j)|}$$

$$\theta = \sum zero\ angles - \sum pole\ angles = \sum_{i=1}^{m} \measuredangle (s+z_i) - \sum_{j=1}^{n} \measuredangle (s+p_j)$$

Problem: Given $F(s) = \frac{(s+1)}{s(s+2)}$ Find F(s) at the point $s = -3 + j4$

Solution: Any complex number can be represented by a vector



For zero (point $s_1 = -1$) the vector is:

$$s - s_1 = s - (-1) = (-3 + j4) - (-1) = -2 + j4 = \sqrt{(-2)^2 + (4)^2} tan^{-1}\left(\frac{4}{-2}\right) = \sqrt{20}\angle 116.6^\circ$$

Magnitude and phase (polar form)

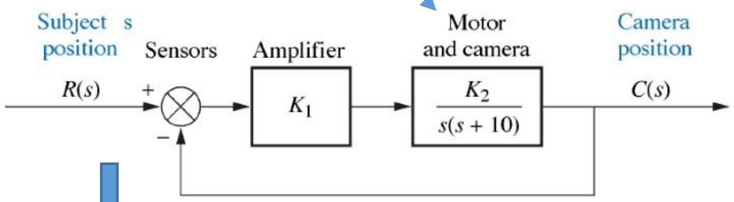
For pole at 0: $5\angle 126.9^\circ$ For pole at -2: $\sqrt{17}\angle 104.0^\circ$

Vector magnitude, $M = \frac{\sqrt{20}}{5\sqrt{17}} = 0.217$

Vector angle, $\theta = \angle(116.6^\circ - 126.9^\circ - 104.0^\circ)$

Vector $M\angle\theta = 0.217\angle -114.3^\circ$

Defining the Root Locus



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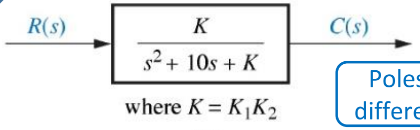


where $K = K_1K_2$

(c)

Poles location for different values of K

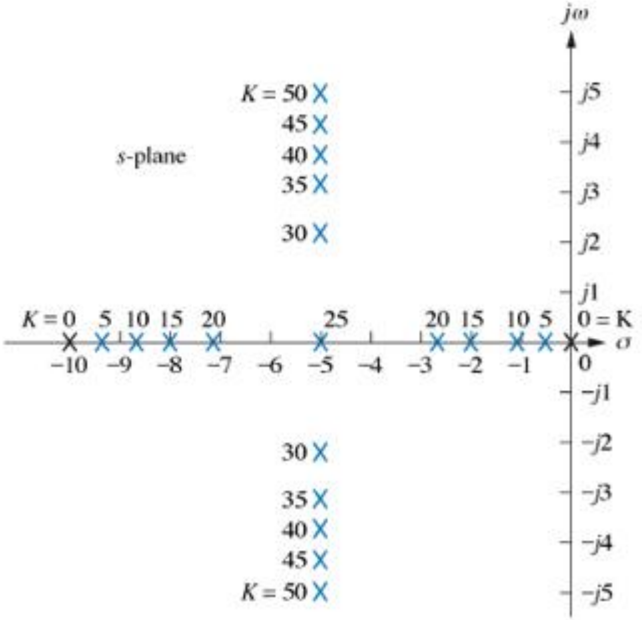
K	Pole 1	Pole 2
0	-10	0
5	-9.47	-0.53
10	-8.87	-1.13
15	-8.16	-1.84
20	-7.24	-2.76
25	-5	-5
30	$-5 + j2.24$	$-5 - j2.24$
35	$-5 + j3.16$	$-5 - j3.16$
40	$-5 + j3.87$	$-5 - j3.87$
45	$-5 + j4.47$	$-5 - j4.47$
50	$-5 + j5$	$-5 - j5$

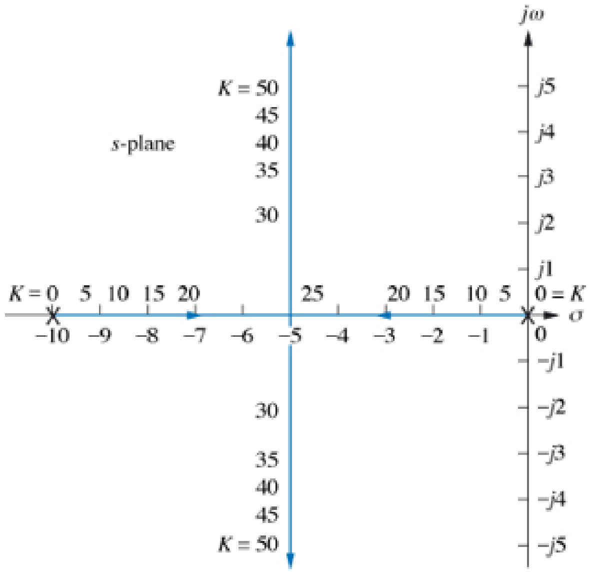
Gain less than 25, over-damped.

Gain = 25, critically damped.

Gain over 25, under-damped.

Stable system, as no pole on right-hand plane.





During underdamped, real parts are same; so settling time (which is related to real part) remains the same.

Damping frequency (imaginary part) increases with gain, resulting in reduction of peak time.

Properties of Root Locus

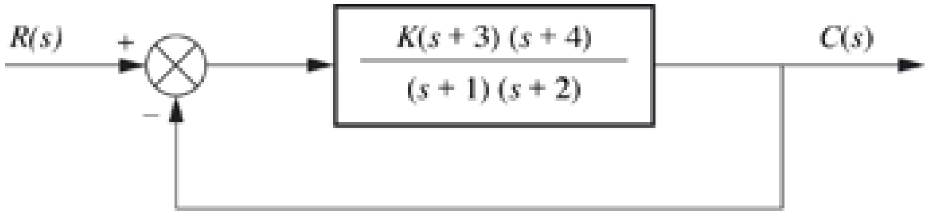
The closed-loop transfer function $T(s) = \frac{KG(s)}{1+KG(s)H(s)}$

Module condition

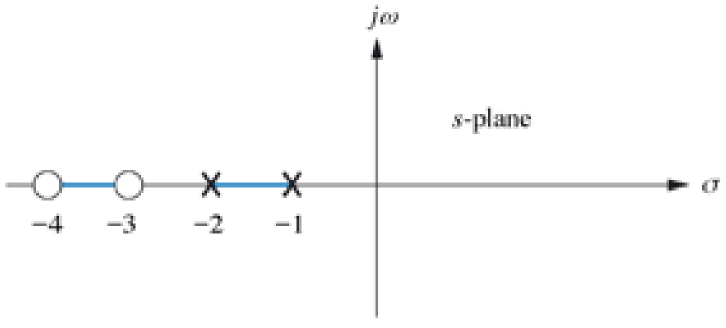
Angle condition

s_0 is a pole if $1 + KG(s_0)H(s_0) = 0 \Rightarrow KG(s_0)H(s_0) = -1 = 1\angle(2k+1)180^0 \quad k = 0, \pm 1, \pm 2, \ldots$

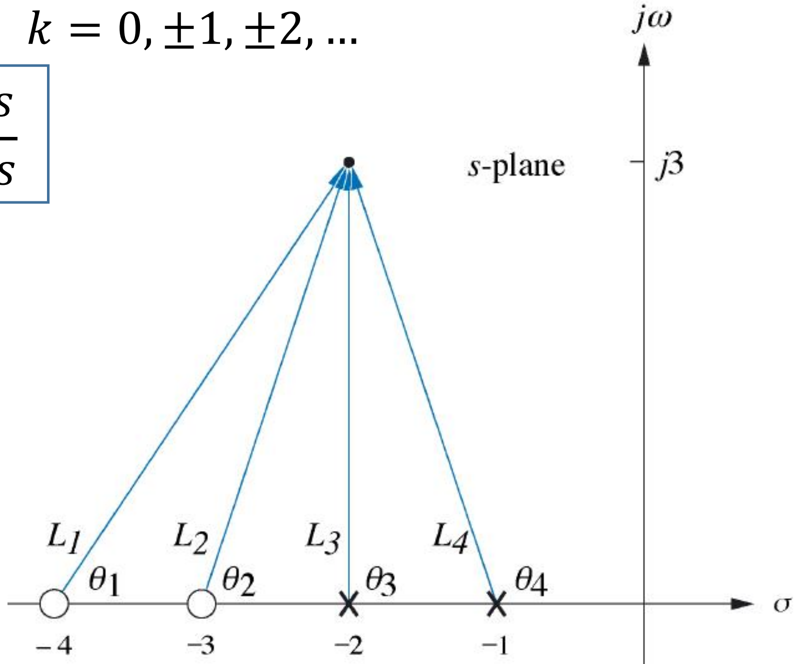
$$KG(s)H(s) = -1 \Rightarrow |KG(s)H(s)| = 1 \Rightarrow \boxed{K = \frac{1}{|G(s)||H(s)|} = \frac{\Pi\ pole\ lengths}{\Pi\ zero\ lengths}}$$



(a)



(b)



Find if the point -2+*j*3 is on root locus for some value of gain, K:

From the angle condition

Σ zero angle - Σ pole angle

$$\theta_1 + \theta_2 - \theta_3 - \theta_4 = 56.31° + 71.57° - 90° - 108.43° = -70.55°$$

Not a multiple of 180^0. So, $-2 + j3$ is not in the root locus (can not be a pole for some value of K).

For the point $-2 + j(\sqrt{2}/2)$ which is on root locus, the gain K is:

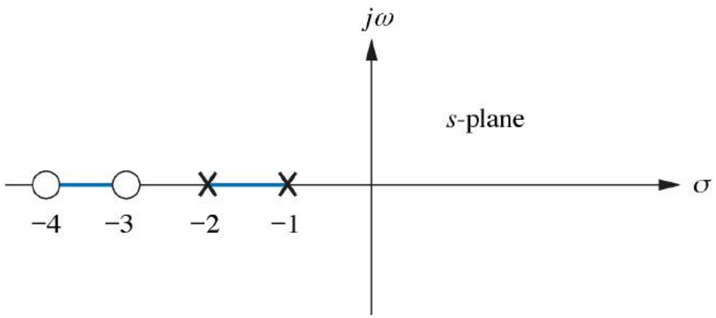
$$K = \frac{\prod pole\ lengths}{\prod zero\ lengths} = \frac{L_3 L_4}{L_1 L_2} = \frac{\frac{\sqrt{2}}{2}(1.22)}{(2.12)(1.22)} = 0.33$$

Sketching the Root Locus[1]

1.Number of branches: Equals the number of closed loop poles.

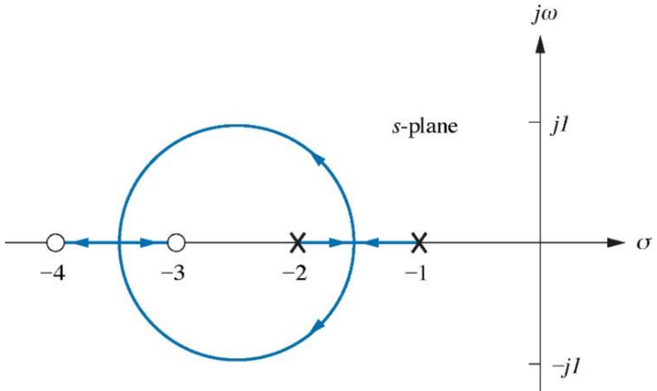
2.Symmetry: Symmetrical about the real axis (conjugate pairs of poles, real coefficients of the characteristic equation polynomial).

3.Real axis segments: For K > 0, root locus exists to the left of an odd number real axis poles and/or zeros (angle condition).



Real-axis segments of the root locus

4.Start and end points: The root locus begins at finite and infinite poles of $G(s)H(s)$ and ends at finite and infinite zeros of $G(s)H(s)$.



Complete root locus for the system

5.Asymptotes: The root locus approaches straight lines as asymptotes as the locus approaches infinity. the equation of the asymptotes is given by:

$$\sigma_a = \frac{\sum finite\ poles - \sum finite\ zeros}{\neq finite\ poles - \neq finite\ zeros}$$

Intersection with Real axis

$$\theta_a = \frac{(2k+1)\pi}{\neq finite\ poles - \neq finite\ zeros} \quad for\ k = 0, \pm 1, \pm 2, \ldots$$

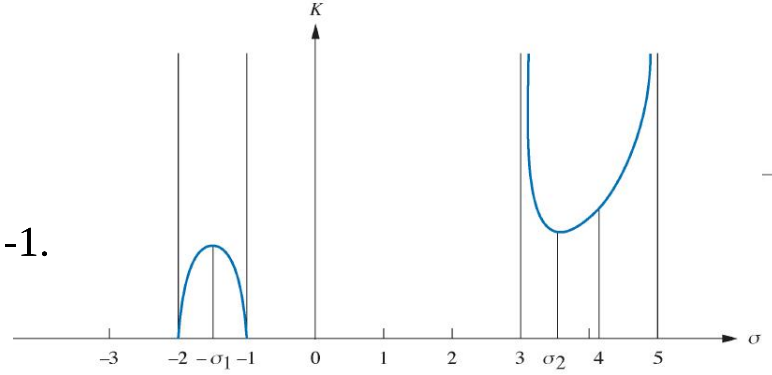
Angle in radian of the asymptote with real axis

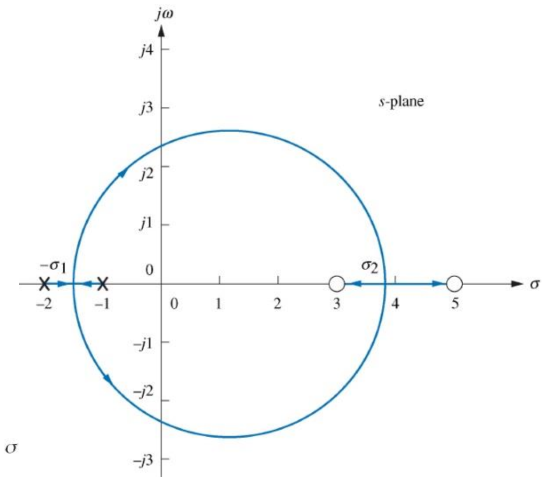
6.Real-Axis Breakaway and Break-in Points:

σ_1: Breakaway point (leave the real axis);
σ_2: Break-in point (return to the real axis); .

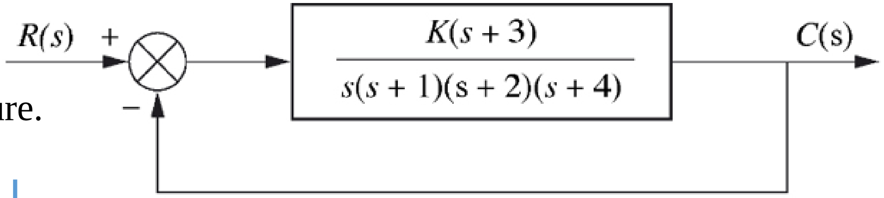
Breakaway point: at *maximum gain on the real axis* between -2 and -1.

Break-in point: at *minimum gain on real axis* (increases when moving towards a zero) between +3 and +5.





Sketching the Root Locus₂



Problem1: *Sketch the Root Locus* for the system shown in the following figure.

Solution1: Calculate asymptotes to find real axis intercept:

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}} = \frac{(-1-2-4)-(-3)}{4-1} = -\frac{4}{3} \quad and$$

- The angles of the lines that intersect at $^{-4}/_{3}$ is given by θ_a:
- For higher values of k, the angles would begin to repeat.

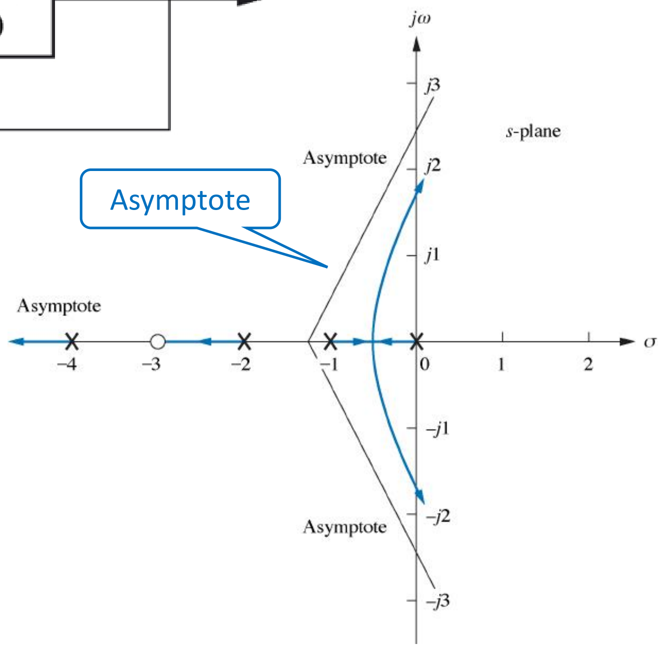
$$\theta_a = \frac{(2k+1)\pi}{\#\text{finite poles} - \#\text{finite zeros}}$$

$$= \pi/3 \quad \text{for } k=0$$

$$= \pi \quad \text{for } k=1$$

$$= 5\pi/3 \quad \text{for } k=2$$

- There are four poles and one finite zero. Root locus begins at poles and ends at zeros.(Three zeros at infinity are at the ends of the asymptotes.)



Problem2: From root-locus graph on figure *find break-in and break-away points*

Method 1:*(transition method)*

$$\sum_{1}^{m} \frac{1}{\sigma + z_i} = \sum_{1}^{n} \frac{1}{\sigma + p_i}$$

z_i and p_i are negative of zero and pole values, respectively, of G(s)H(s).

From figure, we get

$$\frac{1}{\sigma-3} + \frac{1}{\sigma-5} = \frac{1}{\sigma+1} + \frac{1}{\sigma+2}$$

$$\Rightarrow 11\sigma^2 - 26\sigma - 61 = 0$$

$$\Rightarrow \sigma = -1.45, \quad 3.82$$

Method 2: *(Differentiation method)*

$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)} = \frac{K(s^2-8s+15)}{(s^2+3s+2)}$$

Along the real axis ($s = \sigma$) *and* $KG(s)H(s) = -1$

$$\frac{K(s^2-8s+15)}{(s^2+3s+2)} = -1$$

Solving for K

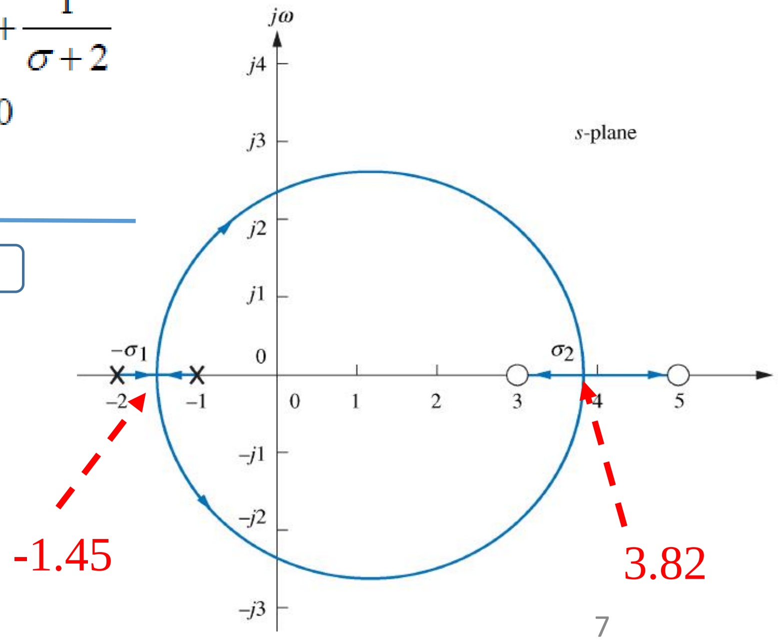
$$K = \frac{-(\sigma^2+3\sigma+2)}{(\sigma^2-8\sigma+15)}$$

Differentiating K with respect to σ (max and min)

$$\frac{dK}{d\sigma} = \frac{(11\sigma^2 - 26\sigma - 61)}{(\sigma^2-8\sigma+15)^2} = 0$$

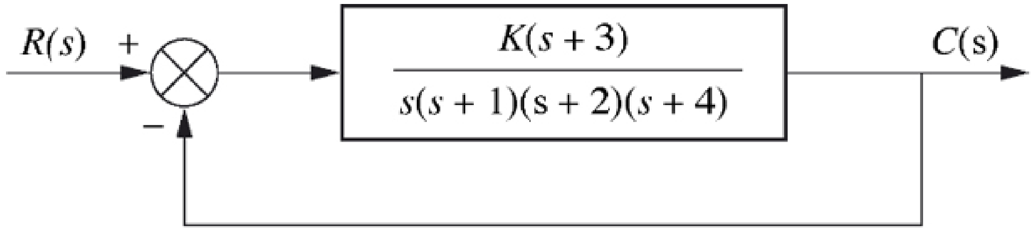
Solving for σ,

$$\sigma = -1.45 \quad and \quad \sigma = 3.82$$



Sketching the Root Locus$_3$

7. Imaginary-Axis Crossing



Stability: the system's poles are in the left half-plane up to a particular value of gain K.

PROBLEM: For the system, find the frequency and gain, K, for which the root locus crosses the imaginary axis. For what range of K is the system stable?

SOLUTION: The closed-loop transfer function

$$T(s) = \frac{K(s+3)}{s^4 + 7s^3 + 14s^2 + (8+K)s + 3K}$$

Characteristic Eq.: $s(s+1)(s+2)(s+4) + K(s+3)$

$$= s^4 + 7s^3 + 14s^2 + (8+K)s + 3K = 0$$

We get Routh table as follows:

s^4	1	14	$3K$
s^3	7	$8+K$	
s^2	(90-K)/7	$3K$	
s^1	$\dfrac{-K^2 - 65K + 720}{90 - K}$		
s^0	$3K$		

A complete *row of zeros* yields the possibility for imaginary-axis roots.

For $K > 0$, only s^1 row can be zero.

$$-K^2 - 65K + 720 = 0$$
$$\Rightarrow K = 9.65$$

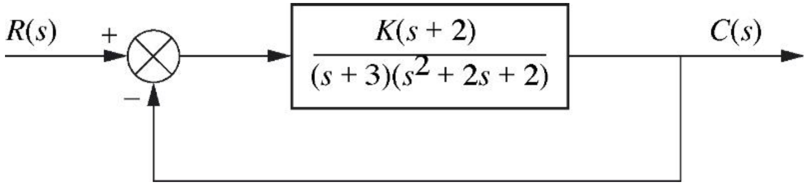
Forming the even polynomial by using the s^2 row (above) with $K = 9.65$,

$$(90-K)s^2 + 21K = 80.35s^2 + 202.7 = 0 \quad \text{Gives} \quad s = \pm j1.59$$

Thus, the root locus crosses the imaginary-axis at $\omega_d = \pm j1.59$ at a gain of K= 9.65 So, the system is stable for $0 \le K < 9.65$

8. Angles of Departure and Arrival

Departure: from complex poles. ***Arrival***: to complex zeros.



Angle of departure: Zero: -2 Poles:-3, -1+j, -1- j

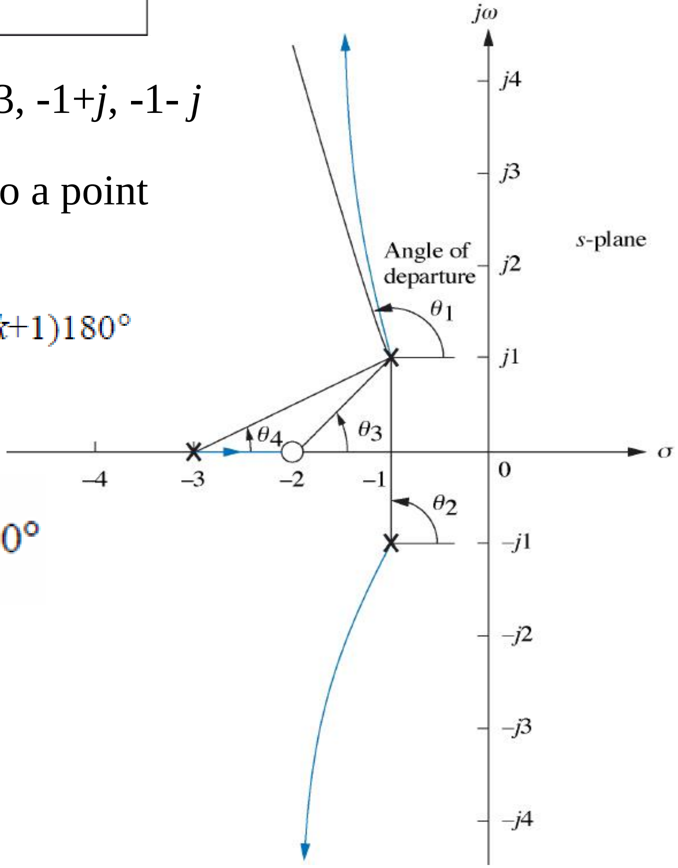
we calculate the sum of angles drawn to a point ε close to the complex pole, -1 + j

Sum (zero angles) – Sum (pole angles) = $(2k+1)180°$

$$-\theta_1 - \theta_2 + \theta_3 - \theta_4 =$$

$$= -\theta_1 - 90° + \underbrace{\tan^{-1}\left(\frac{1}{1}\right)}_{\theta_3} - \underbrace{\tan^{-1}\left(\frac{1}{2}\right)}_{\theta_4} = 180°$$

$$\Rightarrow \theta_1 = -251.6° = 108.4°$$

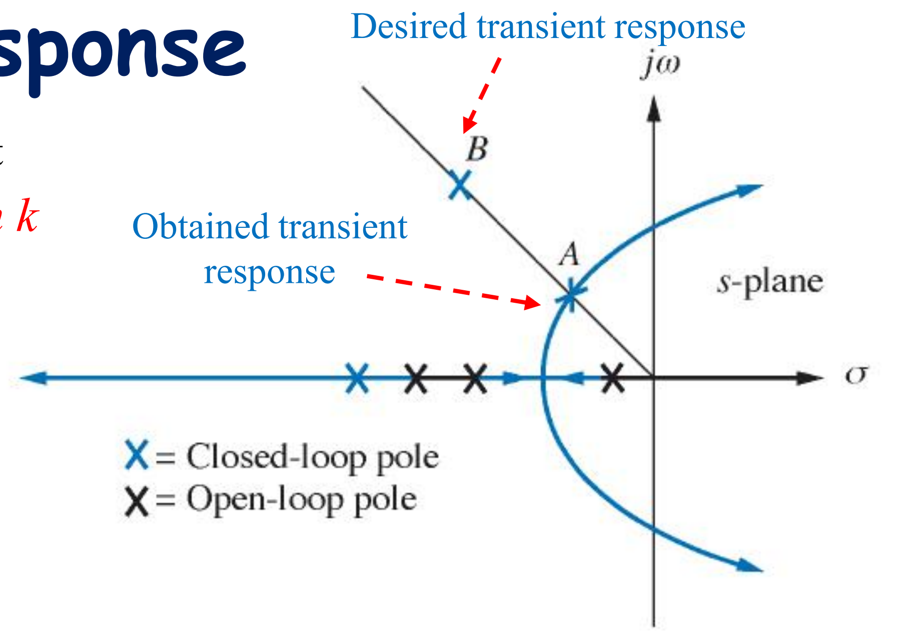


Improving System Response

Speed up the response : move pole from A to B without affecting the percent overshoot

Solution: move the root locus to put the desired pole on it *for some value of gain k* (compensation by adding poles and zeros).

- Dynamic compensator is used if a satisfactory design cannot be obtained by adjustment of gain *k* alone.



(*a*)

a. Sample root locus, showing possible design point via gain adjustment *(A)* and desired design point that cannot be met via simple gain adjustment *(B)*;

Compensators

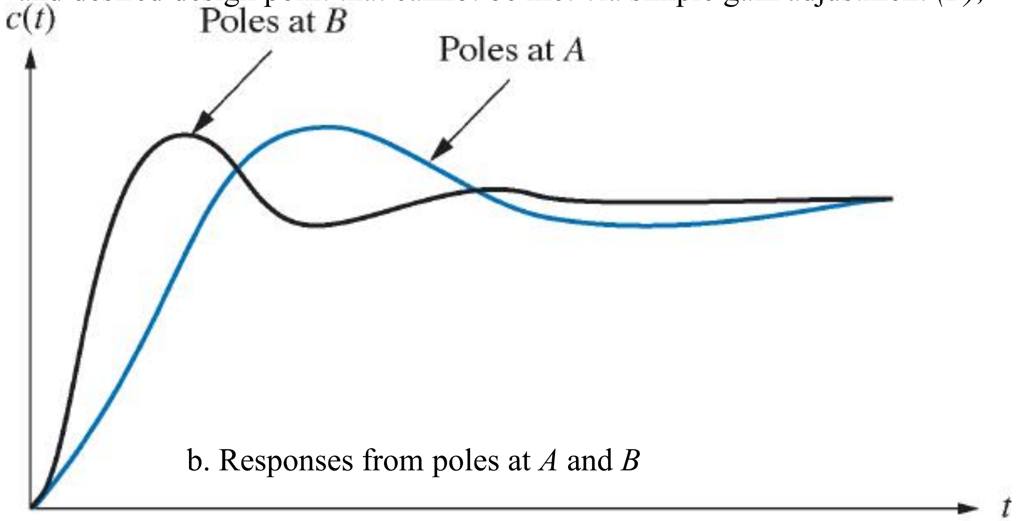
Dynamic *compensators* (function of s) *are designed to improve*:

Transient response by adding an ideal compensator ***PD*** (pure differentiation using active amplifiers) or a ***Lead*** compensator (implemented with passive elements) in the *forward* path or *feedback* path.

Steady-state error by adding an ideal compensator ***PI*** (pure integration using active amplifiers) or a ***Lag*** compensator (implemented with passive elements) in the *forward path* or *feedback path*.

- Compensator transfer function : $C(s) = K\dfrac{s+z}{s+p}$

lead compensation if $z < p$ and ***lag compensation*** if $z > p$.



b. Responses from poles at *A* and *B*

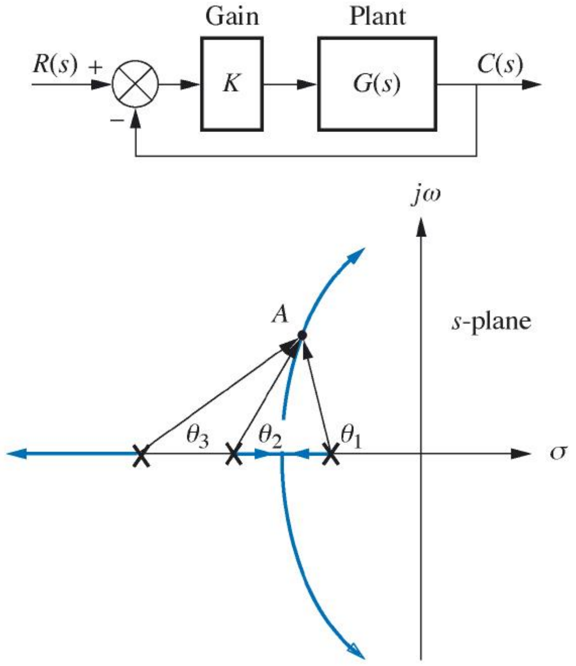
(*b*)

Ideal Integral Compensation (PI)

Improving Steady-State Error

- Steady-state error can be improved (without appreciably affecting the transient response) by placing an open-loop pole at the origin, because this increases the system type by one.

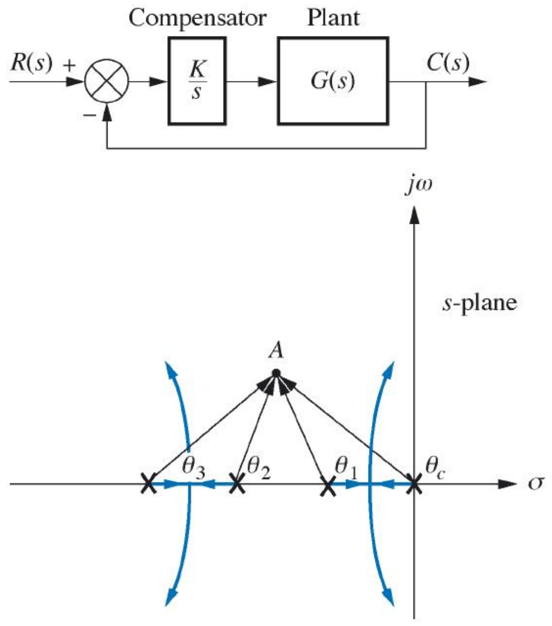
System operating with closed-loop poles at A (desirable transient response)



$-\theta_1 - \theta_2 - \theta_3 = (2k+1)180°$

(a)

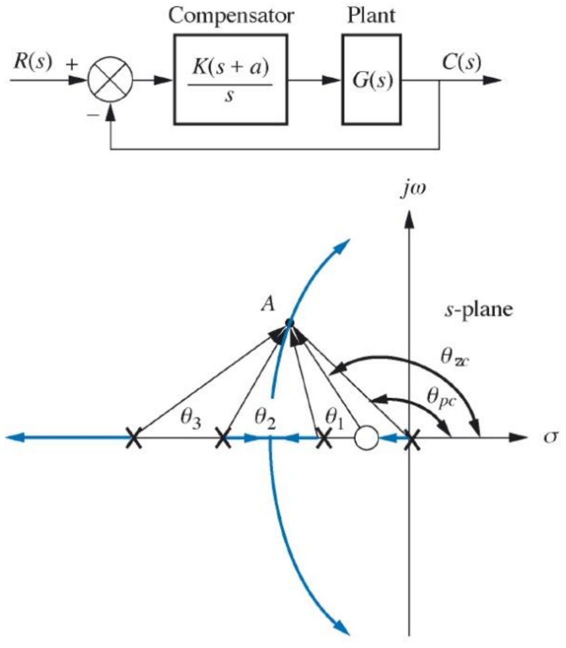
If we add a pole at the origin ➡ Changes root-locus (Point A not on root locus.)



$-\theta_1 - \theta_2 - \theta_3 - \theta_c \neq (2k+1)180°$

(b)

Solution: add a zero close to the pole at the origin to pole cancel out the effect of the added pole on the root-locus.



$-\theta_1 - \theta_2 - \theta_3 - \theta_{pc} + \theta_{zc} \cong (2k+1)180°$

(c)

Pole at *A* is:

a. on the root locus without compensator;
b. not on the root locus with compensator pole added;
c. approximately on the root locus with compensator pole and zero added

we have improved the steady-state error without appreciably affecting the transient response

Example1

Given the system of Figure (a), operating with a *damping ratio of 0.174*, show that the addition of the ideal integral compensator shown in Figure (b) reduces the steady-state error to zero for a step input without appreciably affecting transient response.

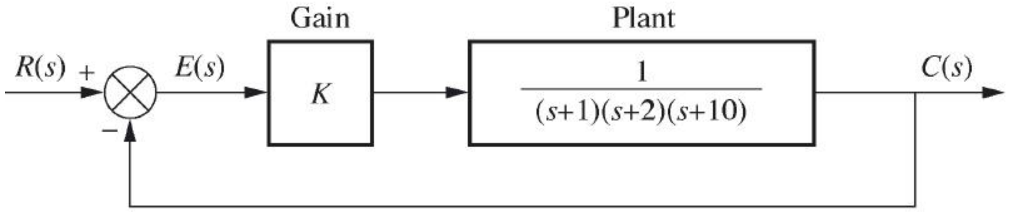


Figure (a)

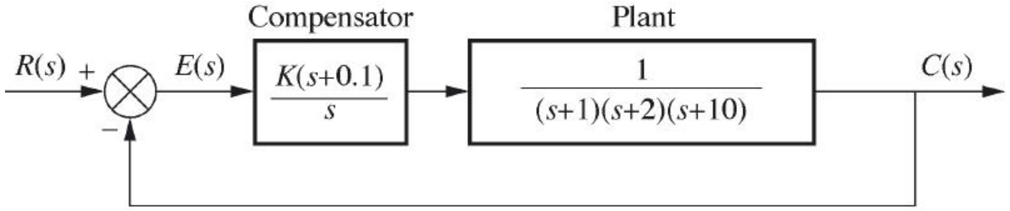


Figure (b)

SOLUTION

For gain $K = 164.6$, searching along the line of $\zeta = 0.174$ for the <u>uncompensated system</u>: <u>dominant poles</u> are $0.694 \mp j3.926$ (third pole at -11.61) Figure (c).

This gain yields Position constant $K_p = \lim_{s\to 0} G(s) = \frac{164.6}{20} = 8.23.$

Hence, the steady-state error is: $e(\infty) = \frac{1}{1+K_p} = \frac{1}{1+8.23} = 0.108$

We add an *ideal integral compensator* with a zero at -0.1.

For gain $K = 158.2$, searching along the line of $\zeta = 0.174$ for the <u>compensated system</u>: <u>dominant poles</u> are $0.678 \mp j3.837$ (forth pole at -0.0902) Figure (e).

Poles and gain are approximately the same ⟹ Same transient response

Damping Ratio unchanged (with K = 158:2). <u>*Steady State Error is ZERO*</u>!.

Closed-loop system
a. before compensation;
b. after ideal integral compensation

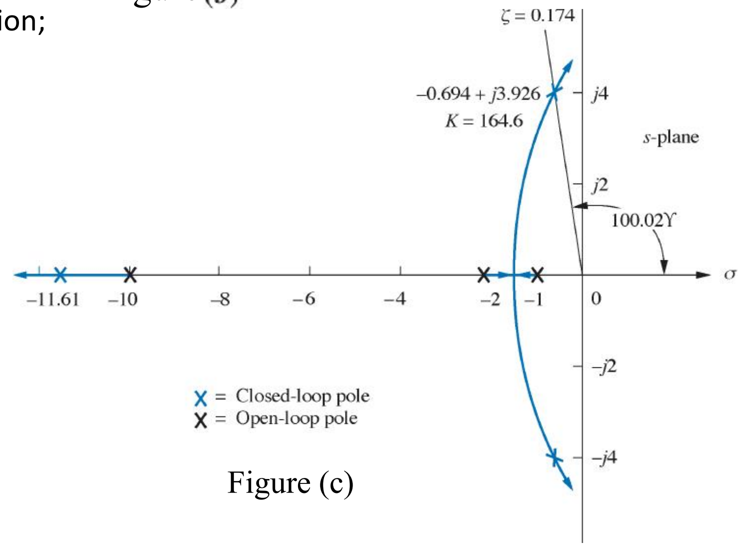


Figure (c)

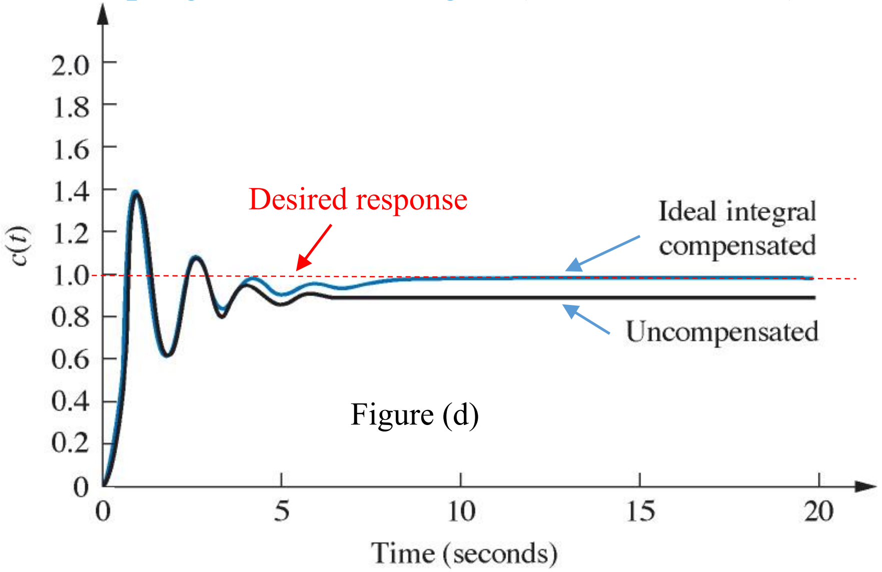


Figure (d)

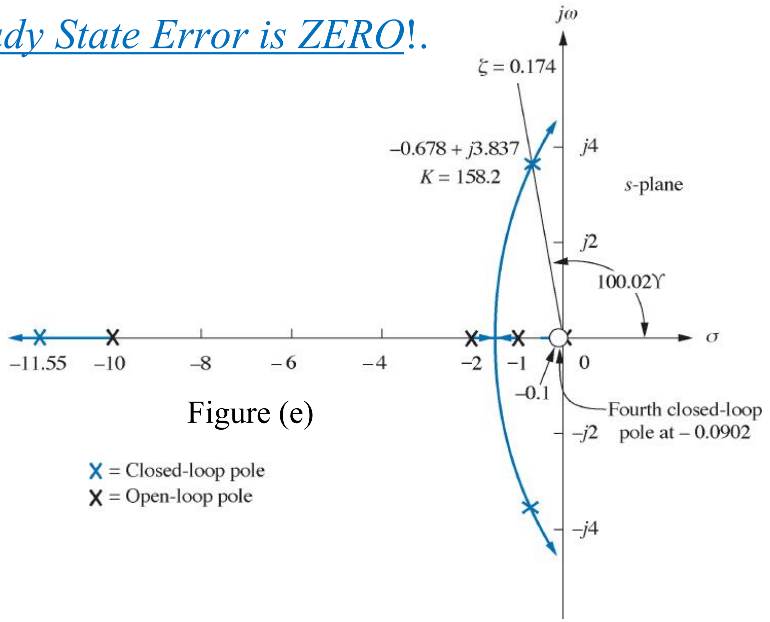


Figure (e)

Lag Compensation

Improving Steady-State Error

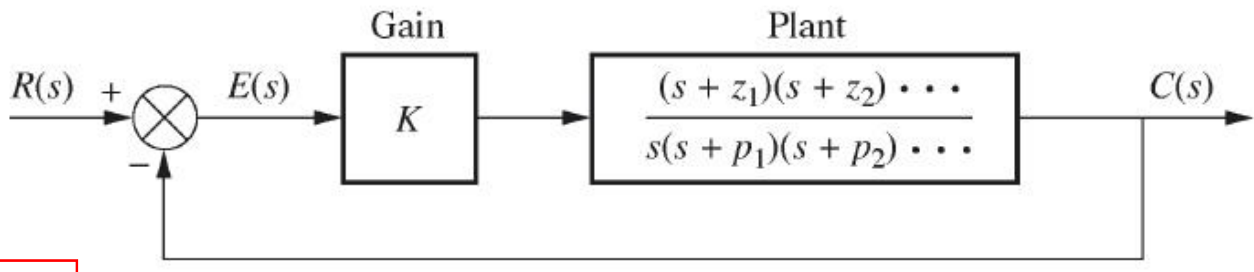
- Similar to the Ideal Integrator, however it has a pole not on the origin but close to the origin (fig c) due to the passive networks.

- Steady State Improvement:

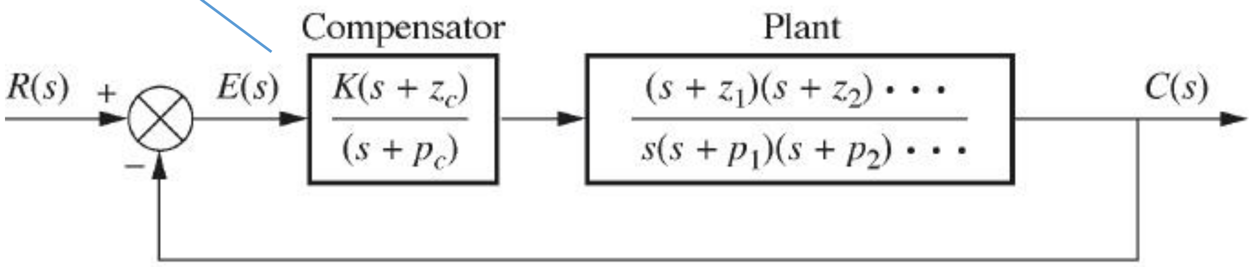
  Before compensation: The static error constant, K_{V_0}, for the system is:

$$K_{v_0} = \lim_{s \to 0} sG(s) = K \frac{z_1 z_2 \dots}{p_1 p_2 \dots}.$$

After compensation: $K_{v_{new}} = \frac{z_c}{p_c} \underbrace{K \frac{z_1 z_2 \dots}{p_1 p_2 \dots}}_{K_{v_0}}$ $\Rightarrow$ $\boxed{k_{v\_new} = \frac{z_c}{p_c} \cdot k_{v\_new}}$



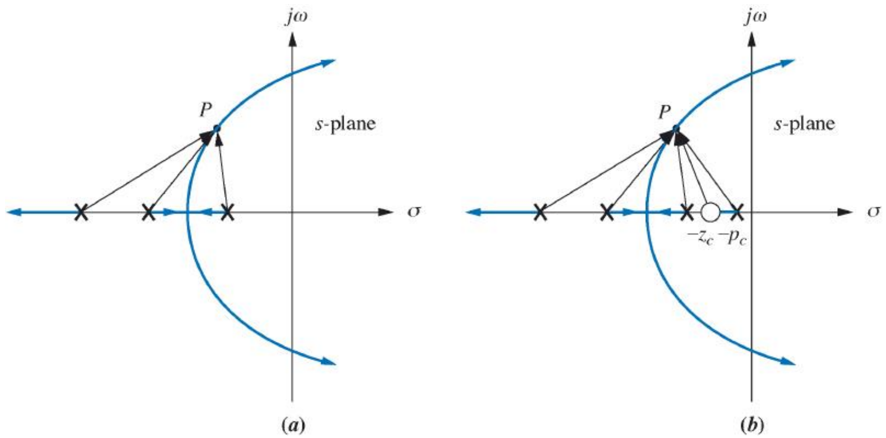
(*a*)



(*b*)

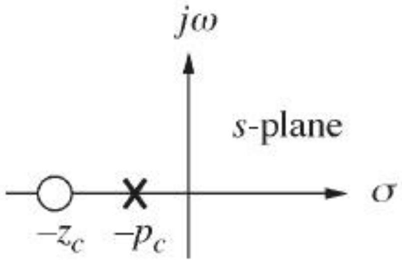
- The effect on the transient response is negligible:

  If the lag compensator pole and zero are close together, the angular contribution of the compensator to point P is approximately zero degrees. point P is still at approximately the same location on the compensated root locus.



Root locus: a. before lag compensation; b. after lag compensation

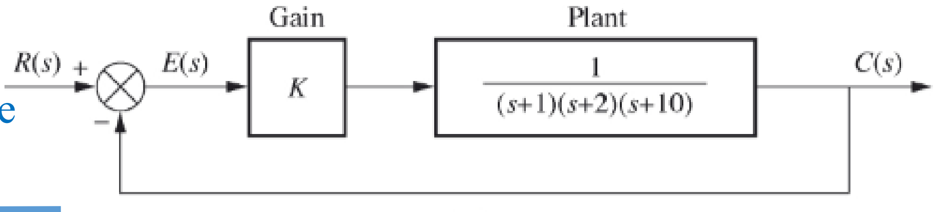
$$G_c(s) = \frac{(s + z_c)}{(s + p_c)}$$



(*c*)

Example2

Compensate the system of Figure (a), whose root locus is shown in Figure (b), to improve the steady-state error by a factor of 10 if the system is operating with a damping ratio of 0.174.

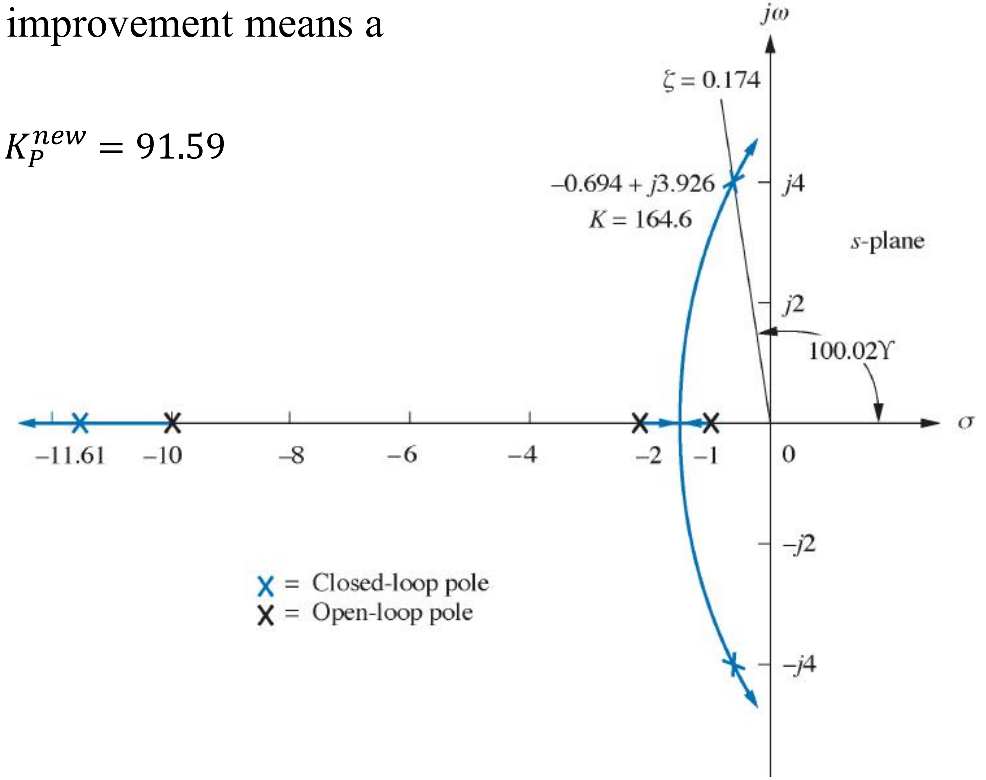


(a)

SOLUTION

- From example 1: *uncompensated system* error was 0.108 with $K_P^{old}= 8.230$. A tenfold improvement means a steady-state error of:

$$e^{new}(\infty) = \frac{e^{old}(\infty)}{10} = \frac{0.108}{10} = 0.0108, \qquad since \;\; e(\infty) = \frac{1}{1+K_P^{new}} \Rightarrow K_P^{new} = 91.59$$



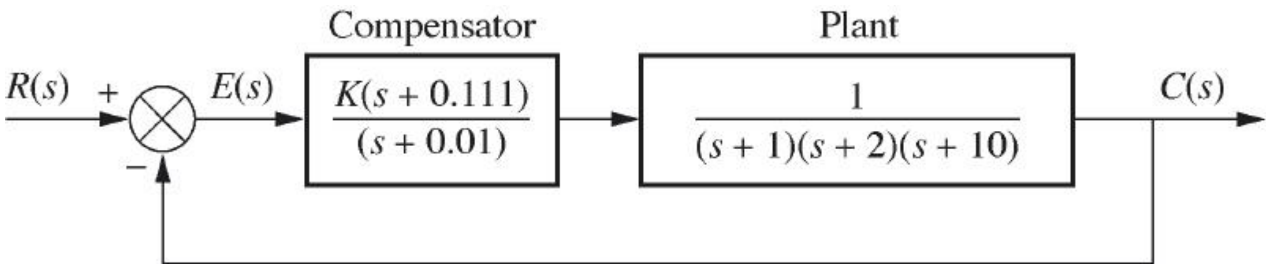
Root locus for *uncompensated system*

- For the *compensated system* $\frac{z_c}{p_c} = \frac{K_P^{new}}{K_P^{old}} = \frac{91.59}{8.23} = 11.13$

Arbitrarily selecting $p_c = 0.01$ ➡ $z_c = 11.13 \cdot p_c = (11.13)(0.01\,)$

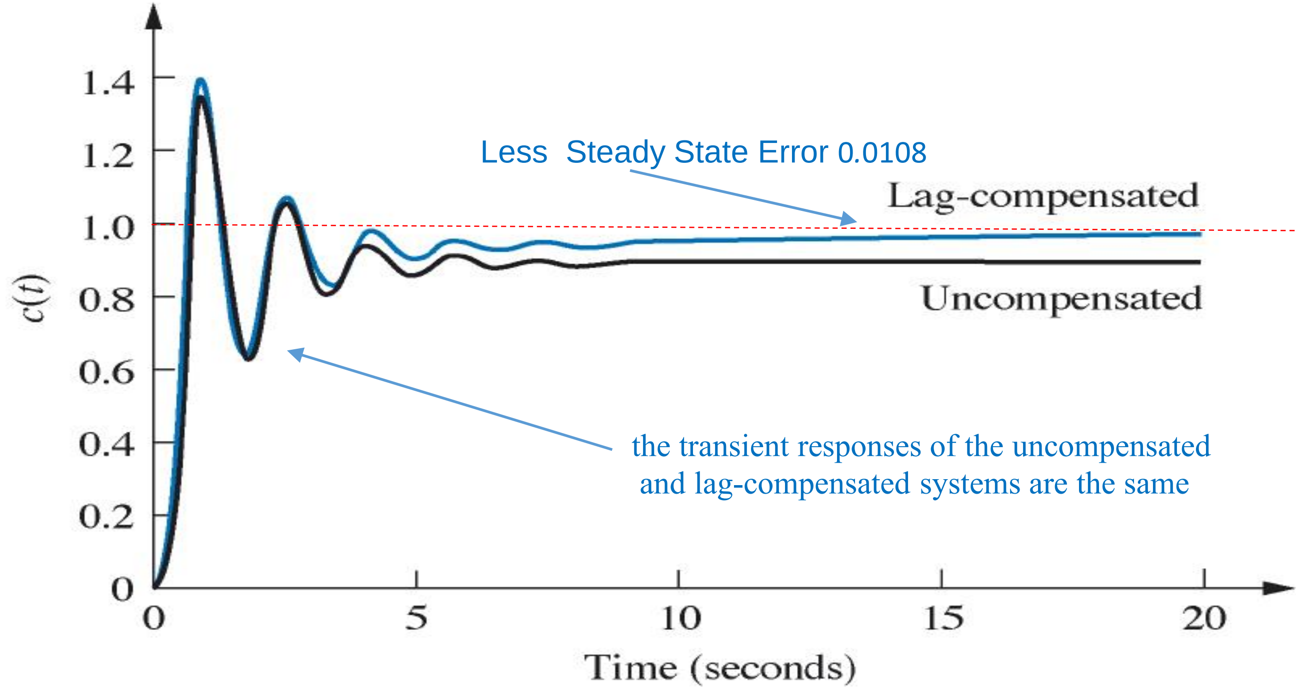
➡ $z_c \approx 0.111$

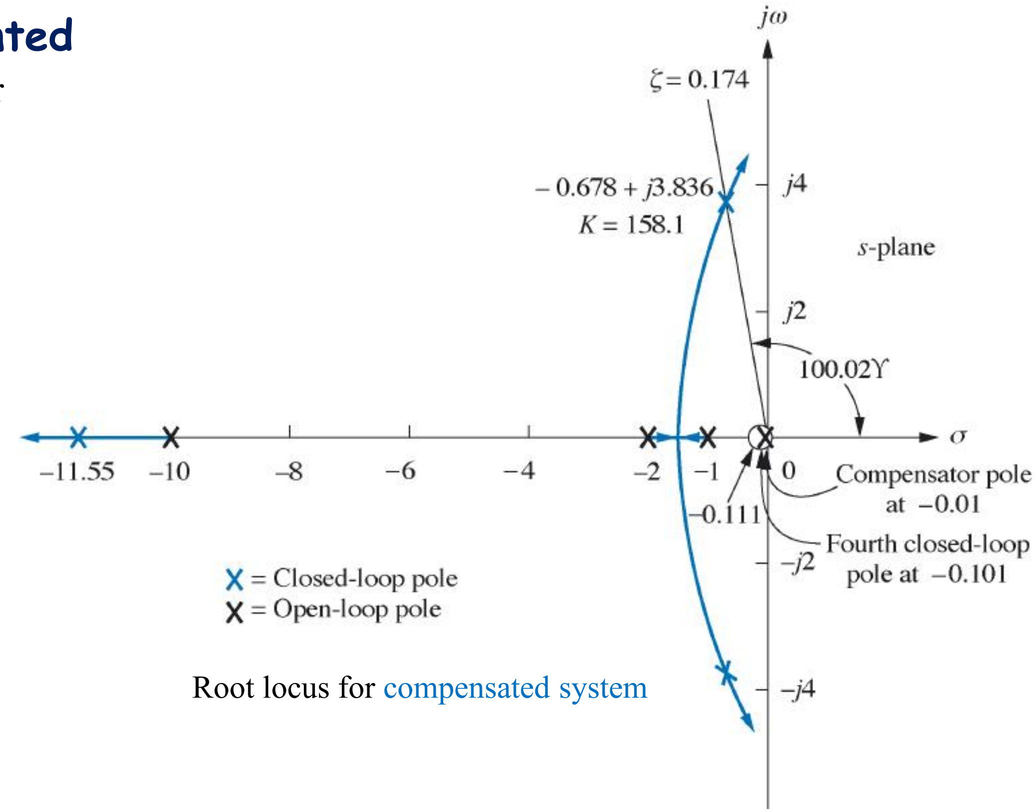
- The compensated system



Example2-Conted

- The transient response of both systems is approximately the same with reduced steady state error





Root locus for compensated system

- Comparison of the Lag-Compensated and the Uncompensated Systems

Parameter	Uncompensated	Lag-compensated
Plant and compensator	$\frac{K}{(s+1)(s+2)(s+10)}$	$\frac{K(s+0.111)}{(s+1)(s+2)(s+10)(s+0.01)}$
K	164.6	158.1
K_p	8.23	87.75
$e(\infty)$	0.108	0.011
Dominant second-order poles	$-0.694 \pm j3.926$	$-0.678 \pm j3.836$
Third pole	-11.61	-11.55
Fourth pole	None	-0.101
Zero	None	-0.111

On the ζ= 0.174 line: (compensated system):

The second-order dominant poles are at - 0.678 ±j3.836 (K=158.1)

The third and fourth closed-loop poles are at -11.55 and - 0.101.

The fourth pole of the compensated system cancels its zero.

Ideal Derivative Compensation (PD)

Improving Transient Response

- The objective is to design a response that has a desirable percent overshoot and a shorter settling time than the uncompensated system. (two approaches).

1. Ideal derivative compensation (Proportional-plus-Derivative (PD) active elements **):** a pure differentiator is added to the forward path of the feedback control system. $G_1(s) = s + z_c$ sensitive to high frequency noise.

2. Lead Compensation: (not pure differentiation) approximates differentiation with a passive network by adding a zero and a more distant pole to the forward-path transfer function.

$$G_1(s) = K\frac{s + z_c}{s + p_c}$$

Distant pole

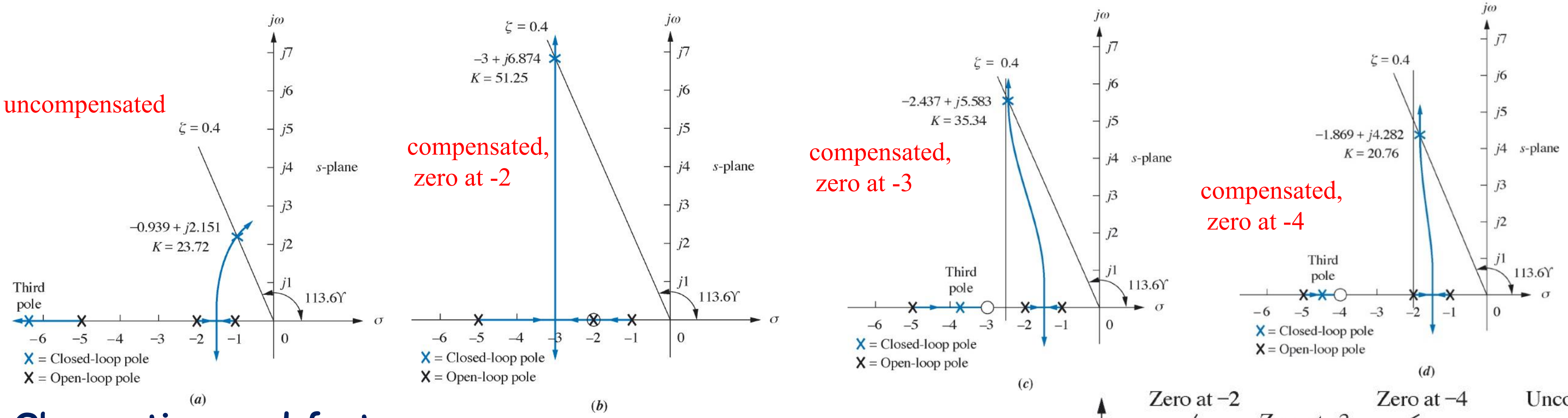
Less sensitive to high frequency noise.

- The transient response of a system can be selected by choosing an appropriate closed-loop pole location on the s-plane.
- If this point is on the root locus, then a simple gain adjustment is all that is required in order to meet the transient response specification.
- If the closed loop root locus doesn't go through the desired point, it needs to be reshaped (add poles and zeros in the forward path).
- One way to speed up the original system is to add a single zero to the forward path. $G_c(s) = s + z_c$

Ideal Derivative Compensation (PD)

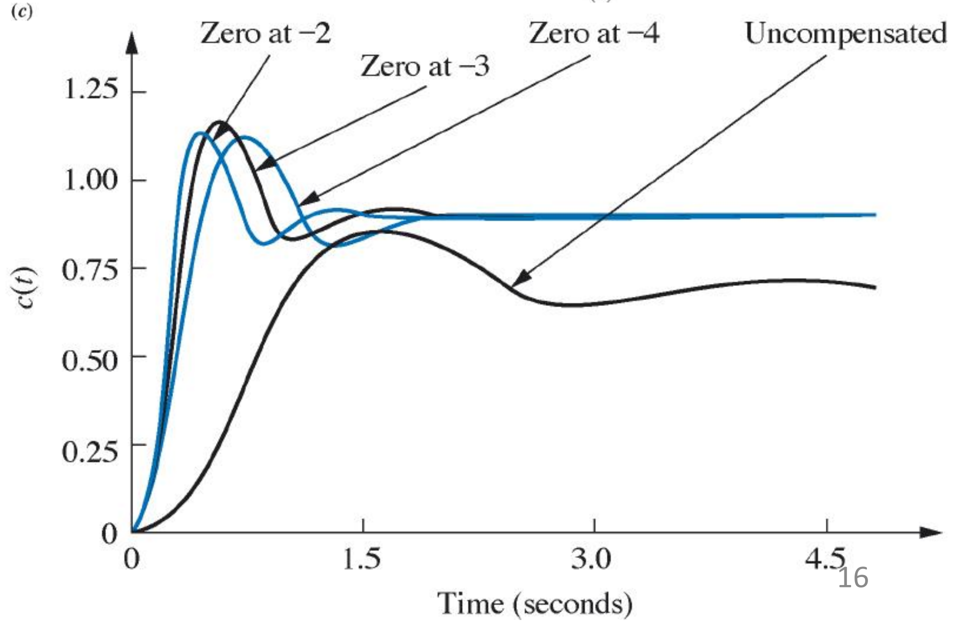
Improving Transient Response

- See how it affects by an example of a system operating with a damping ratio of 0.4:



Observations and facts:

- In each case gain K is chosen such that percent overshoot is same.
- Compensated poles have more negative real part (smaller settling time) and larger imaginary part (smaller peak time).
- Zero placed farther from the dominant poles, compensated dominant poles move closer to the origin.



Example3$_1$

Given the system of Figure (a), design an ideal derivative compensator to yield a 16% overshoot, with a threefold reduction in settling time.

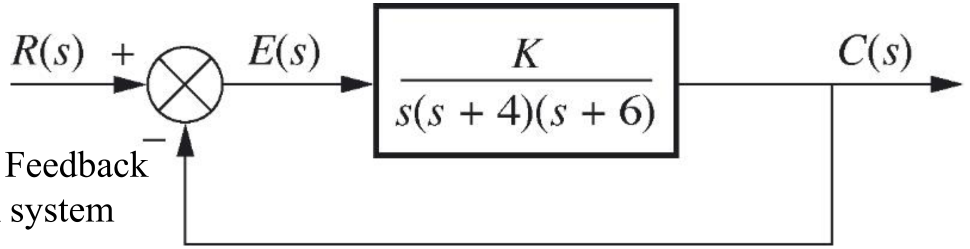


Fig (a) Feedback control system

SOLUTION

The performance of the *uncompensated system* operating with 16% overshoot fig (b).

16% Overshoot ➡ $\zeta = 0.504$, $\xrightarrow[\text{Dominant second-order poles}]{\text{along damping ratio line}}$ -1,205 ±j2.064.

($with\ k = 43.35$ and third pole at -7.59.)

Settling time ➡ $$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{1.205} = 3.320$$

Location of the *compensated system's* dominant poles.(*Desired poles*)

$$T_s^{new} = \frac{T_s^{old}}{3} = 1.107$$ threefold reduction in the settling time

➡ $$\sigma = \frac{4}{T_s^{new}} = \frac{4}{1.107} = 3.613$$ real part of the compensated system's dominant, second-order pole

$$\omega_d = 3.613\ tan(180^o - 120.26\ ^0) = 6.193$$ Imaginary part of the compensated system's dominant pole on line $\zeta = 0.504$

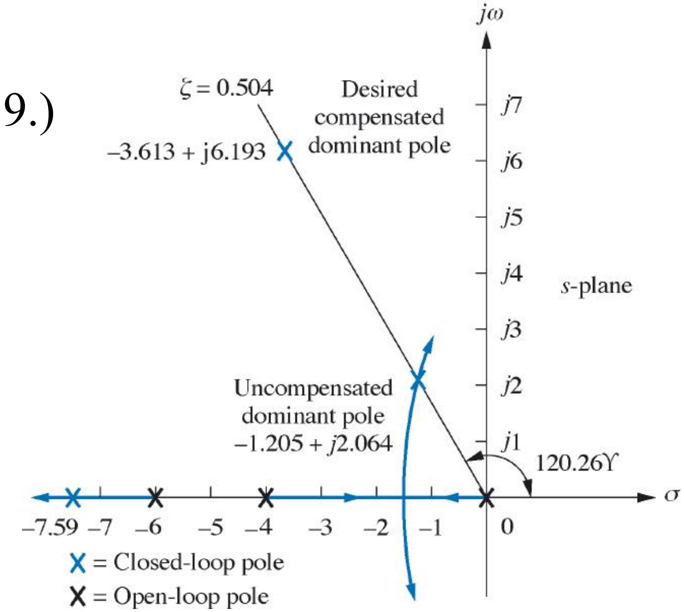


Fig (b) Compensated dominant pole

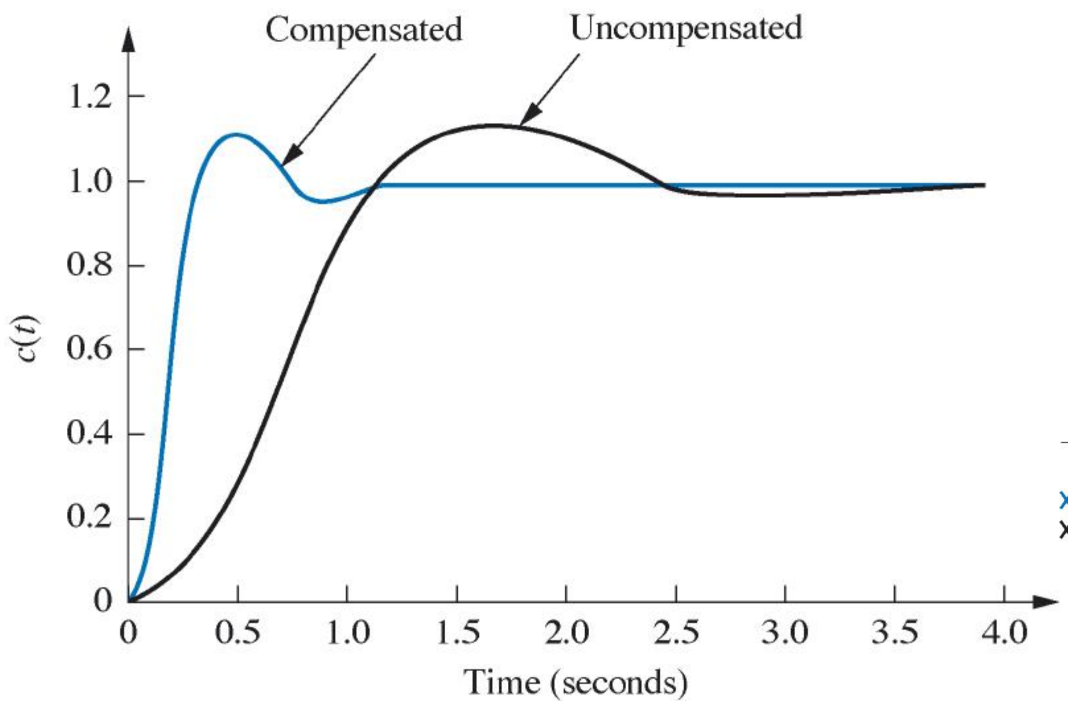
Example3_2

Zero contribution angle > 90^0 → zero position less than desired pole real part.

Design the location of the compensator zero

- The angle contribution of poles for the desired pole location: -275.6^0.
- To achieve -180^0 the angle contribution of the placed zero should be: $-275.6^0 + x = -180^0 \rightarrow x = 95.6^0$
- From the fig (c):

$$\frac{6.193}{3.613-\sigma} = \tan(180-95.6) \Rightarrow \sigma = 3.006$$



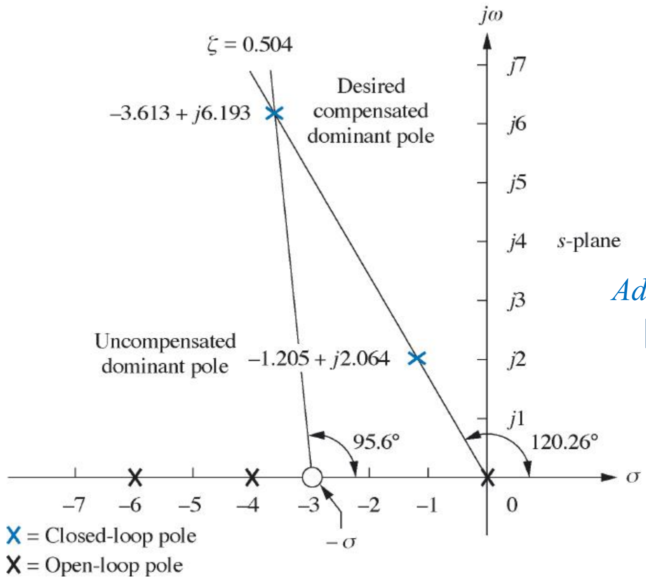


Fig (c) Evaluating the location of the compensating zero

Adding zero

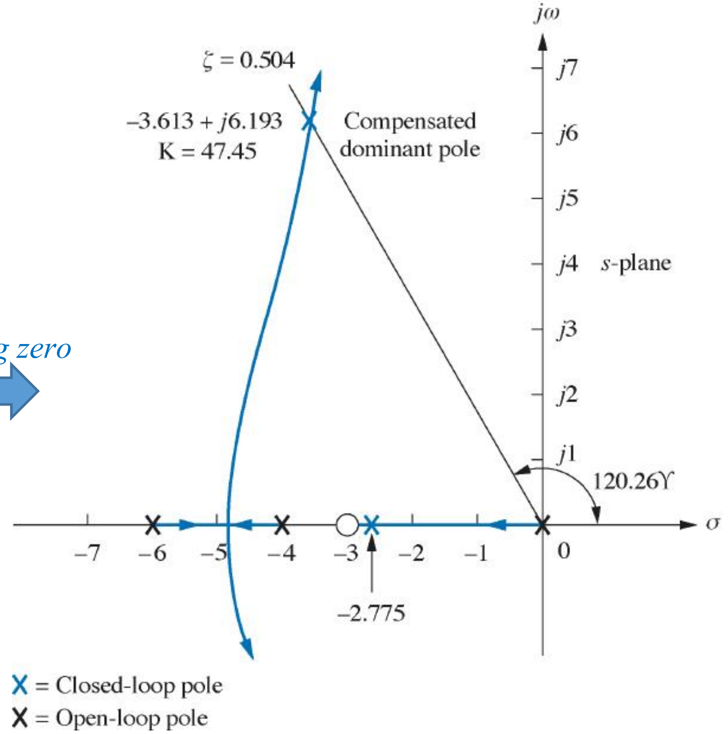
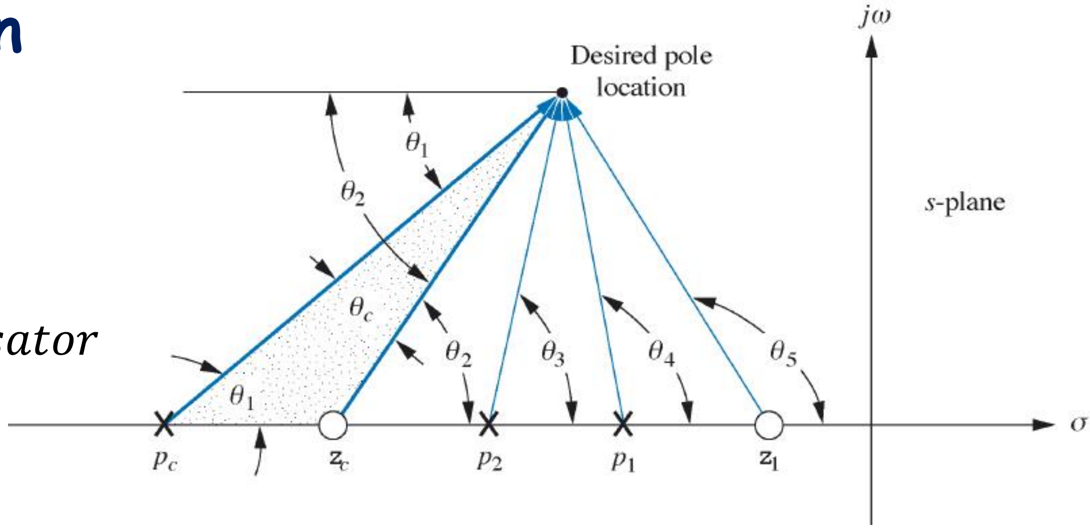


Fig (c) Root locus for the compensated system

Fig (d) Uncompensated and compensated system step responses

Lead Compensation



Basic Idea: The difference between 180° and the sum of the angles must be the angular contribution required of the compensator.

Example: looking at the Figure, we see that:

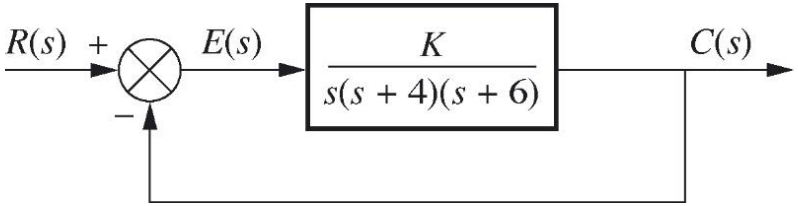
$$\theta_2 - \theta_1 - \theta_3 - \theta_4 + \theta_5 = (2k+1)180^0$$

$$where\ \theta_1 - \theta_2 = \theta_c\ is\ the\ angular\ contribution\ of\ the\ compensator$$

- There are infinitely many choices of z_c and p_c providing same θ_c

Example4$_1$

Design three lead compensators for the system in Figure to reduce the settling time by a factor of 2 while maintaining 30% overshoot.



SOLUTION

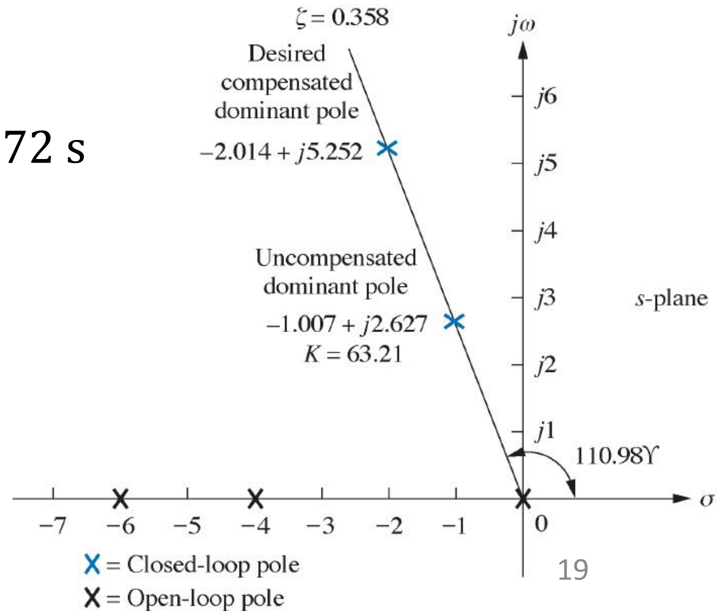
- Characteristics of the uncompensated system operating at 30% overshoot

30% Overshoot $\xrightarrow{\text{damping ratio}}$ $\zeta = 0.358$, $\xrightarrow[\text{along damping ratio line}]{\text{Dominant second-order pair of poles}}$ -1,007 ±j2.627. $\xrightarrow[\text{From pole's real part}]{\text{settling time}}$ $T_s = {}^{4}/_{1.007} = 3.972$ s

- Design point (*Desired Poles location*)

twofold reduction in settling time $\rightarrow$ $T_s = {}^{3.972}/_{2} = 1.986$ s $\xrightarrow{\text{real part of the desired pole location}}$ $-\zeta\omega_n = -{}^{4}/_{T_s} = -2.014$

Imaginary part of the desired pole location $\rightarrow$ $\omega_d = -2.014\tan(110.98^0) = 5.252$



Example4$_2$

- *Lead compensator Design*.

Place the zero on real axis at -5 (arbitrarily as possible solution).
sum the angles (this zero and uncompensated system's poles and zeros),

resulting angle → $\theta_0 = -172.69^0$ → the angular contribution required from the compensator pole → $\theta_c = -180^0 + 172.69^0 = -7.31^0$

location of the compensator pole → From geometry in fig(a)

$$\frac{5.252}{p_c - 2.014} = \tan(7.31^0)$$

compensator pole → $p_c = 42.96$

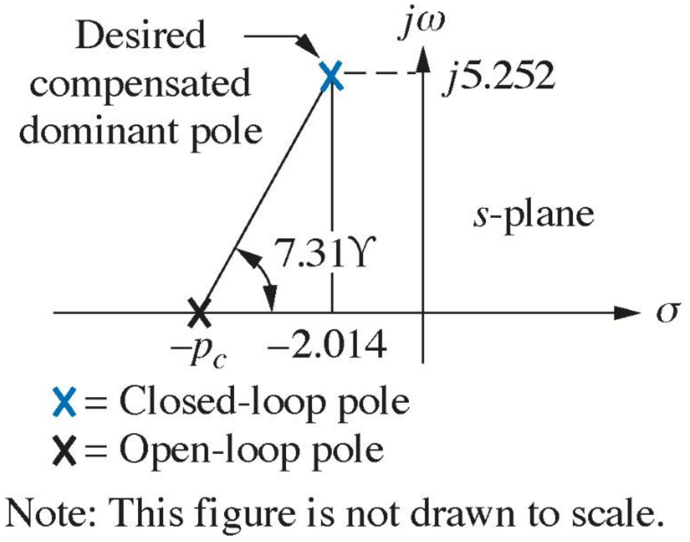
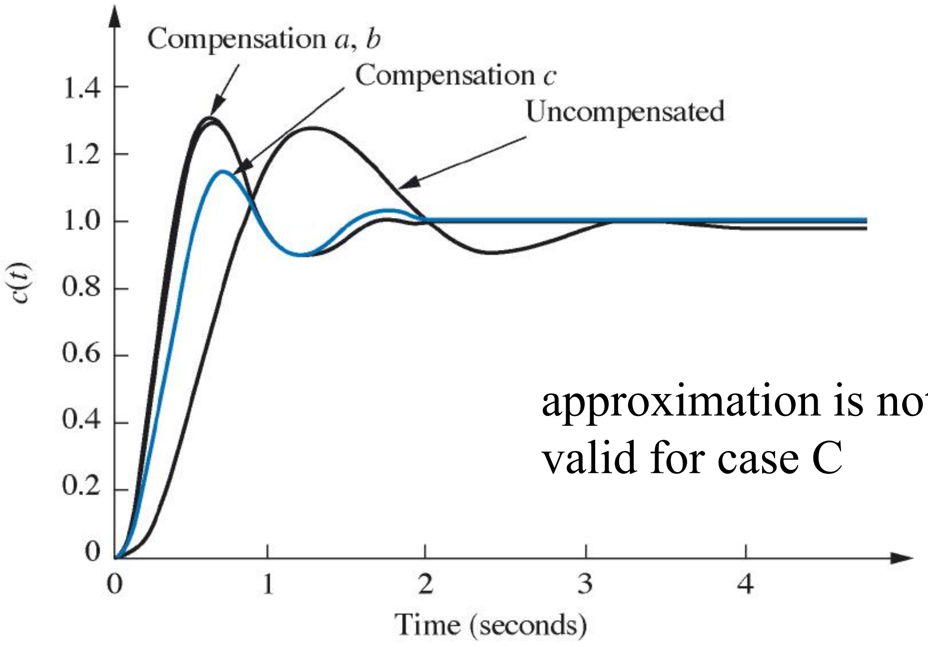


Fig (a) 5-plane picture used to calculate the location of the compensator pole



approximation is not valid for case C

Fig (c) Uncompensated system and lead compensation responses (zeros at a:-5, b:-4 c: -2)

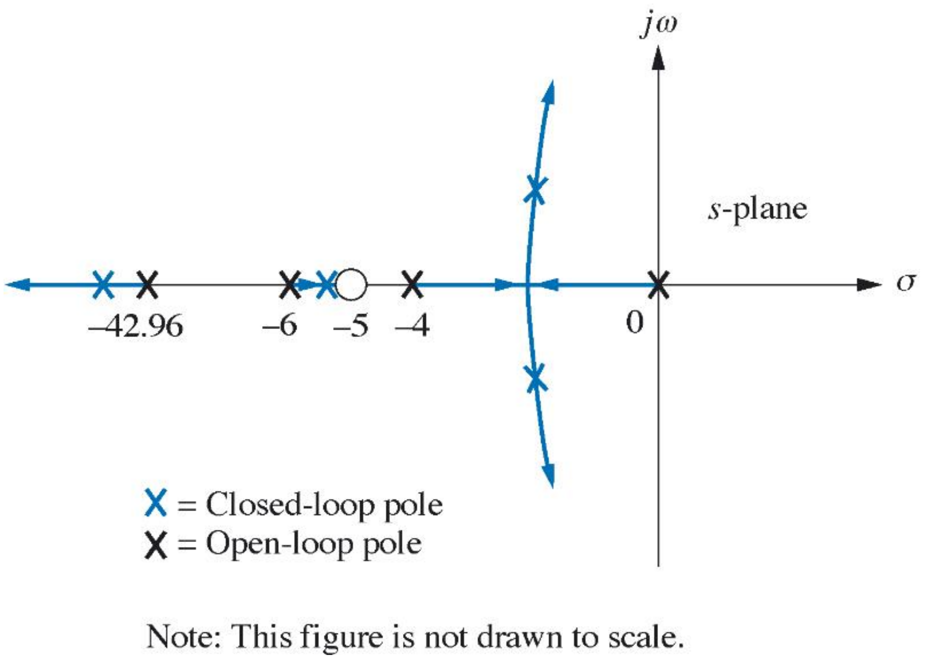


Fig (b) Compensated system root locus

Improving Steady-State Error and Transient Response

- Combine the design techniques to obtain improvement in steady-state error and transient response *independently*.

  - First improve the transient response.(PD or lead compensation).
  - Then improve the steady-state response. (PI or lag compensation).

- Two Alternatives

  - PID (Proportional-plus-Integral-plus-Derivative) (with Active Elements).
  - Lag-Lead Compensator. (with Passive Elements).

PID Controller Design

- Transfer Function of the compensator (two zeros and one pole):

$$G_c(s) = k_1 + \frac{k_2}{s} + k_3\, s = \frac{k_1\, s + k_2 + k_3\, s^2}{s} = \frac{k_3(\, s^2 + \frac{k_1}{k_3}\, s + \frac{k_2}{k_3})}{s}$$

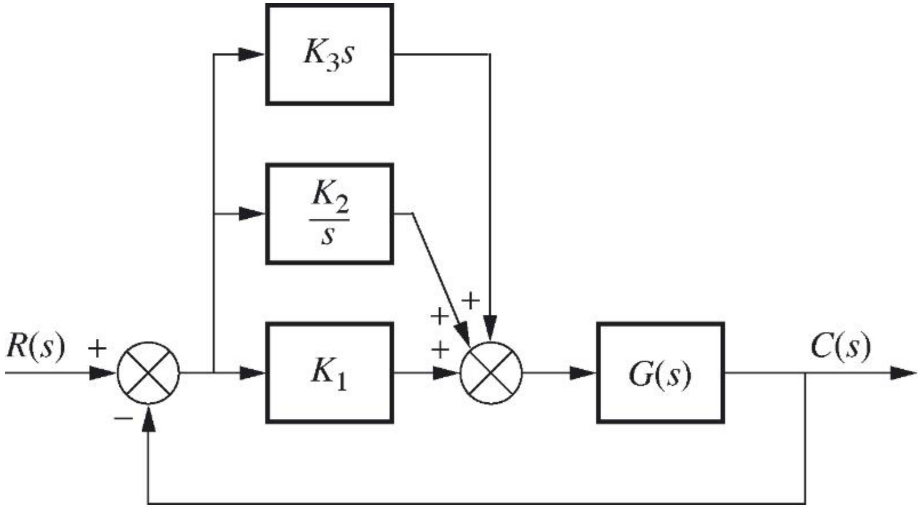


Fig (a) PID controller implementation

- Design Procedure (Fig (a))

1. From the requirements figure out the desired pole location to meet transient response specifications.
2. Design the PD controller to meet transient response specifications.
3. Check validity (all requirements have been met) of the design by simulation.
4. Design the PI controller to yield the required steady-state error.
5. Determine the gains, $k_1, k_2 and\ k_3$ (Combine PD and PI).
6. Simulate the system to be sure all requirements have been met.
7. Redesign if simulation shows that requirements have not been met.

Example5[1]

Given the system of Figure (a), design **a PID** controller so that the system can operate with **a peak time that is two-thirds** that of the uncompensated system at **20% overshoot** and with **zero steady-state error** for a step input.

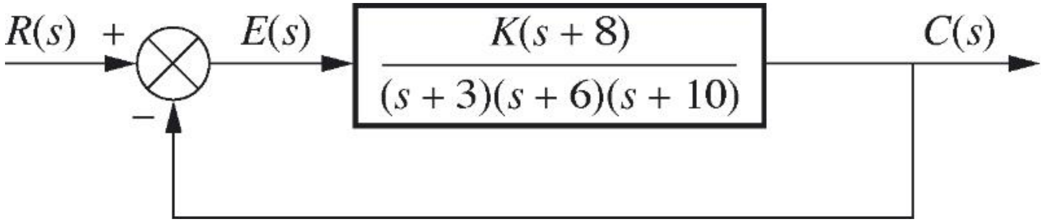


Fig (a) Uncompensated feedback control system

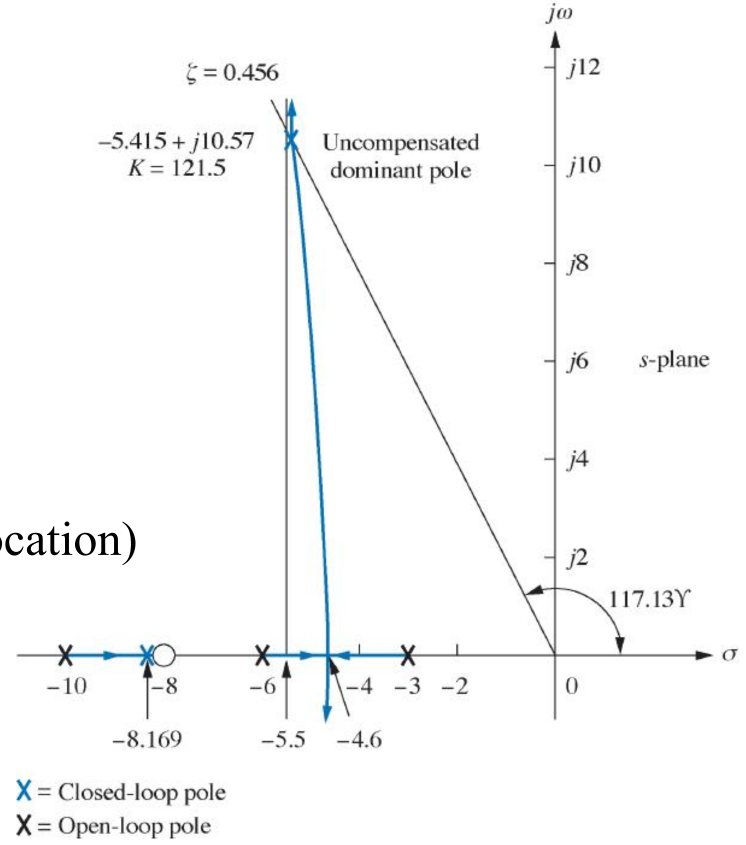
SOLUTION

- Evaluation of the uncompensated system

20% Overshoot → damping ratio → $\zeta = 0.456$, → Dominant second-order pair of poles along damping ratio line → -5,415 ±j10.57 with gain of 121.5.

Peak time → $T_p = \frac{\pi}{\omega_d} = 0.297\ seconds$

between - 8 and -10 for a gain equivalent to that at the dominant poles → A third pole at -8.169



- To reduce the peak time to two-thirds. (find the compensated system's dominant pole location)

The imaginary part →

$$\omega_d = \frac{\pi}{T_p} = \frac{\pi}{(^2/_3)(0.297)} = 15.87$$

The real part →

$$\sigma = \frac{\omega_d}{\tan(117.13^0)} = -8.13$$

$\tan(117.13^0)$= - $\tan(180 - 117.13^0)$

Example5$_2$

- Design of the compensator

  (sum of angles from the uncompensated system's poles and zeros to the desired compensated dominant pole is -198.37^0)

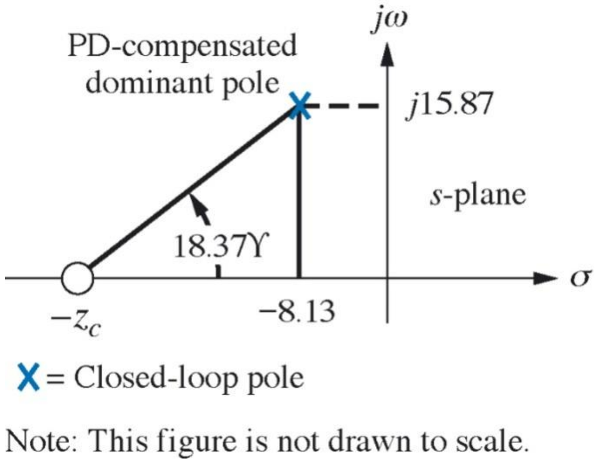


Fig (a) Calculating the PD compensator zero

the required contribution from the compensator zero z_c → $-198.37^0 + \theta_c = -180^0$ → $\theta_c = 18.37^0$

compensating zero's location. From geometry in Fig(a) → $\dfrac{15.87}{z_c - 8.13} = tan18.37^0$ → $z_c = 55.92$ → (the PD controller.) $G_{PD}(s) = (s + 55.92)$

gain at the design point → $k = 5.34$

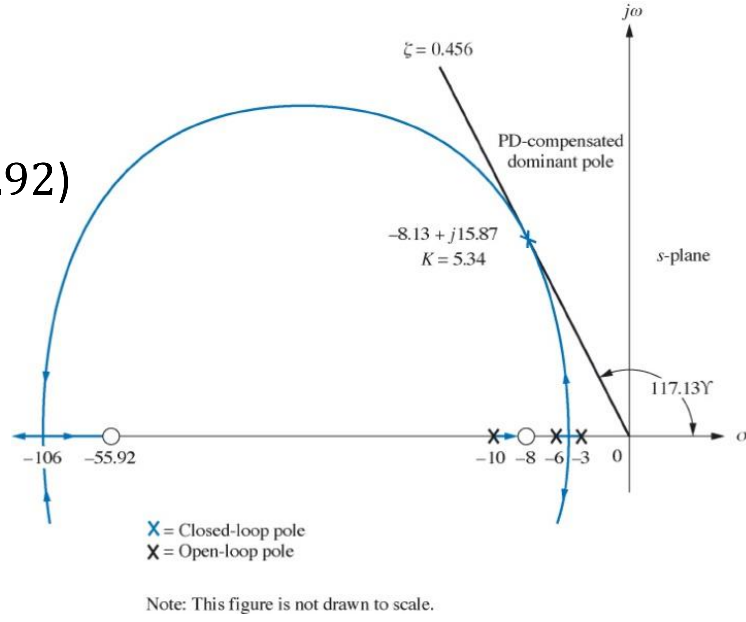


Fig (b) Root locus for PD-compensated system

- The PD-compensated system is simulated. Fig (b) (next slide) shows the reduction in peak time and the improvement in steady-state error over the uncompensated system. (step 3 and 4)

Example5$_3$

- A PI controller is used to reduce the steady-state error to zero (for PI controller the zero is placed at -0.5 close to the origin)

PI controller is used as

$$G_{PI}(s) = \frac{s + 0.5}{s}$$

Searching the 0.456 damping ratio line we find the dominant, second-order poles

$$-7.516 \pm j\,14.67 \ \textit{with associated gain}\ k = 4.6$$

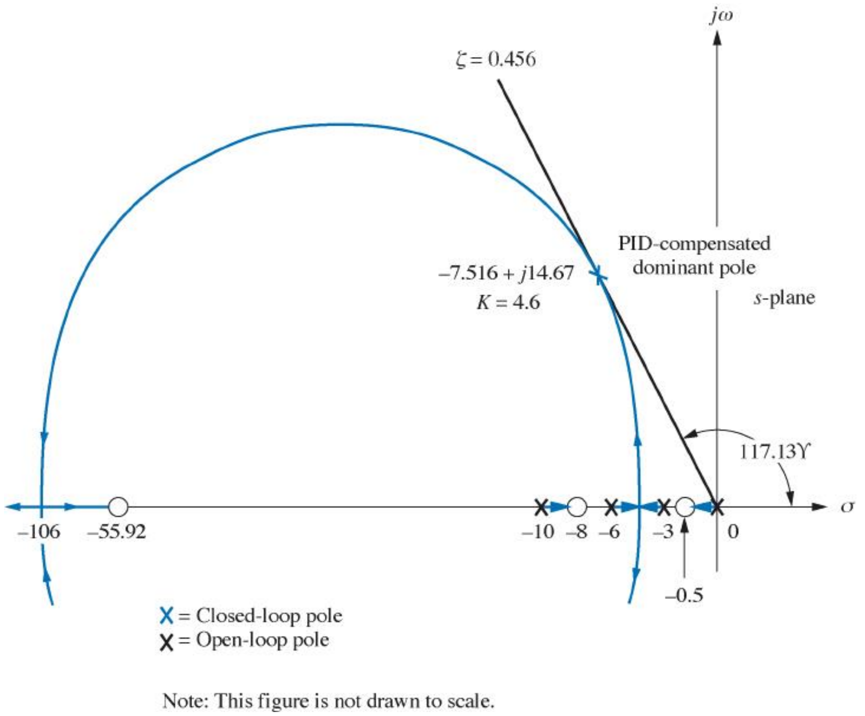


Fig (a) Root locus for PID-compensated system

- Now we determine the gains (the PID parameters),

$$G_{PID}(s) = \frac{k(s + 55.92)(s + 0.5)}{s} = \frac{4.6(s + 55.92)(s + 0.5)}{s}$$

$$= 256.5 + 128.6\frac{1}{s} + 4.6\,s = k_1 + k_2\frac{1}{s} + k_3 s$$

Matching: $k_1 = 256.5$, $k_2 = 128.6$, $k_3 = 4.6$

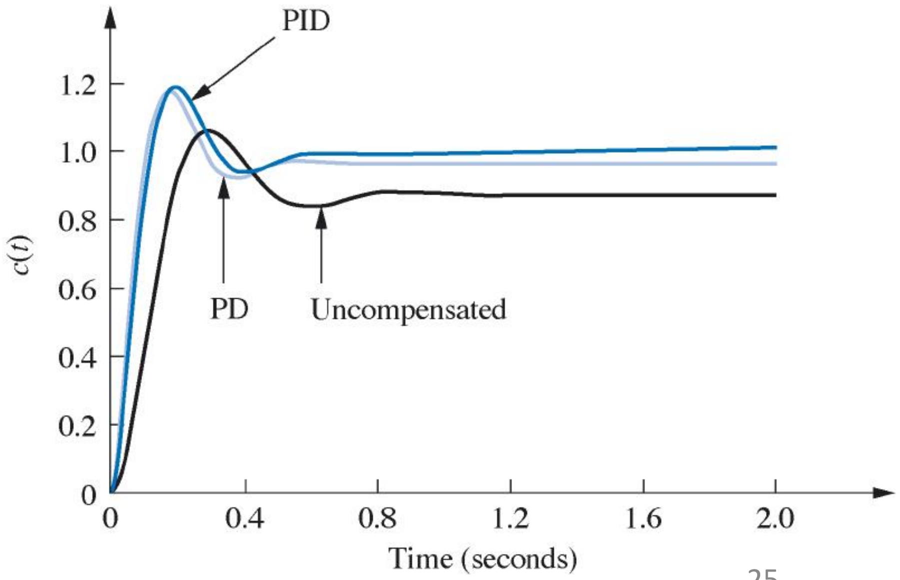


Fig (b) Step responses for uncompensated, PD compensated, and PID compensated systems

Lag-Lead Compensator Design
(Cheaper solution then PID)

- First design the lead compensator to improve the transient response. Next we design the lag compensator to meet the steady-state error requirement.

- Design procedure:

1. Determine the desired pole location based on specifications. (Evaluate the performance of the uncompensated system).

2. Design the lead compensator to meet the transient response specifications.(zero location, pole location, and the loop gain).

3. Evaluate the steady state performance of the lead compensated system to figure out required improvement.(simulation).

4. Design the lag compensator to satisfy the improvement in steady state performance.

5. Simulate the system to be sure all requirements have been met. (If not met redesign)

Example6_1

Design a lag-lead compensator for the system of Figure so that the system will operate with 20% overshoot and a twofold reduction in settling time. Further, the compensated system will exhibit a tenfold improvement in steady-state error for a ramp input.

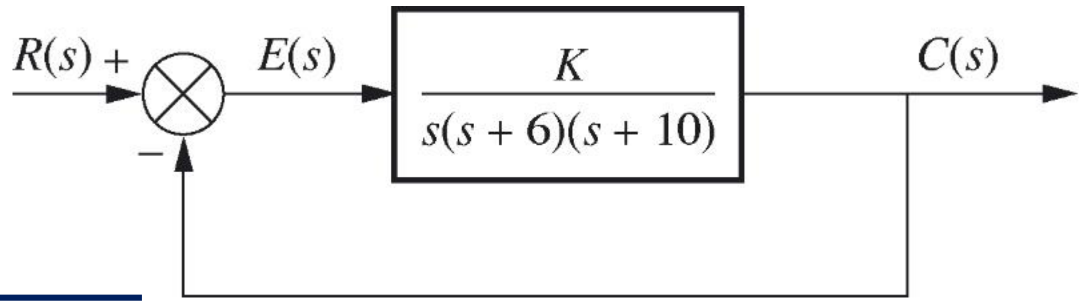


Fig (a) Uncompensated system

SOLUTION

- **Step 1:** Evaluation of the uncompensated system

20% Overshoot $\xrightarrow{\text{damping ratio}}$ $\zeta = 0.456$, $\xrightarrow[\text{along damping ratio line}]{\text{Dominant second-order pair of poles}}$ -1,794 ±j3.501 with gain of 192.1.

- **Step 2 :** Lead compensator design (selection of the location of the compensated system's dominant poles).

Twofold reduction of settling time $\xrightarrow[\text{the real part of the dominant pole}]{}$ $-\zeta\,\omega_n = -2(1.794) = -3.588$ $\xrightarrow{\text{the imaginary part of the dominant pole}}$ $\omega_d = \zeta\,\omega_n \tan(117.13^0) = 7.003$

lead compensator design. $\xrightarrow[\text{Arbitrarily select a location for the lead compensator zero.}]{}$ $z_c = -6$

- compensator zero coincident with the open-loop pole to eliminate a zero and leave the lead-compensated system with three poles. (same number that the uncompensated system has)

Example6$_2$

- Finding the location of the compensator pole.
- Sum the angles to the design point from the uncompensated system's poles and zeros and the compensator zero and get -164.65°.
- The difference between 180° and this quantity is the angular contribution required from the compensator pole (—15.35°).
- Using the geometry shown in Figure (b)

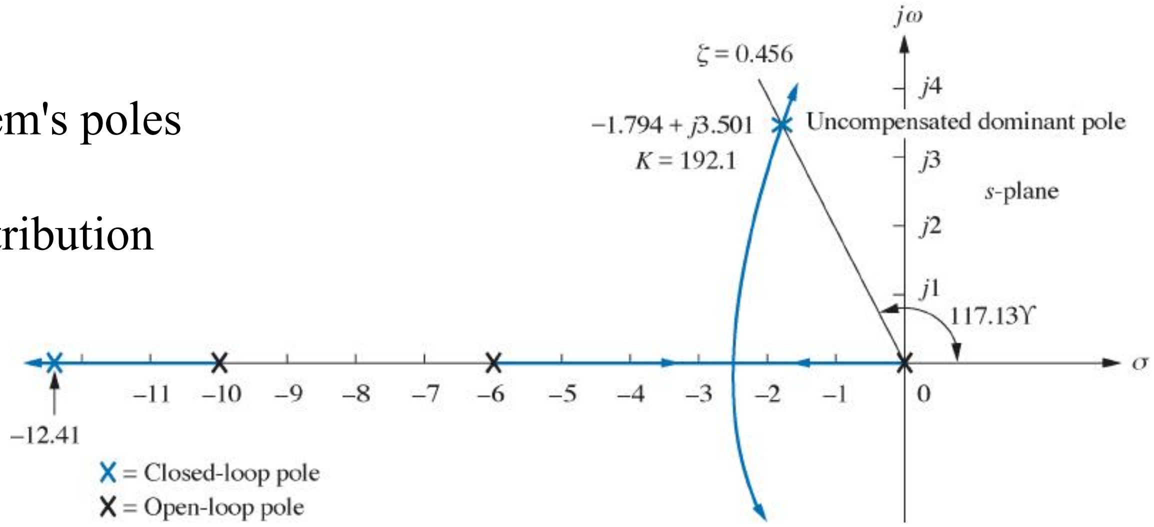


Fig (a) Root locus for uncompensated system

Compensator pole.

$$\frac{7.003}{p_{c-3.588}} = \tan(15.35^{\circ}) \quad \Rightarrow \quad p_c = -29.1$$

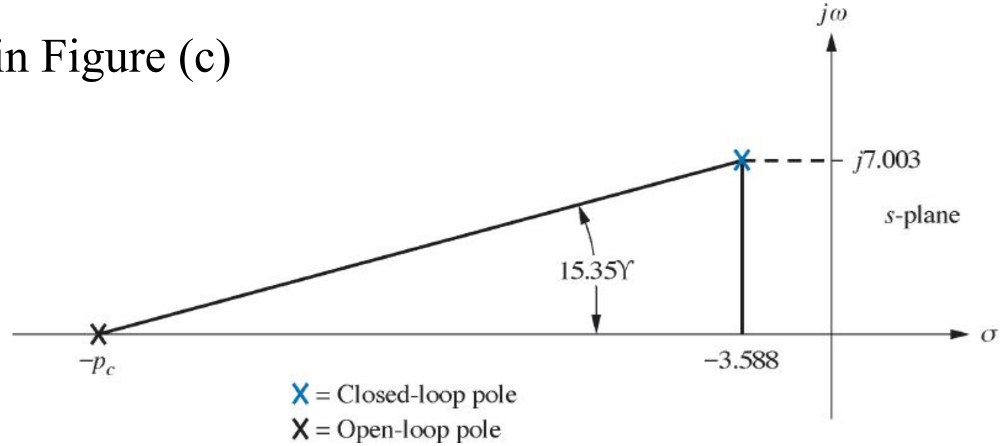


Fig (b) Evaluating the compensator pole

- The complete root locus for the lead-compensated system is sketched in Figure (c)

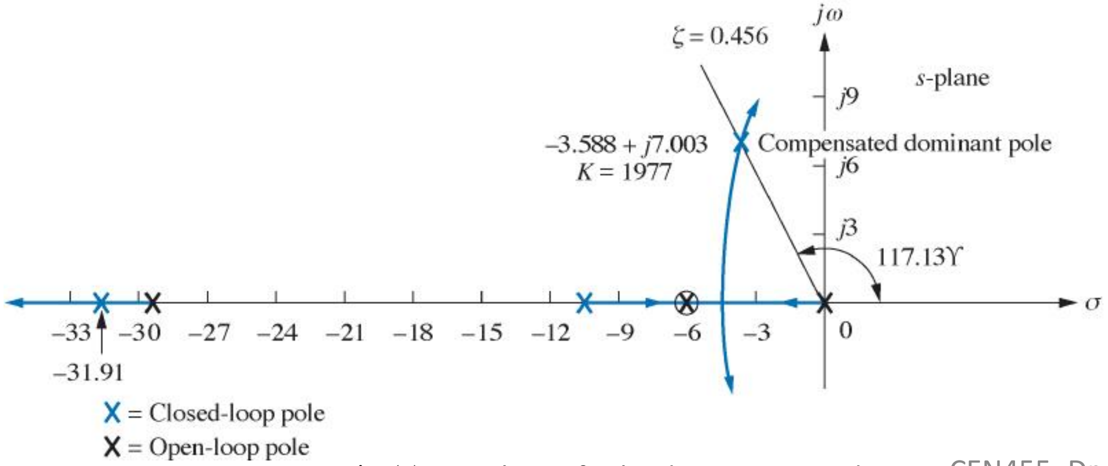


Fig (c) Root locus for lead-compensated system

Example6$_3$

- **Steps 3 and 4:** Check the design with a simulation. (The result for the lead compensated system is shown in Figure(a) and is satisfactory.)

- **Step 5:** design the lag compensator to improve the steady-state error.

uncompensated system's open-loop transfer function

$$G(s) = \frac{192.1}{s\,(s+6)(s+10)} \xrightarrow{\text{static error constant}} k_{vO} = 3.201$$

inversely proportional to the steady-state error

lead-compensated system's open-loop transfer function

$$G_{LC}(s) = \frac{1977}{s\,(s+10)(s+29.1)} \xrightarrow{\text{static error constant}} k_v = 6.794$$

the addition of lead compensation has improved the steady-state error by a factor of 2.122

$$\frac{6.794}{3.201} = 2.122 \xrightarrow[\text{lag compensator factor for steady-state error improvement}]{\text{Need of tenfold improvement}} k_{vN} = \frac{10}{2.122} = 4.713$$

Step 6: We arbitrarily choose the lag compensator pole at 0.01,

lag compensator zero

$$\Rightarrow z_c = p_c \frac{k_{vN}}{k_{vO}} = 0.01\frac{4.713}{3.201} = 0.04713 \xrightarrow{\text{lag compensator}} G_{Lag}(s) = \frac{(s+0.04713)}{(s+0.01)} \xrightarrow{\text{lag-lead-compensated Open loop TF}} G_{LLC}(s) = \frac{K\,(s+0.04713)}{s\,(s+10)(s+29.1)(s+0.01)}$$

- The uncompensated system pole at - 6 canceled the lead compensator zero at -6.
- Drawing the complete root locus for the lag-lead-compensated system and by searching along the 0.456 damping ratio line

closed-loop dominant poles

$$\Rightarrow p_c = -3.574 \pm j\,6.976 \quad \textit{with a gain of } 1971.$$

Example6$_4$

- Fig (b) shows the complete draw of the lag-lead-compensated root locus.
- The lag-lead compensation has indeed increased the speed of the system (settling time or the peak time).

Step 7: The final proof of our designs is shown by the simulations of Figure (b)

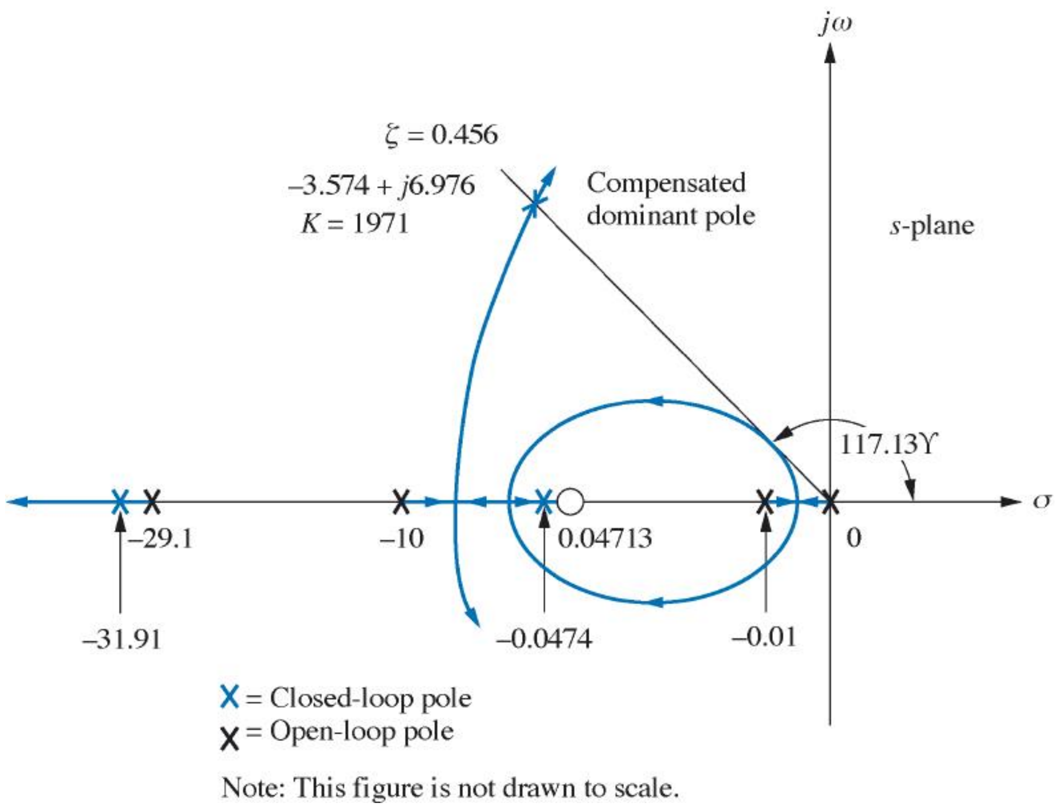


Fig (a) Root locus for lag-lead-compensated system

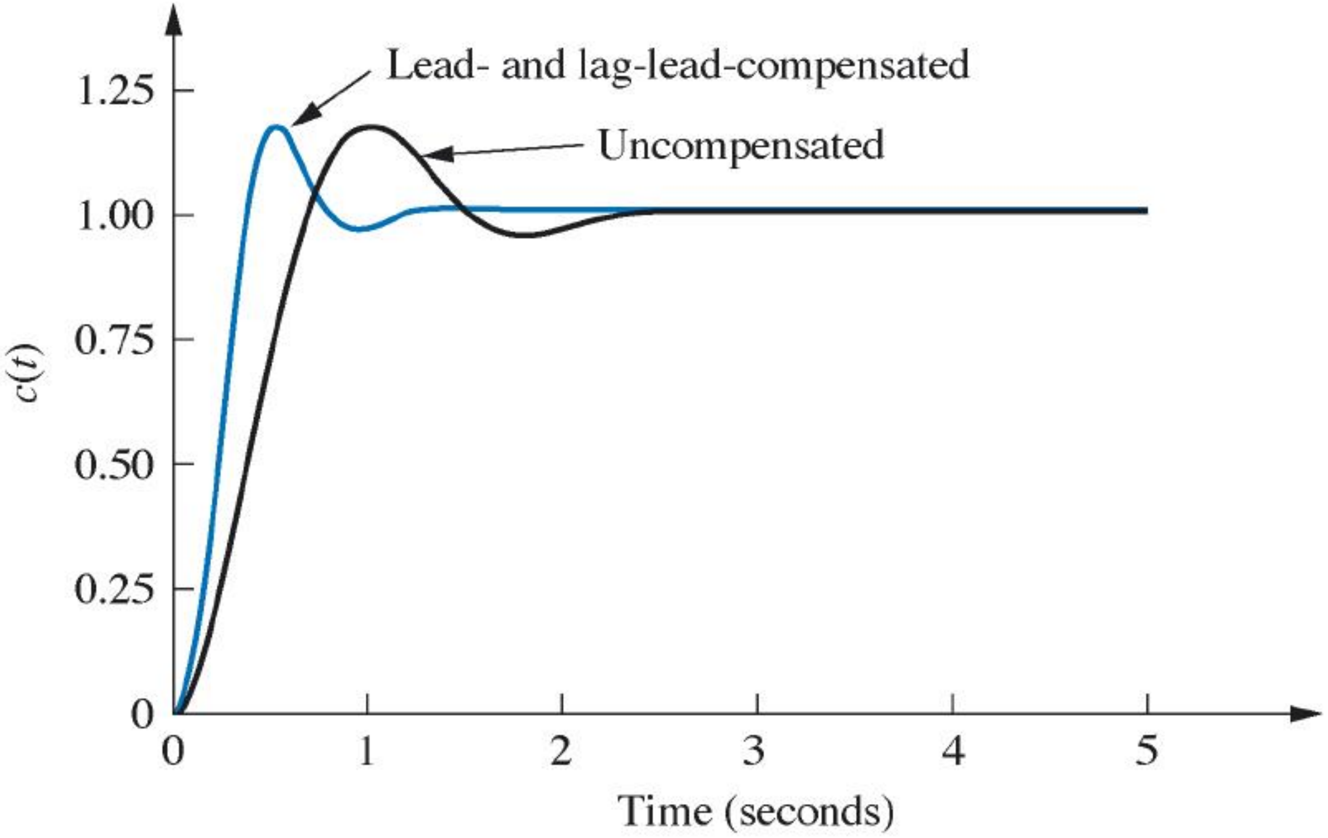
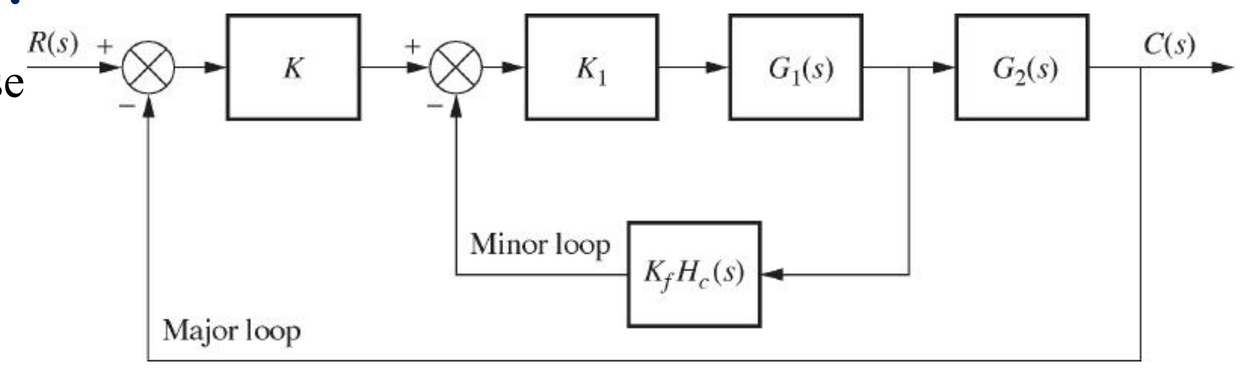


Fig (b) Improvement in step response for lag-lead-compensated system

Feedback Compensation

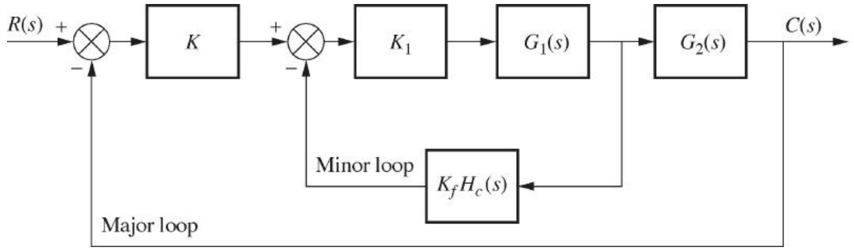
Compensator $H_c(s)$ is used at the minor feedback to reshape the root-locus and improve transient response and steady-state response independently ($G_2(s)$ can be unity).

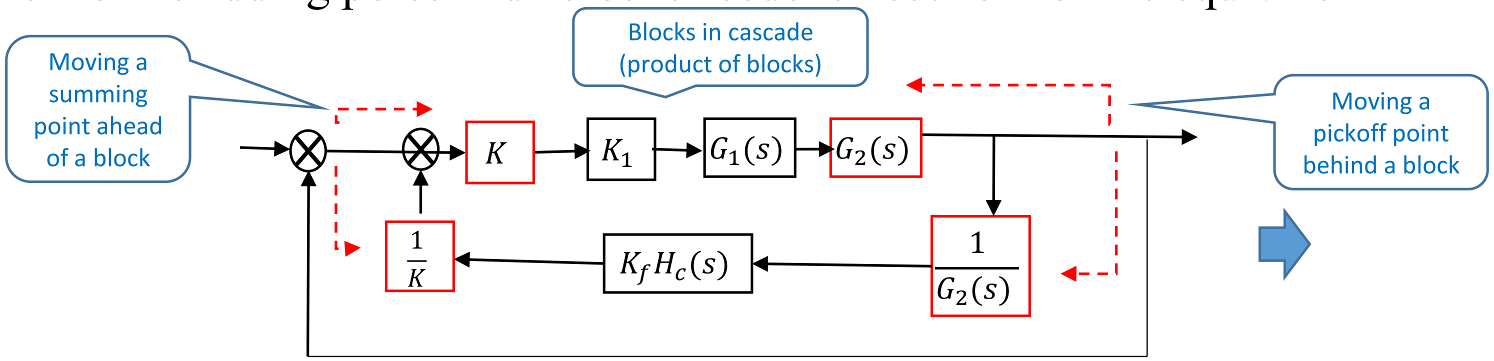
- Can be more complicated than cascade.
- Can provide faster response.
- Can be used in cases where noise is a concern if we use cascade compensators.
- May not require additional gain.

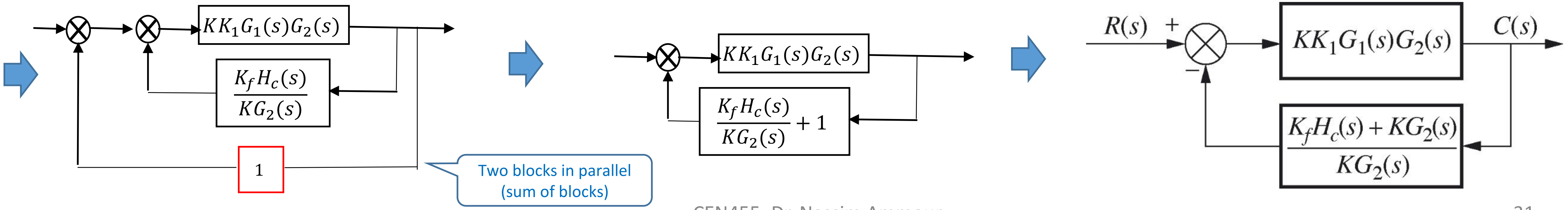


The design of feedback compensation consists of finding the gains, such as K, K_1 *and* K_f.

Similar to cascade compensation. Consider compensation as adding poles and zeros to feedback section for the equivalent system.







Example7$_1$

Given the system of Figure (a), design rate feedback compensation, as shown in Figure (b), to reduce the settling time by a factor of 4 while continuing to operate the system with 20% overshoot.

SOLUTION

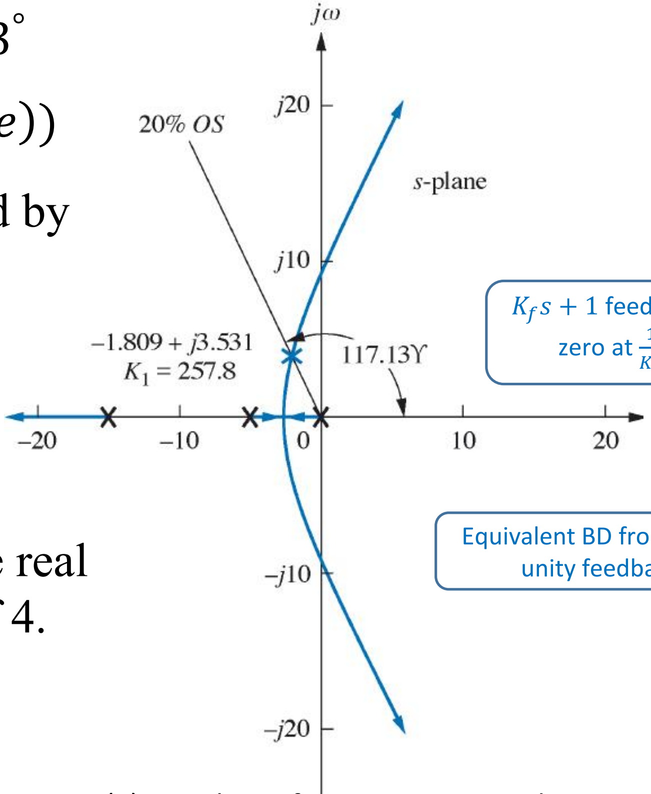
- First design a PD compensator.

- For the uncompensated system, Search along the 20% overshoot line (ζ = 0.456)

The angle of the 20% overshoot line $\Rightarrow$ $180^{\circ} - \arccos(\zeta) = 117.13^{\circ}$

the dominant poles $\Rightarrow$ $-1.809 \pm j\,3.531 \quad (see\ fig\ (e))$

- The settling time is 2.21 seconds and must be reduced by a factor of 4 to 0.55 second.



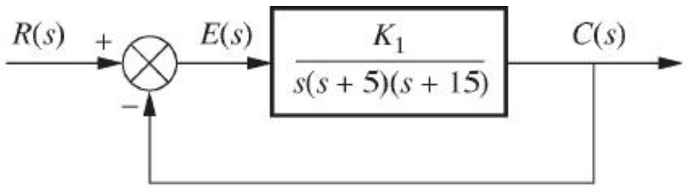
(e) Root locus for uncompensated system

- Next determine the location of the dominant poles for the compensated system.

- To achieve a fourfold decrease in the settling time, the real part of the desired pole must be increased by a factor of 4.

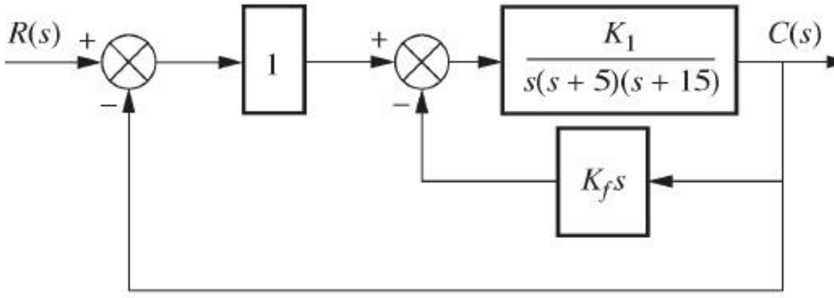
Real part of Compensated pole $\Rightarrow$ $4(-1.809) = -7.236$

Imaginary part of Compensated pole $\Rightarrow$ $w_d = -7.236\tan(117.13^{\circ}) = 14.12$



(a)

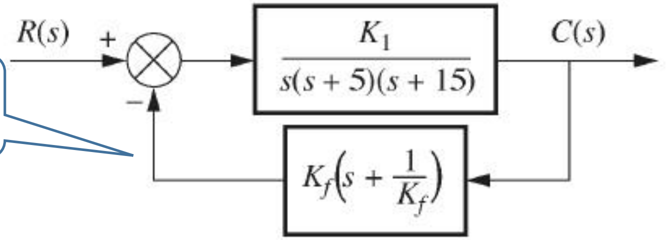
System



(b)

system with rate feedback compensation

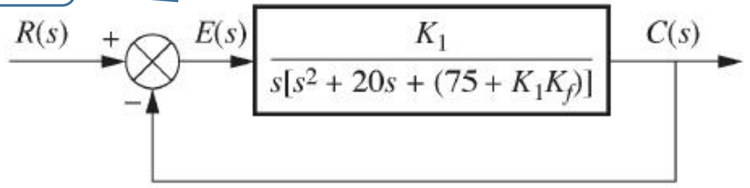
$K_f s + 1$ feedback, zero at $\frac{1}{K_f}$



(c)

equivalent compensated system;

Equivalent BD from fig (b) unity feedback



(d)

equivalent compensated system showing unity feedback

Example7_2

Compensated dominant pole position → $p_c = -7.236 \pm j\ 14.12$

- Sum of the angles from the uncompensated system's poles (add zero to yields 180°)

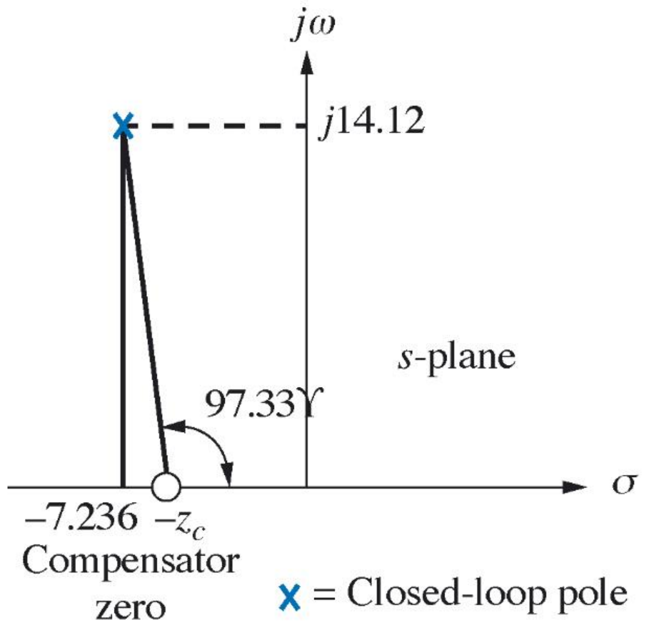
compensator zero contribution

$$\theta = -277.33^{\circ} \quad \Rightarrow \quad \theta_z - 277.33° = -180° \rightarrow \quad \theta_z = +97.33^{\circ}$$

- Using the geometry shown in Figure (a)

Compensator's zero location

$$\frac{14.12}{7.236 - z_c} = \tan(180^{\circ} - 97.33^{\circ}) \quad \Rightarrow \quad z_c = 5.42$$



(a) Finding the compensator zero

- The root locus for the equivalent compensated system (fig (c) previous slide) is shown in Figure (b)

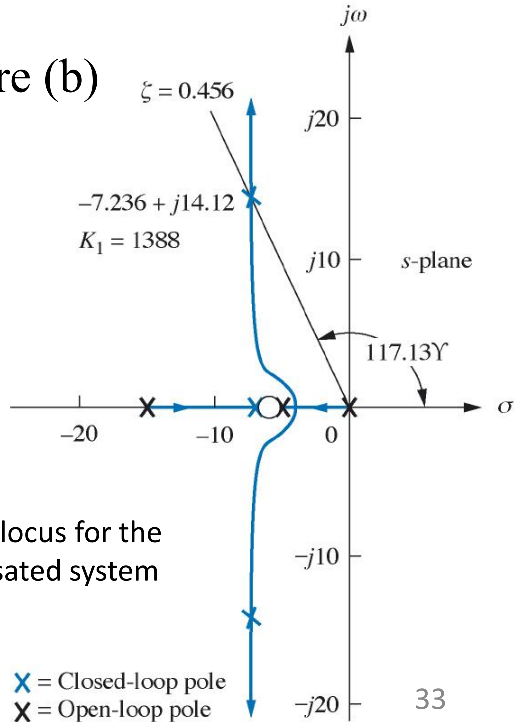
The gain at the design point, → K_1 = 1388

Since K_f is the reciprocal of the compensator zero,

$$z_c = \frac{1}{K_f} \quad \Rightarrow \quad K_f = \frac{1}{z_c} = \frac{1}{5.42} = 0.185 \quad \Rightarrow \quad K_1 K_f = 256.7$$

- steady-state error characteristic (fig (d) slide 32)

$$K_v = \lim_{s \to 0} sG(s) = \frac{K_1}{75 + K_1 K_f} = 4.18$$



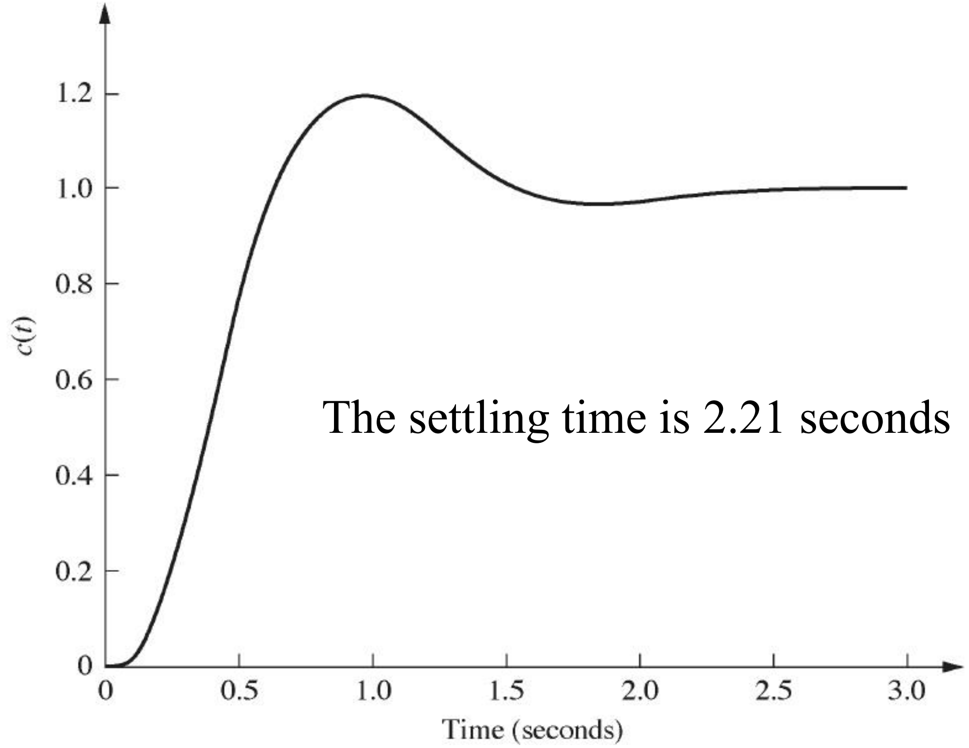
(b) Root locus for the compensated system

Example7$_3$

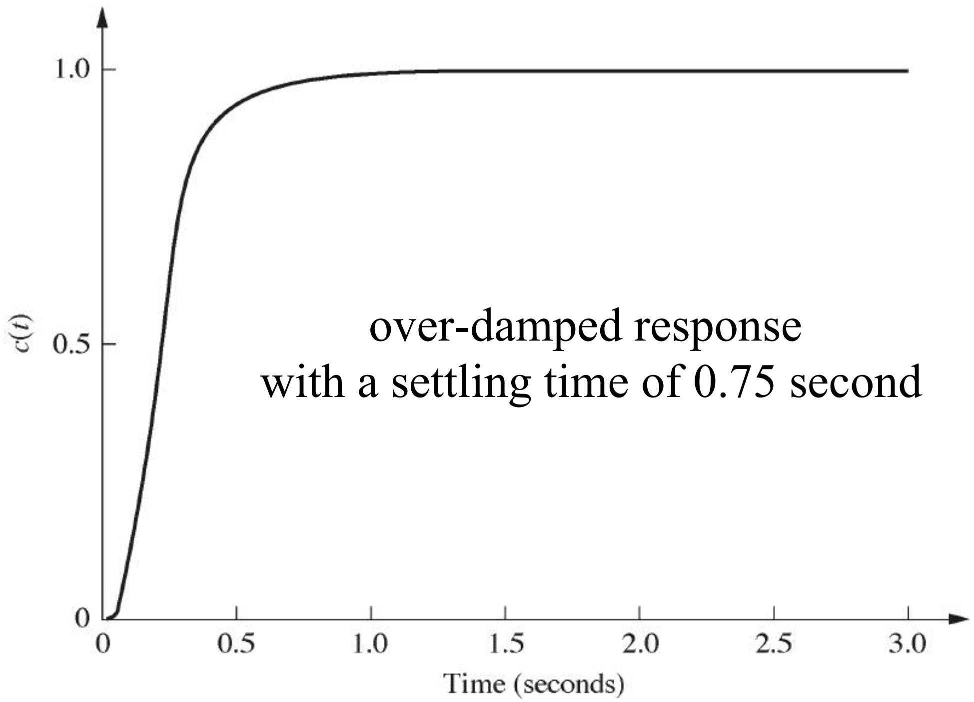
- The closed-loop transfer function is (fig (d) slide 32)

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{K_1}{s^3 + 20\,s^2 + (75 + K_1 K_f)s + K_1}$$

- The results of the simulation are shown in Figure (a) and (b)



(a) Step response for uncompensated system



(b) Step response for the compensated system

Physical Realization of Compensation

Active-Circuit Realization

- $Z_1(s)$and $Z_2(s)$are used as a building block to implement the compensators and controllers, such as PID controllers.

- The transfer function of an inverting operational amplifier

$$\frac{V_0(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}$$

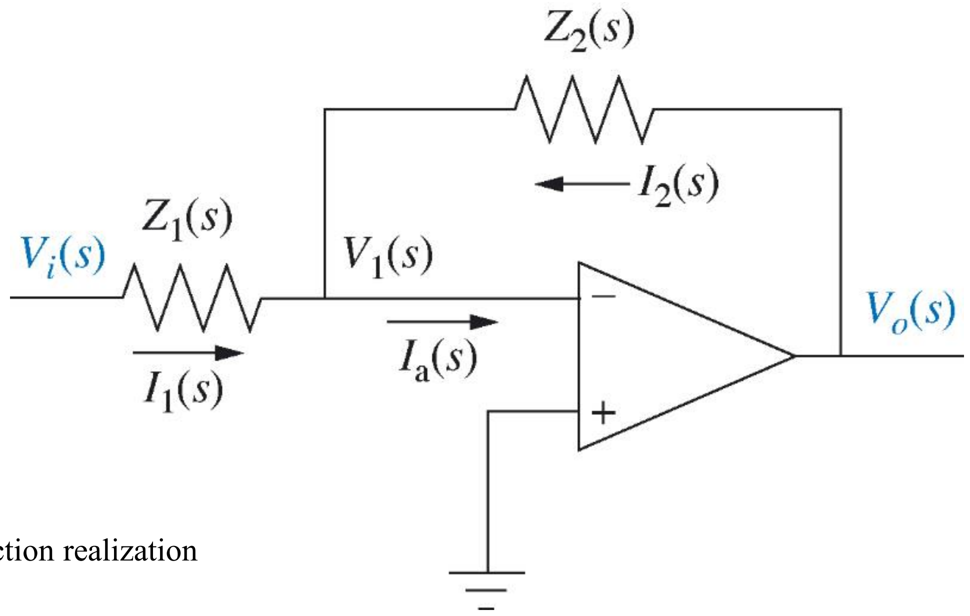


Fig (a) Operational amplifier for transfer function realization

- Table1 summarizes the realization of PI, PD, and PID controllers as well as lag, lead, and lag-lead compensators using Operational amplifiers.

- Fig (a) : lag-lead compensator can be formed by cascading the lag compensator with the lead compensator.

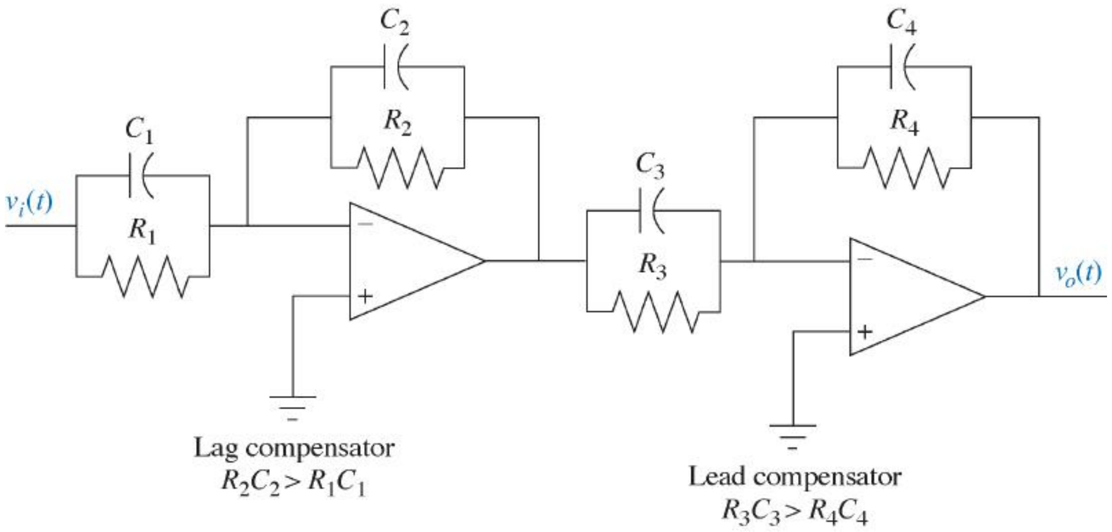


Fig (a) Lag-lead compensator implemented with operational amplifiers

Function	$Z_1(s)$	$Z_2(s)$	$G_c(s) = -\frac{Z_2(s)}{Z_1(s)}$
Gain	R_1	R_2	$-\frac{R_2}{R_1}$
Integration	R	C	$-\frac{\frac{1}{RC}}{s}$
Differentiation	C	R	$-RCs$
PI controller	R_1	R_2 C	$-\frac{R_2}{R_1}\frac{\left(s+\frac{1}{R_2C}\right)}{s}$
PD controller	C R_1	R_2	$-R_2C\left(s+\frac{1}{R_1C}\right)$
PID controller	C_1 R_1	R_2 C_2	$-\left[\left(\frac{R_2}{R_1}+\frac{C_1}{C_2}\right)+R_2C_1s+\frac{\frac{1}{R_1C_2}}{s}\right]$
Lag compensation	C_1 R_1	C_2 R_2	$-\frac{C_1}{C_2}\frac{\left(s+\frac{1}{R_1C_1}\right)}{\left(s+\frac{1}{R_2C_2}\right)}$ where $R_2C_2 > R_1C_1$
Lead compensation	C_1 R_1	C_2 R_2	$-\frac{C_1}{C_2}\frac{\left(s+\frac{1}{R_1C_1}\right)}{\left(s+\frac{1}{R_2C_2}\right)}$ where $R_1C_1 > R_2C_2$

Table 1 Active realization of controllers and compensators, using an operational amplifier

Example8

Implement the PID controller of Example 5

SOLUTION

- The transfer function of the PID controller is $G_c(s) = \dfrac{4.6(s+55.92)(s+0.5)}{s}$
- which can be put in the form $G_c(s) = s + 56.42 + \dfrac{27.96}{s}$
- Comparing the PID controller in Table 1 with this equation we obtain the following three relationships:

$$\frac{R_2}{R_1} + \frac{C_1}{C_2} = 56.42 \qquad R_2C_1 = 1 \qquad \frac{1}{R_1C_2} = 27.96$$

- Rhmbd sgdqd `qd ent q t mj mnv mr `mc sgqdd dpt `shnmr
we arbitrarily select a practical value:

$C_2 = 0.1\ \mu F$ ⇨ $R_1 = 357.65\ k\Omega,\ R_2 = 178.891\ k\Omega\ \ and\ \ C_1 = 5.59\mu F$

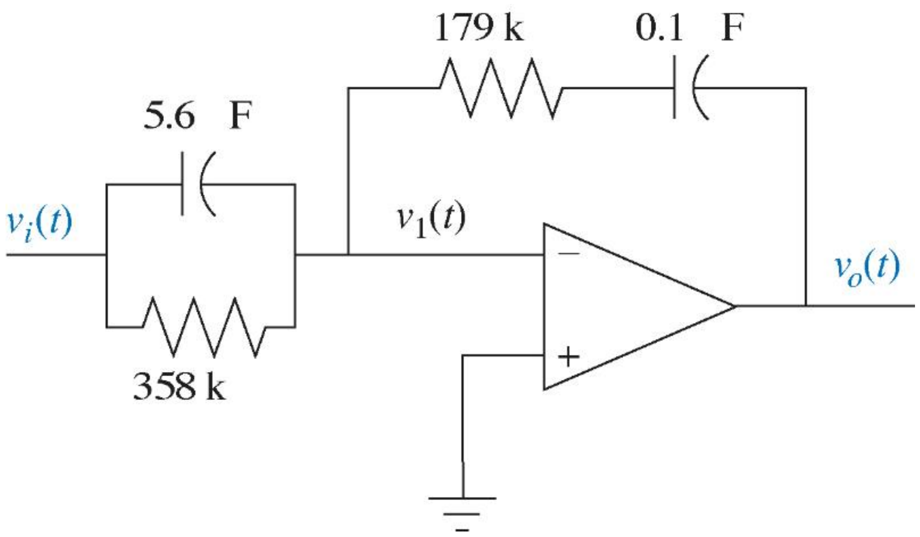


Fig (a) PID controller

- The complete circuit is shown in Figure (a) where the circuit element values have been rounded off.

Passive-Circuit Realization

- Lag, lead, and lag-lead compensators can also be implemented with passive networks (Table 2) .

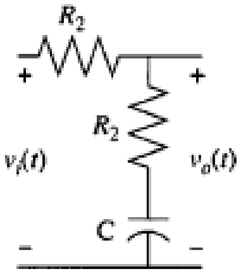
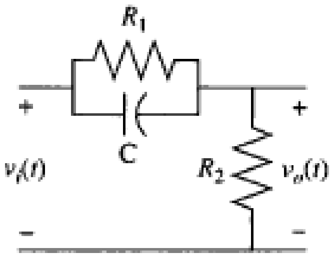
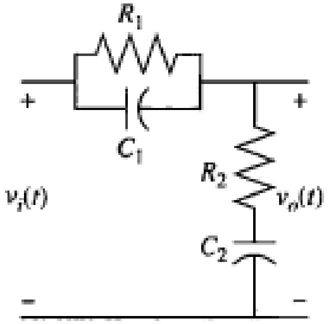
Function	Network	Transfer function, $\frac{V_o(s)}{V_i(s)}$
Lag compensation		$\frac{R_2}{R_1+R_2}\frac{s+\frac{1}{R_2C}}{s+\frac{1}{(R_1+R_2)C}}$
Lead compensation		$\frac{s+\frac{1}{R_1C}}{s+\frac{1}{R_1C}+\frac{1}{R_2C}}$
Lag-lead compensation		$\frac{\left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{1}{R_2C_2}\right)}{s^2+\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_2C_1}\right)s+\frac{1}{R_1R_2C_1C_2}}$

TABLE 2 Passive realization of compensators

Example9

Realize the lead compensator designed in Example 4 (Compensator b zero at -4).

SOLUTION

- The transfer function of the lead compensator is $G_c(s) = \dfrac{s+4}{s+20.09}$

- Comparing the transfer function of a lead network shown in Table 2 with The equation, we obtain the following two relationships:

$$\frac{1}{R_1 C} = 4 \qquad and \qquad \frac{1}{R_1 C} + \frac{1}{R_2 C} = 20.09$$

- Since there are three network elements and two equations, we may select one of the element values arbitrarily

$$C = 1\ \mu F \Rightarrow R_1 = 250\ k\Omega \quad and \quad R_2 = 62.2\ k\Omega$$