Chapter 6

Digital Control Systems

Introduction

- In almost all applications, both the plant and the actuator are analog systems.
- In digital control systems, analog compensators (analog circuits) are replaced with a digital computer (or micro-Controller, microprocessor).

A/D converter converts analog signals to digital signals.
D/A converter converts digital signals to analog signals.

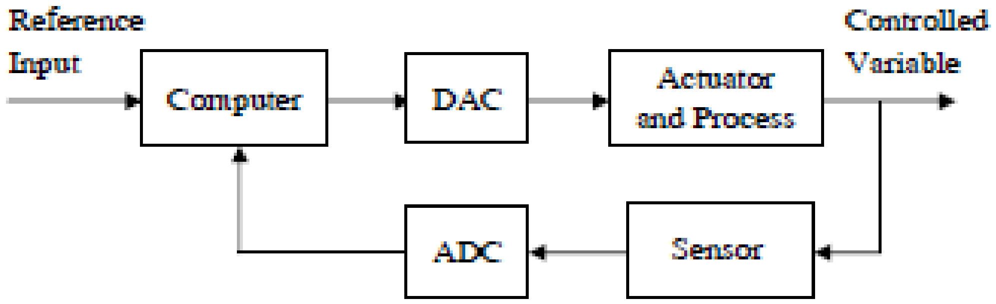


Figure: Configuration of a digital control system.

- **Digital control offers distinct advantages over analog control**

1. **Accuracy:** Digital signal represented by 12 bits or more represents high precision and small error compared to analog signal.
2. **Reduced cost:** A single digital computer can replace numerous analog controllers with a subsequent reduction in cost.
3. **Flexibility in response to design changes**: Modifications can be implemented with software changes rather than expensive hardware modifications.
4. **Noise immunity:** Digital systems exhibit more noise immunity than analog systems.

Difference equations (discrete-time model describing a system)

$$y(k+n) = a_{n-1}y(k+n-1) + \cdots + a_1 y(k+1) + a_0 y(k) = b_n u(k+n) + \cdots + b_0 u(k)$$

Difference equation of order n. If the forcing function $u(k)$ is equal to zero, the equation is said to be homogeneous.

If the equation is linear and the coefficients a_i, b_i are constant, the difference equation is linear time invariant LTI.

Signal Conversion

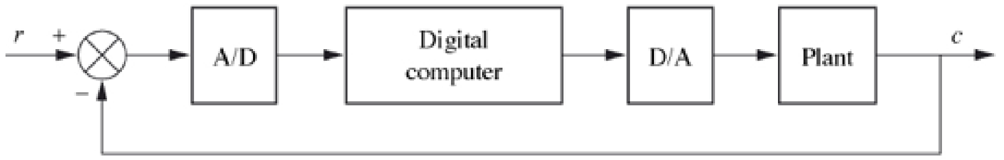
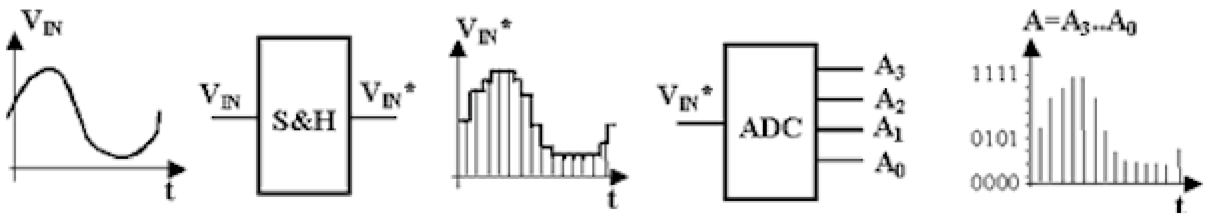


Fig.1. Placement of the digital computer within the loop; with A/D and D/A converters

Analog-to-Digital Conversion (ADC model)

- The analog signal is sampled and held at periodic intervals by a ***zero-order-holder*** (Z-O-H) (staircase approximation to the analog signal.). The sampling rate must be at least twice the bandwidth of the signal (Nyquist sampling rate).

- The ADC converts the sample to a digital number (quantization and coding). ***Quantization error*** is due to the quantization process that rounds off the analog voltage to the next higher or lower level.



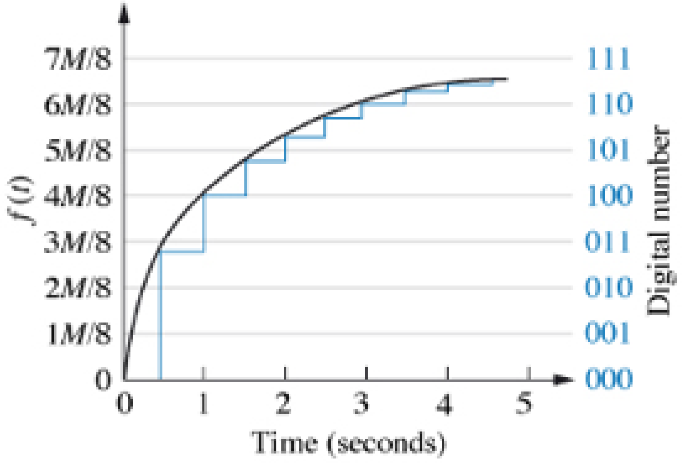
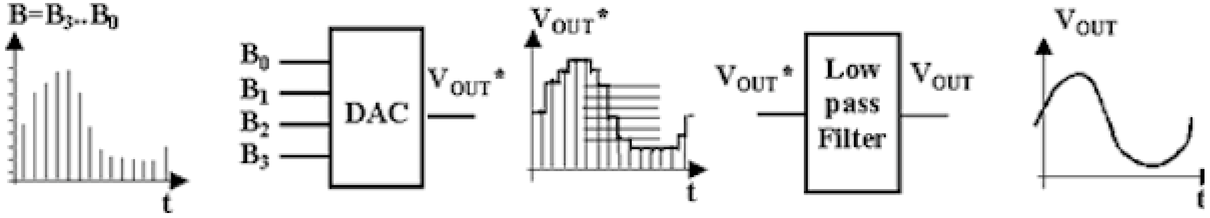


Fig.2. Analog-to-digital conversion

Digital-to-Analog Conversion (DAC model)

- Properly weighted voltages are summed together to yield the analog output.

- the switches are electronic and are set by the input binary code.



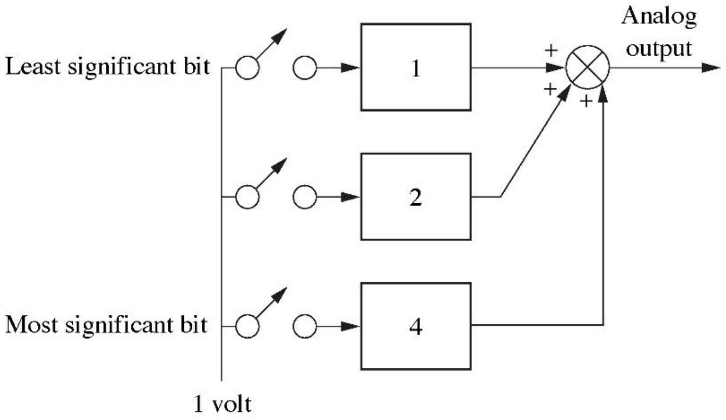


Fig.3. Digital-to-analog converter

Z-Transform

- The z-transform is an important tool in the analysis and design of discrete-time systems. it plays a role similar to that served by Laplace transforms in continuous-time.
- Z-transform of a digital signal $f(k)$ is:

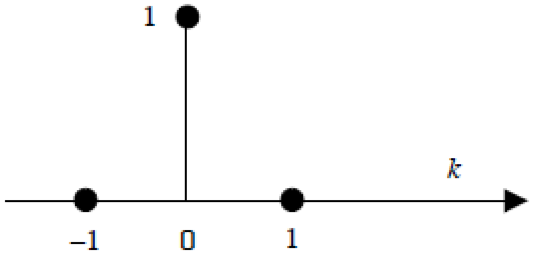
$$F(z) = \sum_{k=-\infty}^{+\infty} f(k) \cdot z^{-k}$$

To overcome the nonlinearity problem, S-domain is transformed to another domain where the operator is linear: Z- domain by setting $z = e^{sT}$.

Examples :

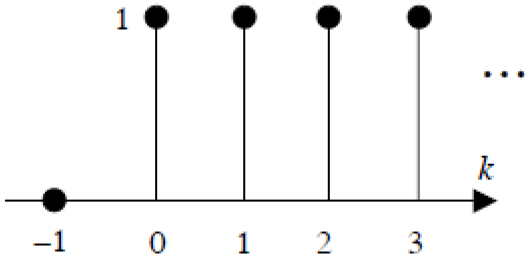
UNIT IMPULSE

$$u(k) = \delta(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$



$$U(z) = 1$$

SAMPLED STEP



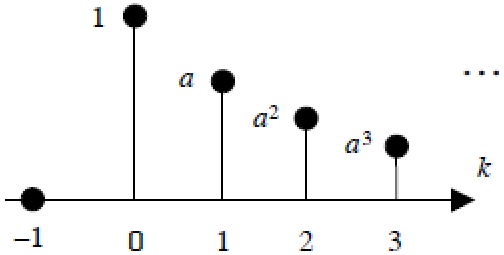
$$U(z) = 1 + z^{-1} + z^{-2} + z^{-3} + \dots + z^{-k} + \dots$$

$$= \sum_{k=0}^{\infty} z^{-k}$$

$$U(z) = \frac{1}{1 - z^{-1}}$$

$$= \frac{z}{z-1}$$

EXPONENTIAL



$$u(k) = \begin{cases} a^k, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

$$U(z) = 1 + az^{-1} + a^2 z^{-2} + \dots + a^k z^{-k} + \dots$$

$$U(z) = \frac{1}{1 - (a/z)}$$

$$= \frac{z}{z-a}$$

3.1 Partial table of z- and s-transforms

TABLE 13.1 Partial table of z- and s-transforms

	$f(t)$	$F(s)$	$F(z)$	$f(kT)$
1.	$u(t)$	$\frac{1}{s}$	$\frac{z}{z-1}$	$u(kT)$
2.	t	$\frac{1}{s^2}$	$\frac{Tz}{(z-1)^2}$	kT
3.	t^n	$\frac{n!}{s^{n+1}}$	$\lim_{a\to 0}(-1)^n \frac{d^n}{da^n}\left[\frac{z}{z-e^{-aT}}\right]$	$(kT)^n$
4.	e^{-at}	$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$	e^{-akT}
5.	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$	$(-1)^n \frac{d^n}{da^n}\left[\frac{z}{z-e^{-aT}}\right]$	$(kT)^n e^{-akT}$
6.	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$\frac{z \sin \omega T}{z^2 - 2z\cos \omega T + 1}$	$\sin \omega kT$
7.	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	$\frac{z(z-\cos \omega T)}{z^2 - 2z\cos \omega T + 1}$	$\cos \omega kT$
8.	$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$	$\frac{ze^{-aT}\sin \omega T}{z^2 - 2ze^{-aT}\cos \omega T + e^{-2aT}}$	$e^{-akT}\sin \omega kT$
9.	$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$	$\frac{z^2 - ze^{-aT}\cos \omega T}{z^2 - 2ze^{-aT}\cos \omega T + e^{-2aT}}$	$e^{-akT}\cos \omega kT$

3.2 Some Properties

	Theorem	Name
1.	$z\{af(t)\} = aF(z)$	Linearity theorem
2.	$z\{f_1(t) + f_2(t)\} = F_1(z) + F_2(z)$	Linearity theorem
3.	$z\{e^{-aT}f(t)\} = F(e^{aT}z)$	Complex differentiation
4.	$z\{f(t - nT)\} = z^{-n}F(z)$	Real translation
5.	$z\{tf(t)\} = -Tz\dfrac{dF(z)}{dz}$	Complex differentiation
6.	$f(0) = \lim_{z\to\infty} F(z)$	Initial value theorem
7.	$f(\infty) = \lim_{z\to 1}(1 - z^{-1})F(z)$	Final value theorem

The Inverse Z-Transform (IZT)

- Three methods for finding the inverse z-transform (the sampled time function from its z-transform) will be described:

1. partial-fraction expansion.
2. Residue.
3. the power series method.

- Since the z-transform came from the sampled waveform, the inverse z-transform will yield only the values of the time function at the sampling instants.

IZT via Partial-Fraction Expansion

- The Laplace transform consists of a partial fraction that yields a sum of terms leading to exponentials, that is, *A/(s + a)*.

- Knowing that: $\frac{z}{z-a} \rightarrow a^{kT}$ or $\frac{z}{z-e^{-bT}} \rightarrow \frac{1}{s+b} \rightarrow e^{-b\,k\,T}$

$$F(z) = \frac{N(z)}{D(z)} \Longrightarrow \frac{F(z)}{z} = \frac{A}{z-a} + \frac{B}{z-b} + \cdots \Longrightarrow F(z) = \frac{A\,z}{z-a} + \frac{B\,z}{z-b} + \cdots$$

- The inverse is: $f(kT) = A\,a^k + B\,b^k + \cdots$

Example

Given the function F(z) find the sampled time function. $F(z) = \frac{0.5\,z}{(z-0.5)(z-0.7)}$

SOLUTION

- First we divide F(z) by z (z-functions have often the term z in their numerator) , then we perform a partial-fraction expansion

$$\frac{F(z)}{z} = \frac{0.5}{(z-0.5)(z-0.7)} = \frac{A}{z-0.5} + \frac{B}{z-0.7} = \frac{-2.5}{z-0.5} + \frac{2.5}{z-0.7}$$

- Next, multiply through by *z*. $F(z) = \frac{0.5\,z}{(z-0.5)(z-0.7)} = \frac{-2.5\,z}{z-0.5} + \frac{2.5\,z}{z-0.7}$

- Thus the inverse z-transform $f(k) = -2.5(0.5)^k + 2.5(0.7)^k$

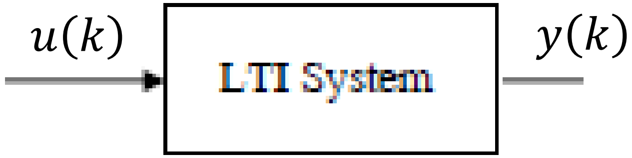
- The ideal sampled time function is: $f^*(t) = \sum_{k=0}^{\infty} f(kT)\cdot\delta(t-Kt) = \sum_{k=0}^{\infty}\left(-2.5(0.5)^k + 2.5(0.7)^k\right)\cdot\delta(t-Kt)$

Time Response and Transfer Function of a discrete system

- The *response* of an LTI discrete-time system to an input sequence is given by the *convolution* of the input sequence and the impulse response sequence of the system (transfer function of the system $\mathrm{h}(\mathrm{k})$).

System's transfer function

$$y(k) = h(k) * u(k) = \sum_{i=0}^{\infty} h(k-i)u(i)$$



- The *z-transform of the convolution* of two time sequences is equal to the *product of their z-transforms*.

$$Y(z) = \sum_{k=0}^{\infty} y(k)z^{-k} = \sum_{k=0}^{\infty}\left[\sum_{i=0}^{\infty} h(k-i)u(i)\right] z^{-k} \xrightarrow{j=k-i} Y(z) = \sum_{i=0}^{\infty}\sum_{j=-i}^{\infty} u(i)h(j)\, z^{-(i+j)} = \left[\sum_{i=0}^{\infty} u(i)z^{-i}\right]\left[\sum_{j=0}^{\infty} h(j)z^{-j}\right]$$

Causality property

$$\Rightarrow \boxed{Y(z) = H(z)\, U(z)}$$

$H(z)$ is the transfer function of the system

Example: Given the discrete-time system $y(k+1) - y(k) = u(k+1)$, find the system transfer function and the system response to a sampled unit step.

The transfer function: $zY(z) - Y(z) = zU(z) \Rightarrow H(z) = \dfrac{Y(z)}{U(z)} = \dfrac{z}{z-1}$

unit step repose: $Y(z) = H(z)U(z) = \dfrac{z}{z-1}\dfrac{z}{z-1} = z\dfrac{z}{(z-1)^2} = zR(z)$ $\xrightarrow{\text{Inverse Z-transform}}$ $y(k) = \begin{cases} k+1 & k \geq 0 \\ 0 & k < 0 \end{cases}$ Time advanced ramp

unit step's z-transform

z-transform of unit ramp

Modeling the Digital Computer

Modeling the Sampler

- The objective is to derive a mathematical model for the digital computer (transfer function) as represented by a sampler and zero-order hold.
- When signals are sampled, the Laplace transform can be replaced by another related transform called ***the z-transform***.

- The sampling model is a switch turning on and off at a uniform sampling rate T. It can also be considered to be a product of the time waveform to be sampled $f(t)$ and a sampling function $s(t)$ with pulse width T_w.

- The time equation of the sampled waveform $f_{Tw}^{*}(t)$

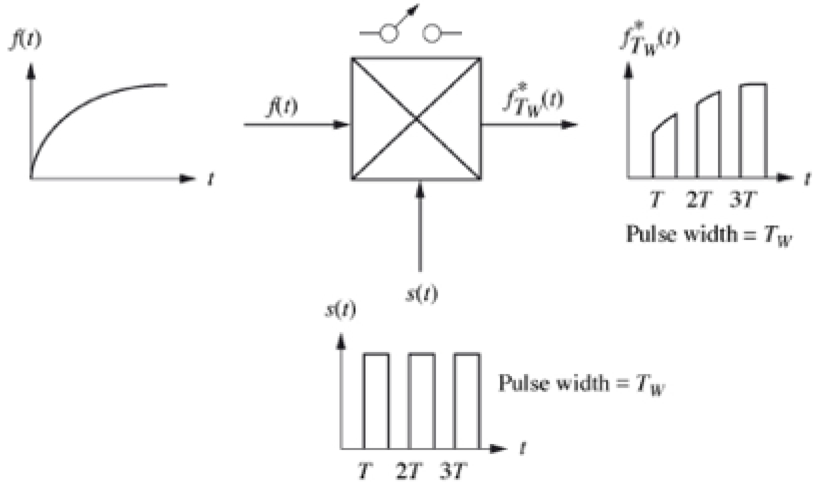


FIG.1. uniform-rate sampling (switch opening and closing), product of time waveform and sampling waveform

Hold ideal sampler

$$f_{Tw}^{*}(t) = f(t) \cdot s(t) = f(t) \cdot \sum_{k=-\infty}^{+\infty} [u(t-kT) - u(t-kT-T_w)] = T_w \cdot f^{*}(t)$$

Using the Laplace transform,

Replace $e^{-T_w s}$ with its series expansion

Laplace inverse transform

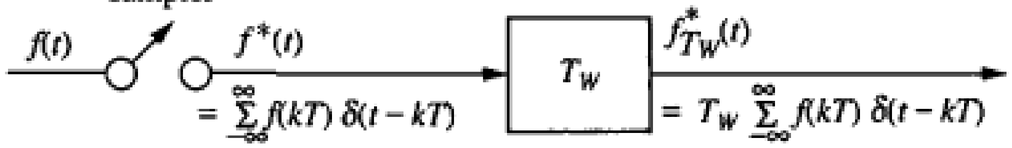
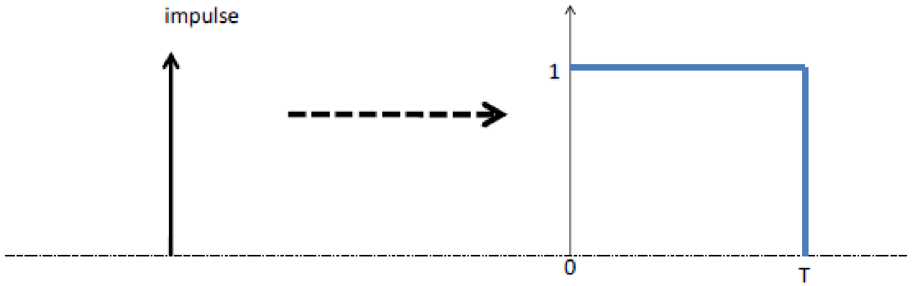


Fig.2. Model of sampling with a uniform rectangular pulse train

Modeling the Digital Computer

Modeling the zero-order hold

- The digital computer is modeled by cascading two elements: (1) an ideal sampler and (2) a zero-order hold.
- The zero-order hold follows the sampler and holds the last sampled value of *f(t)*.
- Using an impulse input at zero time, the output is a step that starts at $t = 0$ and ends at $t = T$



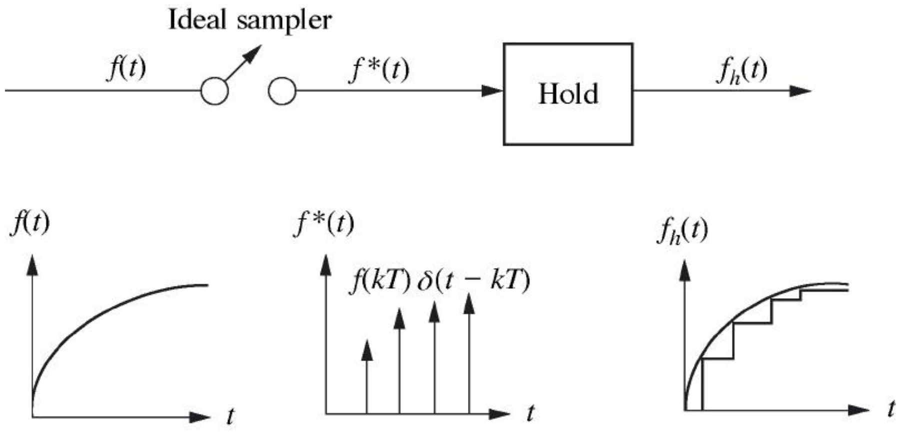


FIGURE (1) Ideal sampling and the zero-order hold

The output is: $y(t)=h(t)=u(t)-u(t-T)$ → Laplace transform is:

$$\mathcal{L}\{u(t)-u(t-T)\}=\frac{1}{s}-\frac{1}{s}\,e^{-Ts}=\frac{1-e^{-Ts}}{s}$$

Proof

$$\mathcal{L}\{u(t)-u(t-T)\}=\int_0^\infty u(t)e^{-st}dt-\int_0^\infty u(t-T)e^{-st}dt=\int_0^\infty e^{-st}dt-\int_T^\infty e^{-st}dt=\left(-\frac{1}{s}e^{-st}\right)\Big|_0^\infty-\left(-\frac{1}{s}e^{-st}\right)\Big|_T^\infty$$

$u(t-T) = 1$ for $t \geq T$

$$\Rightarrow\quad \mathcal{L}\{u(t)-u(t-T)\}=\frac{1}{s}-\frac{1}{s}\,e^{-Ts}=\frac{1-e^{-Ts}}{s}$$

$$\Rightarrow\quad \boxed{ZOH(s)=G_h(s)=\frac{1-e^{-Ts}}{s}}$$

Example: Converting G1(s) in Cascade with z.o.h. to G(z)

Given a z.o.h. in cascade with $G_1(s) = \dfrac{s+2}{s+1}$ or $G(s) = \dfrac{1-e^{-Ts}}{s}\dfrac{s+2}{s+1}$

find the sampled-data transfer function, *G(z),* if the sampling time, *T,* is 0.5 second.

SOLUTION

- Knowing that $Z^{-1} = e^{-sT}$ ➡ $G(s) = (1-e^{-Ts})\dfrac{G_1(s)}{s}$ ➡ $G(z) = (1-z^{-1})\mathbb{Z}\left[\dfrac{G_1(s)}{s}\right] = \dfrac{z-1}{z}\mathbb{Z}\left[\dfrac{G_1(s)}{s}\right]$

- the impulse response (inverse Laplace transform) of $\dfrac{G_1(s)}{s}$

$$G_2(s) = \frac{G_1(s)}{s} = \frac{s+2}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} = \frac{2}{s} - \frac{1}{s+1}$$

- Taking the inverse Laplace transform $g_2(t) = 2 - e^{-t}$ ➡ $g_2(kT) = 2 - e^{-kT}$ ➡ (From Table) $G_2(z) = \dfrac{2z}{z-1} - \dfrac{z}{z-e^{-T}}$

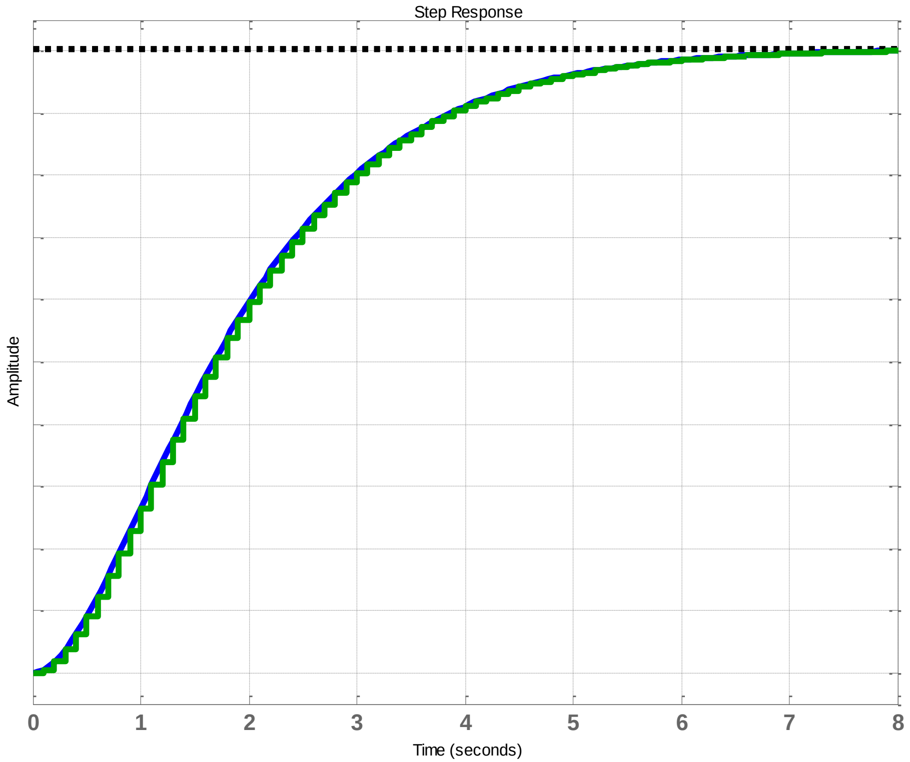
- Substituting $T = 0.5s$ $\quad G_2(z) = \mathbb{Z}\left[\dfrac{G_1(s)}{s}\right] = \dfrac{2z}{z-1} - \dfrac{z}{z-0.607} = \dfrac{z^2 - 0.213z}{(z-1)(z-0.607)}$

➡ $$G(z) = \frac{z-1}{z}Z\left[\frac{G_1(s)}{s}\right] = \frac{z-0.213}{z-0.607}$$

Available Commands for Continuous/Discrete Conversion

The commands c2d, d2c, and d2d perform continuous to discrete, discrete to continuous, and discrete to discrete (resampling) conversions, respectively.

```
sysd = c2d(sysc,Ts)  % Discretization w/ sample period Ts
sysc = d2c(sysd)     % Equivalent continuous-time model
sysd1= d2d(sysd,Ts)  % Resampling at the period Ts
```




```
>>  sys=tf(1,[1 2 1])

Transfer function:
      1
-------------
s^2 + 2 s + 1

>> Ts=0.1; sysd=c2d(sys,Ts)

Transfer function:
0.004679 z + 0.004377
---------------------
z^2 - 1.81 z + 0.8187

Sampling time (seconds): 0.1
>> step(sys,sysd)
```

MATLAB Code :

Continuous/Discrete system

```
>> T=1;Num=1;Den=[1 0 0];
>> sysc=tf(Num,Den);
>> sysd=c2d(sysc,T,'zoh')
Transfer function:
 0.5 z + 0.5
-------------
z^2 - 2 z + 1
Sampling time (seconds): 1
```

Block Diagram Reduction

- *Objective*: find the closed-loop *sampled-data* transfer function of an arrangement of subsystems that have a *computer* in the loop. When manipulating block diagrams for sampled-data systems, the rule is:

$$Z\{G_1(s)G_2(s)\} \neq G_1(z)G_2(z) \qquad \{G_1(s)G_2(s)\}^* \neq G_1(s)^*\, G_2(s)^* \qquad \{G_1(s)G_2(s)^*\}^* = G_1(s)^*\, G_2(s)^*$$

Sampled data systems

Their z-transforms

$C(z) = R(z)G(z)$ → (a)

no sampler between $G_1(s)$ and $G_2(s)$

single transfer function $G_1(s)G_2(s) = G_{12}(s)$

$C(z) = R(z)G_{12}(z)$ → (b)

cascaded of two subsystems each like in (a)

$C(z) = R(z)\, G_1(z)\, G_2(z)$ → (c)

the continuous signal entering the sampler is $R(s)G_1(s)$

$RG_1(z) = Z\{R(s)G_1(s)\}$

$C(z) = RG_1(z)\, G_2(z)$ → (d)

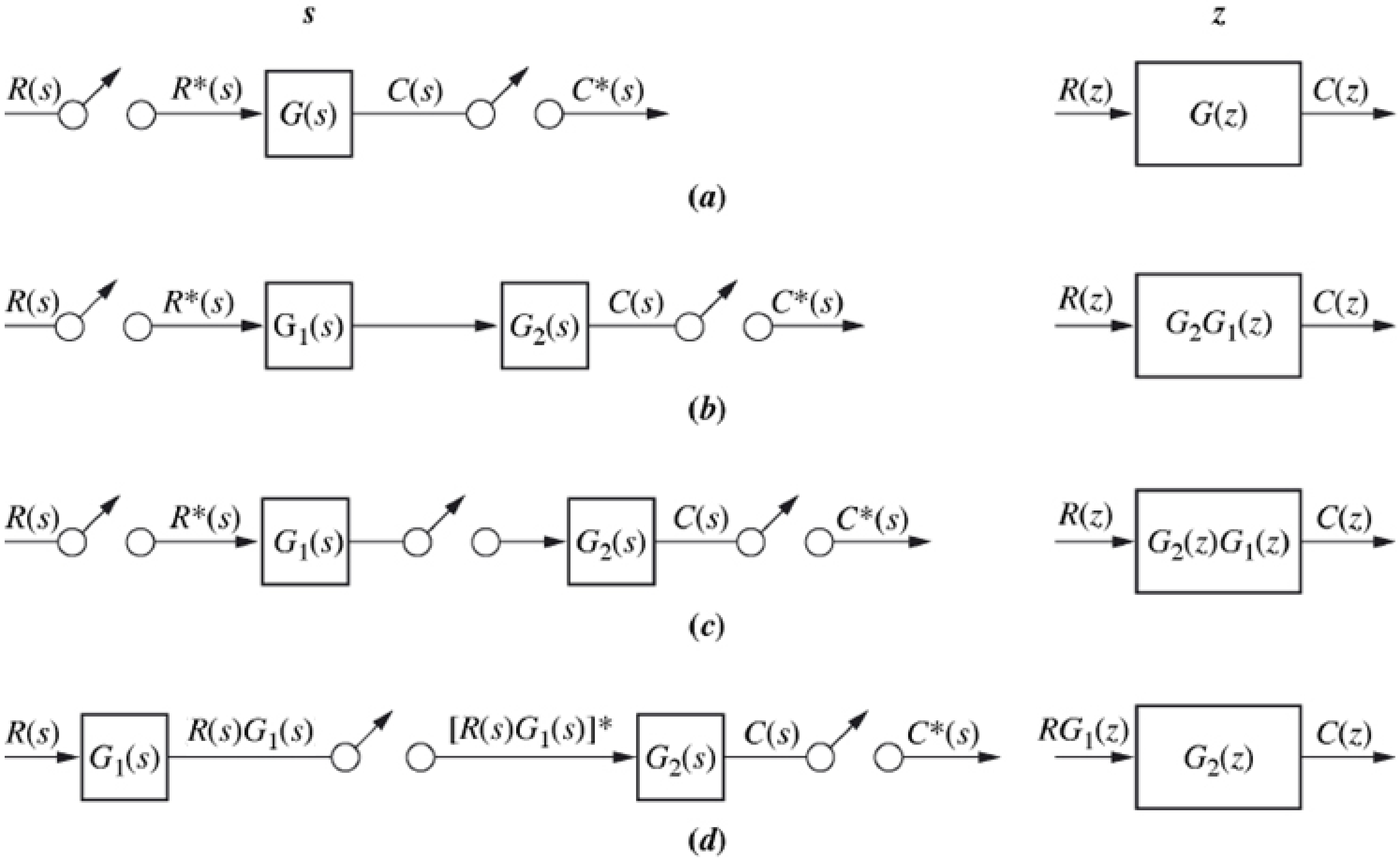
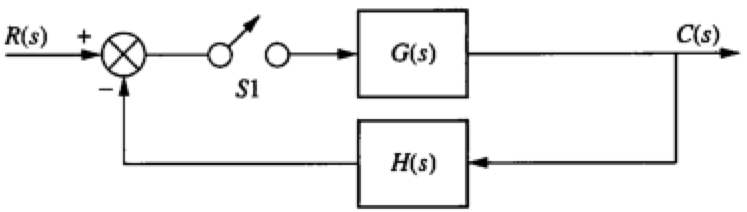


Figure Sampled-data systems and their z-transforms

Example



Find the z-transform of the system shown in Figure

SOLUTION

- The objective is to reduce the block diagram of Figure (a) and reducing it to the one shown in Figure (f).

1. place a phantom sampler at the output of any subsystem that has a sampled input (is not an input to other subsystem)

2. add phantom samplers S2 and S3 at the input to a summing junction whose output is sampled (synchronized samplers).

3. move sampler S1 and G(s) to the right past the pickoff point (to yield a sampler at the input of G(s)H(s))

   $G(s)H(s)$ with samplers S1 and S3 becomes $GH(z)$

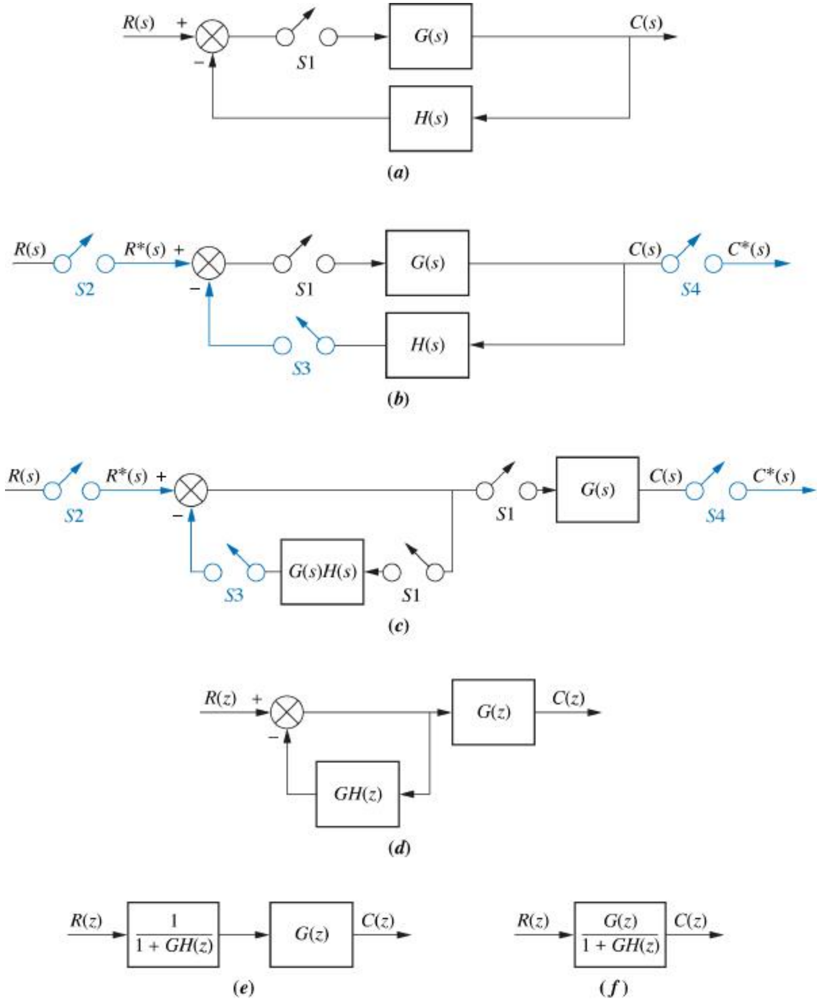
   $G(s)$ with samplers S1 and S4 becomes $G(z)$

   Converting $R^*(s)$ to $R(z)$ and $C^*(s)$ to $C(z)$

   Now we have the system represented totally in the z-domain

4. using the feedback formula, we obtain the first block (Fig(e))

5. multiplication of the cascaded sampled-data systems yields the final result (Fig(f))



Note: Phantom samplers are shown in color.

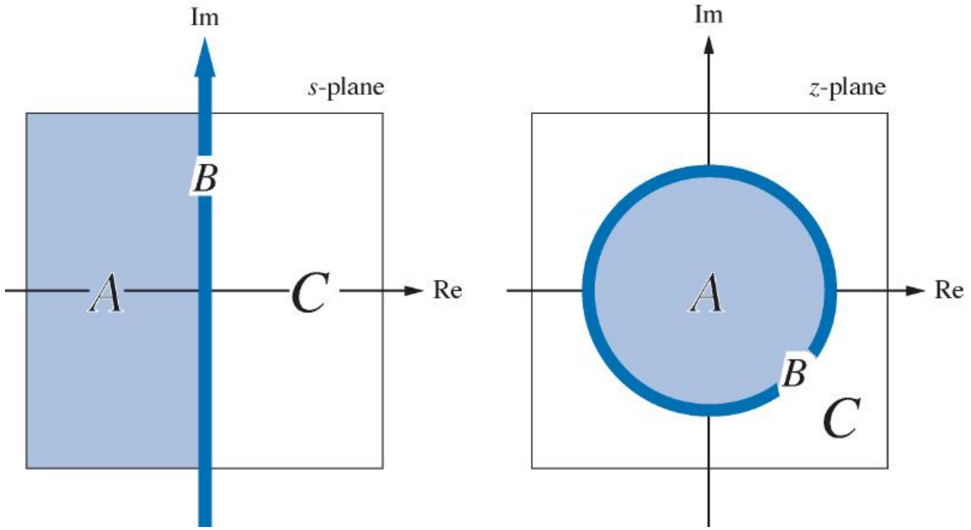
FIGURE Steps in block diagram reduction of a sampled-data system

6 Stability

- The stability of digital system can be analyzed in Z-Domain or in S-Domain.
- Changes in *sampling rate* not only change the nature of the response from over-damped to underdamped, but also can turn a *stable system into an unstable one*.

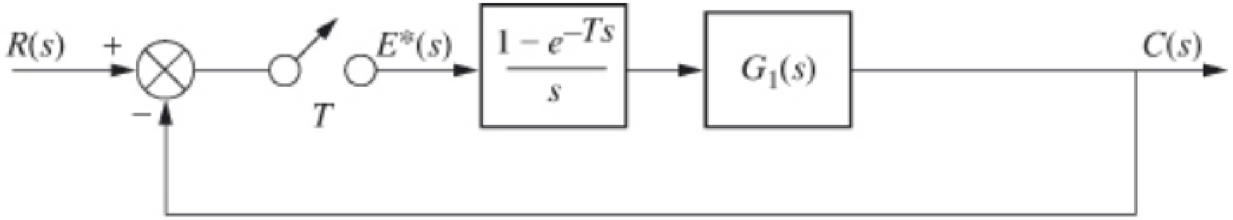
Digital System stability via Z-Plane

- In the S-plane, the region of stability is the left half-plane.
- If the transfer function, *G(s),* is transformed into a sampled-data transfer function, *G(z),* the region of stability on the z-plane can be evaluated from $Z = e^{Ts}$.
- Letting $s = \alpha + j\omega$ we obtain: $Z = e^{Ts} = e^{T(\alpha + j\omega)} = e^{\alpha T} e^{j\omega T} = e^{\alpha T}(\cos \omega T + j sin\omega T) = e^{\alpha T}\angle\omega T$
- From the above equation, we can deduce that the stable domain that corresponds to α<0, lies inside the unity circle, the $j\omega$ (α=0) axis lies on the unity circle, and the unstable domain α>0 lies outside the unity circle.



- Thus, a digital system is stable if and only if all poles of the closed-loop transfer function T(z) are inside the unity circle.
- The digital system is marginally stable if poles of multiplicity one of the closed-loop transfer function *T*(*z*) are on the unity circle and other are inside the unity circle.

Example



Study the stability of the closed-loop system in the figure.

Where $G_1(s) = \frac{1}{s+2}$ and T=0.5s

SOLUTION

$$G(z) = (1 - z^{-1})\mathbb{Z}\left[\frac{G(s)}{s}\right] = \frac{0.316}{z - 0.368}$$

$$T(z) = \frac{G(z)}{1 + G(z)} = \frac{0.316}{z - 0.05}$$

since the pole is inside the unity circle then the system is stable.

MATLAB Code:

```
>> T=1;Num=1;Den=[1 2];
>> T=0.5;Num=1;Den=[1 2];
>> sysc=tf(Num,Den);
>> sysd=c2d(sysc,T,'zoh')

Transfer function:
  0.3161
----------
z - 0.3679
 Sampling time (seconds): 0.5
```

```
>> sysclD=feedback(sysd,1)

Transfer function:
  0.3161
-----------
z - 0.05182

Sampling time (seconds): 0.5
```

- We can check the stability with regards to the sampling period T:

$$G(s) = \frac{(1 - e^{-Ts})}{s(s+2)} \Rightarrow G(z) = 0.5\frac{1 - e^{-2T}}{z - e^{-2T}}$$

$$\Rightarrow T(z) = \frac{G(z)}{1 + G(z)} = \frac{0.5(1 - e^{-2T})}{z - (1.5e^{-2T} - 0.5)} \Rightarrow$$

The pole is $(1.5e^{-2T} - 0.5)$

The system is stable for all T >0.

- Let $\mathrm{G}(s) = \frac{10}{s+1} \Rightarrow G(z) = 10\,\frac{(1 - e^{-T})}{z - e^{-T}} \Rightarrow T(z) = \frac{10\,(1 - e^{-T})}{z - (11e^{-T} - 10)} \Rightarrow$

The pole is $(11e^{-T} - 10)$

The system is stable for:

0 < T< 0.2.

$$for\ T = 0 \quad \to 11e^{-0} - 10 = 1 \qquad and \qquad for\ T = 0.2 \quad \to 11e^{-0.2} - 10 = -1$$

- The pole is $(11e^{-T} - 10)$, monotonically decreases from +1 to -1 for 0 < T < 0.2.
- For 0.2 < T <∞, $(11e^{-T} - 10)$ monotonically decreases from - 1 to -10.
- Thus, the pole of *T(z)* will be inside the unit circle, and the system will be stable if 0 < T < 0.2.
- In terms of frequency, where f= 1/T, the system will be stable as long as the sampling frequency is 1/0.2 = 5 hertz or greater.

Stability via S-plane (Routh-Hurwitz criterion)

- Find the transformation from z-Domain to s-domain $G(s) = G(Z)|_{Z=e^{sT}}$ (nonlinear operator).
- The most used transformation is the *Bilinear Transformation*, where: $Z = \frac{s+1}{s-1}$ (mapping from s-domain to z-domain)

Example: Let the characteristic equation of a system be: $D(z) = z^3 - z^2 - 0.2\,z + 0.1 = 0$

In s-domain for $Z = \frac{s+1}{s-1}$, this is equivalent to : $s^3 - 19\,s^2 - 45\,s - 17 = 0.$

TABLE 13.3 Routh table for Example 13.8

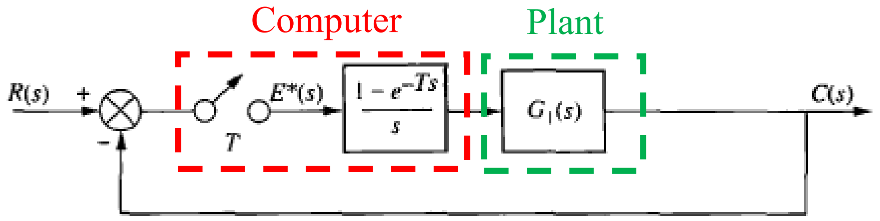
s^3	1	−45
s^2	−19	−17
s^1	−45.89	0
s^0	−17	0

Thus system is unstable and has 1 pole outside the unity circle.
No pole on the unity circle and two poles inside the unity circle.

Steady-State Errors

Effect of sampling upon the steady-state error

- Consider the digital system where the digital computer is represented by the sampler and zero-order hold. The transfer function of the plant is represented by $G_1(s)$.

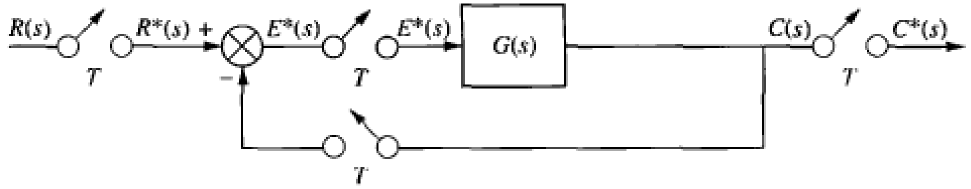


(a)

- we have: $E(z) = R(z) - C(z),$

Or from (d): $E(z) = \dfrac{R(z)}{1+G(z)}$ ←

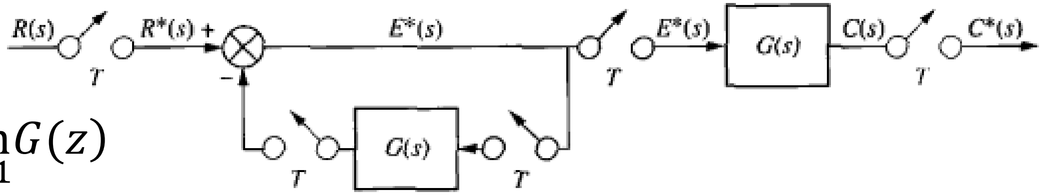
$$E(z) = R(z) - G(z)E(z)$$
$$E(z) + G(z)E(z) = R(z)$$
$$E(z)[1+G(z)] = R(z)$$



(b)

- Using the final value theorem (for discrete signals see slide 9):

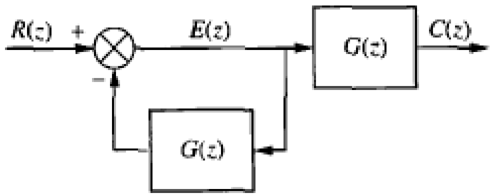
$$e_{ss}^* = e^*(\infty) = \lim_{z\to 1}(1-z^{-1})\,E(z) = \lim_{z\to 1}(1-z^{-1})\,\frac{R(z)}{1+G(z)} \quad (1)$$



(c)

1. Unit Step Input: $\boldsymbol{R(z) = \frac{z}{z-1}}$ Using formula (1)

$$e_{ss} = \lim_{z\to 1}\left(\frac{z-1}{z}\right)\frac{\left(\frac{z}{z-1}\right)}{1+G(z)} = \frac{1}{1+\lim_{z\to 1} G(z)} \Rightarrow e_{ss} = \frac{1}{1+K_p}$$ Where $K_p = \lim_{z\to 1} G(z)$



(d)

2. Unit Ramp Input: $\boldsymbol{R(z) = \frac{T\,z}{(z-1)^2}}$ Using formula (1)

$$e_{ss} = e(\infty) = \frac{1}{K_v}$$ Where $K_v = \frac{1}{T}\lim_{z\to 1}(z-1)G(z)$

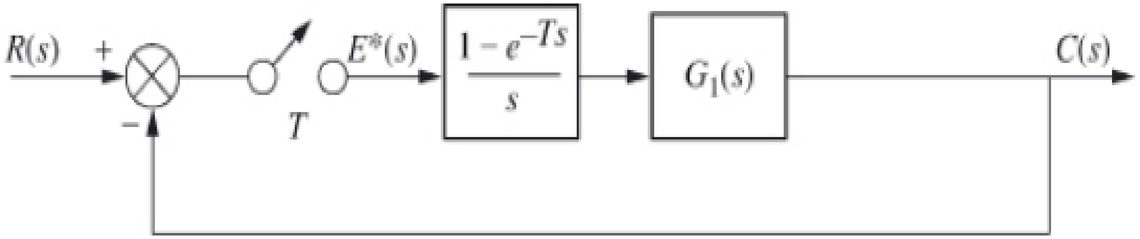
3. Unit Parabolic Input: $\boldsymbol{R(z) = \frac{T^2\,z(z+1)}{2\,(z-1)^3}}$ Using formula (1)

$$e_{ss} = e(\infty) = \frac{1}{K_a}$$ Where $K_a = \frac{1}{T^2}\lim_{z\to 1}(z-1)^2 G(z)$

Steady-State Errors

- The equations developed above for $e^*(\infty)$, K_p, K_v, and K_a are similar to the equations developed for analog systems.
- Multiple pole placement at the origin of the S-plane reduced steady-state errors to zero in the analog case.
- Multiple pole placement at $z = 1$ reduces the steady-state error to zero for digital systems. $s = 0$ maps into $z = 1$ under $z = e^{Ts}$

Example



For step, ramp, and parabolic inputs, find the steady-state error for the feedback control system shown in Figure if: $G_1(s) = \frac{10}{s\,(s+1)}$

SOLUTION

First find *G(s)*, the product of the z.o.h. and the plant. $G(s) = \frac{10(1-e^{-Ts})}{s^2(s+1)} = 10(1-e^{-Ts})\left[\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right]$

The z-transform is then: $G(z) = 10(1-z^{-1})\left[\frac{Tz}{(z-1)^2} - \frac{z}{z-1} + \frac{z}{z-e^{-T}}\right]$ ⇨ $G(z) = 10\left[\frac{T}{z-1} - 1 + \frac{z-1}{z-e^{-T}}\right]$

Thus:

1. For a step input: $K_p = \lim_{z\to 1} G(z) = \infty \rightarrow e_{ss} = e^*(\infty) = \frac{1}{1+K_p} = 0$

2. For a ramp input: $K_v = \frac{1}{T}\lim_{z\to 1}(z-1)G(z) = 10 \rightarrow e_{ss} = e^*(\infty) = \frac{1}{K_v} = 0.1$

3. For a parabolic input: $K_a = \frac{1}{T^2}\lim_{z\to 1}(z-1)^2 G_1(z) = 0 \rightarrow e_{ss} = e^*(\infty) = \frac{1}{K_a} = \infty$

Transient Response on the Z-Plane

- On the s-plane: vertical lines were lines of constant settling time, horizontal lines were lines of constant peak time, and radial lines were lines of constant percent overshoot.

$$T_r = \frac{1.8}{\omega}, \qquad T_s = \frac{4}{\sigma}, \qquad T_p = \frac{\pi}{\omega}, \qquad \%OS = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}$$

- The transformation to z-domain: $z = e^{sT}$, for $s = \sigma + j\omega$ we obtain

$$z = e^{sT} = e^{(\sigma + j\omega)T} = e^{\sigma T} e^{j\omega T} = r e^{j\omega T}$$

- ***Constant settling time*** are concentric circles of radius r.

for $T_s = const = \frac{4}{\sigma} \rightarrow \sigma = const \rightarrow r = const.$

- ***Constant peak time*** for $s = \sigma + j\omega$ we obtain

$T_p = const = \frac{\pi}{\omega} \rightarrow \omega = \frac{\pi}{T_p} = const$` m → Radial lines at an angle $\omega T = \theta_1 = \pi \frac{T}{T_p}$

- ***Constant percent overshoot*** we obtain curves on the z-plane. radial lines in the S-plane are represented by:

$$\frac{\sigma}{\omega} = -\tan(sin^{-1}\xi) = -\frac{\xi\omega_n}{\omega_n\sqrt{1-\xi^2}} = \frac{\xi}{\sqrt{1-\xi^2}}$$

Radius and phase depend on ξ

$$s = \sigma + j\omega = -\omega_n\xi + j\omega_n\sqrt{1-\xi^2}$$

Transforming to Z-plane

$$z = e^{sT} = e^{-\zeta\omega_n T} e^{j\omega_n\sqrt{1-\xi^2}T}$$

- For a desired damping ratio, ξ, curves can then be used as constant percent overshoot curves on the z-plane through a range of ωT (see previous slide).

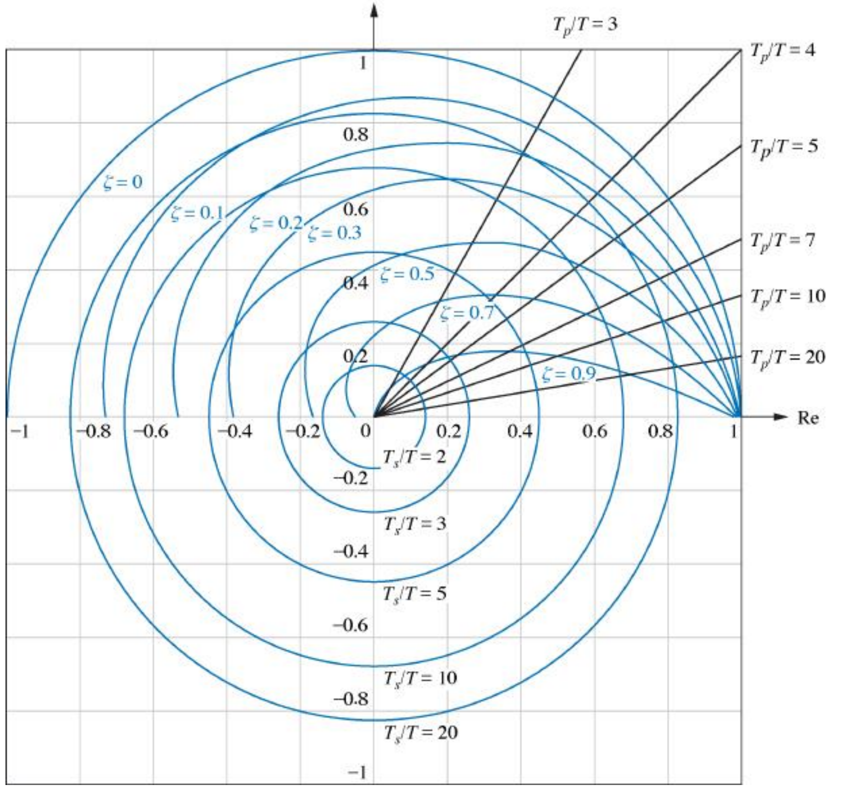


Fig (a) Constant damping ratio, normalized (to the sampling interval) settling time, and normalized peak time plots on the z-plane

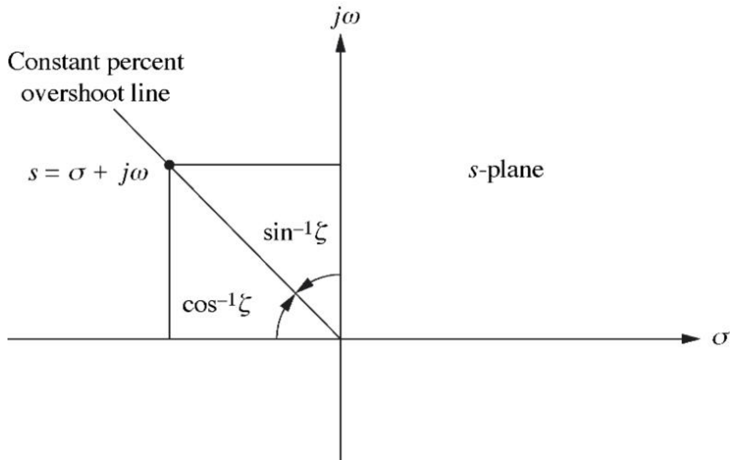
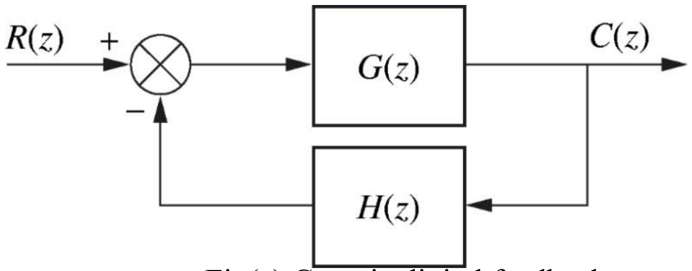


Fig. The s-plane sketch of constant percent overshoot line

Design Gain (P-Controller) via Root Locus

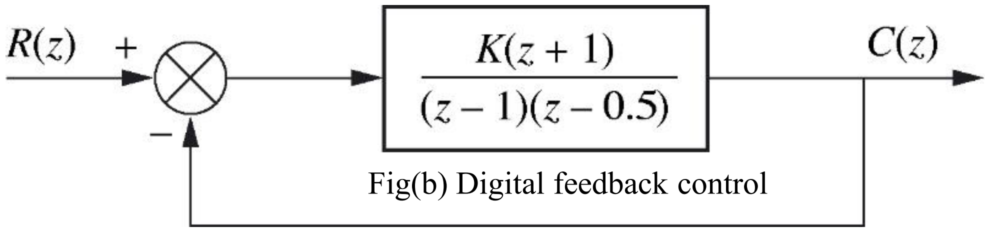
- Plot root locus (*in z plane use the same rules as in s plane*) and determine the gain required for stability (*within the unit circle*) as well as the gain required to meet a transient response requirement (*find the intersection of the root locus with the appropriate curves as they appear on the z-plane*).



Fig(a) Generic digital feedback

Example (Stability Design via Root Locus)

Sketch the root locus for the system shown in Figure (a). Also, determine the range of gain, *K,* for stability from the root locus plot.



Fig(b) Digital feedback control

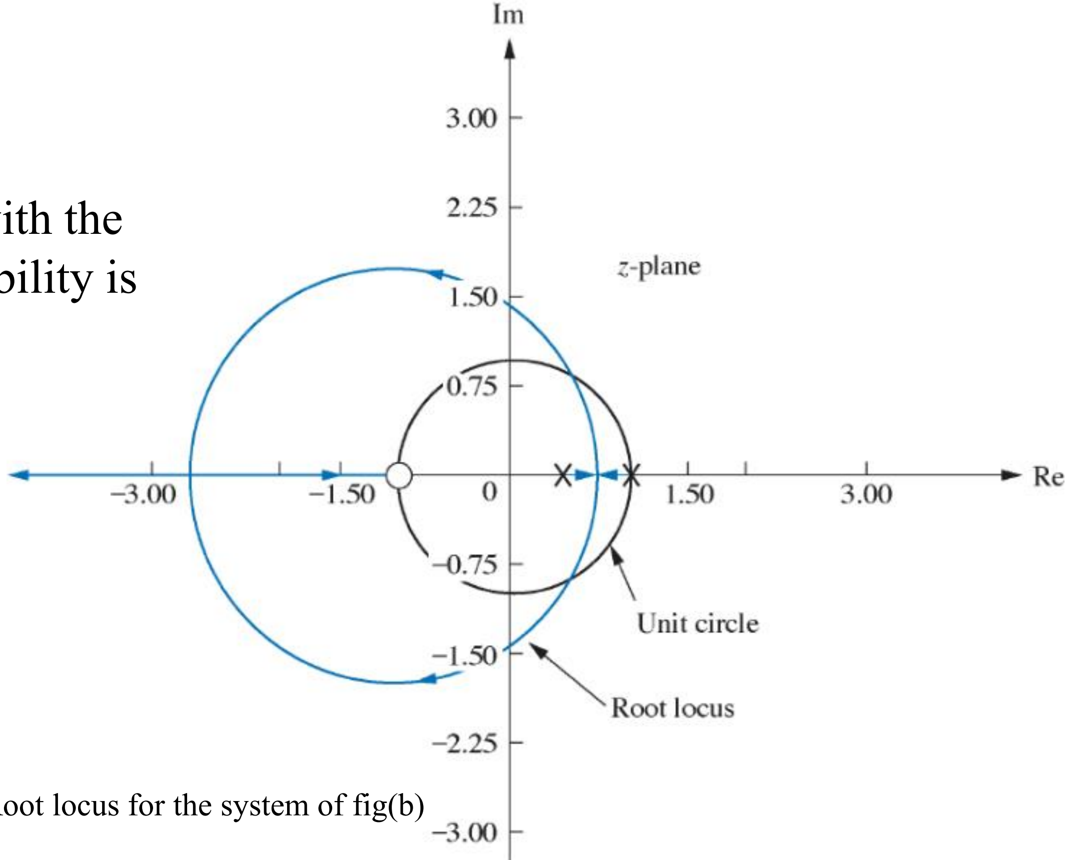
SOLUTION

- Sketch the root locus (use z as s), the results in Figure (b) using Matlab.
- Search along the unit circle for 180°, the intersection of the root locus with the unit circle is $1\angle 60^\circ$ with gain $K = 0.5$. Hence, the range of gain for stability is $0 < K < 0.5$.
- In general, if the open-loop transfer function is given by:

$$K\,G(z) = K\frac{z+\gamma}{(z+\alpha)(z+\beta)}$$

then

The root locus is a circle with center $z_0 = (-\gamma, 0)$ with the radius $r = \sqrt{(\gamma-\alpha)(\gamma-\beta)}$



Fig(c) Root locus for the system of fig(b)

Example (Transient Response Design via Gain Adjustment)

For the system of the previous example, find the value of gain, *K,* to yield a damping ratio of 0.7.

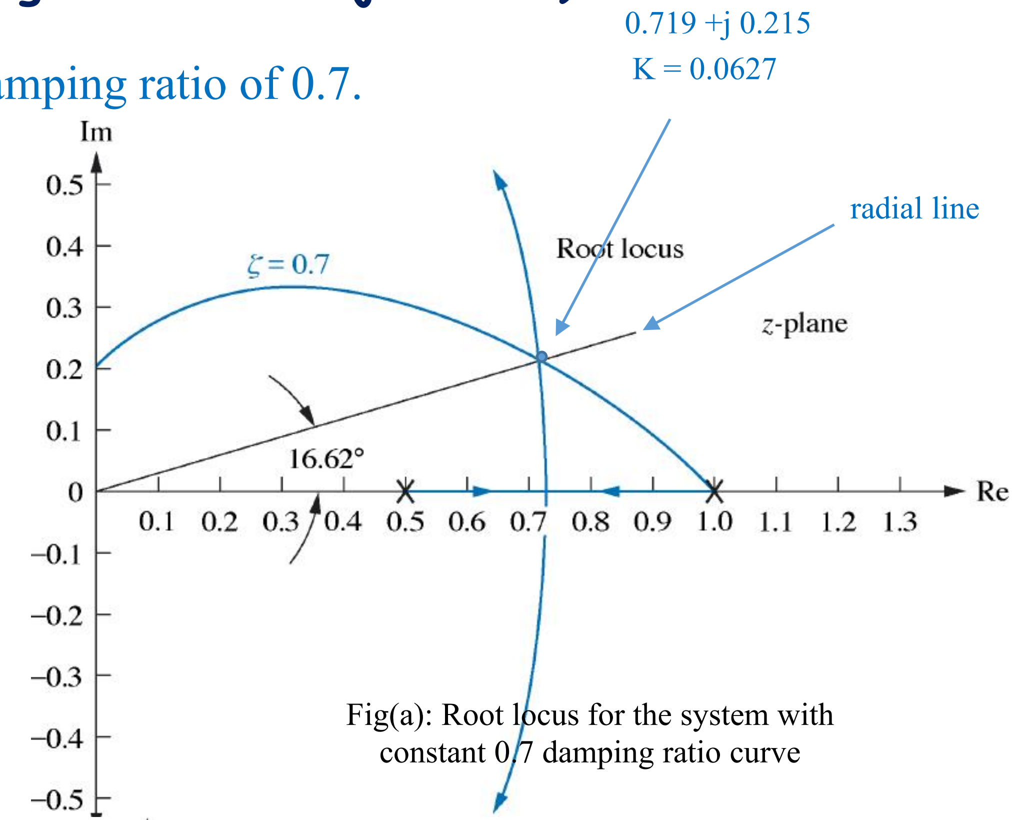
SOLUTION

- Figure (a) shows the constant damping ratio curves superimposed over the root locus for the system as determined from the last example.
- We obtain the gain by searching along a 16.62° radial line for 180° (intersection point of the 0.7 damping ratio curve with the 16.62° radial line). At this point K = 0.0627 at 0.719 +j 0.2153.
- We can now check our design by finding the unit sampled step response of the system Using our design, K = 0.0627.

Input: $R(z) = z/(z - 1)$, sampled unit step

the sampled output $$C(z) = \frac{R(z)G(z)}{1+G(z)} = \frac{0.0627\,z^2 + 0.0627\,z}{z^3 - 2.4373\,z^2 + 2\,z - 0.5627}$$

- Since the overshoot is approximately 5%, the requirement of a 0.7 damping ratio has been met Figure (b).



Fig(a): Root locus for the system with constant 0.7 damping ratio curve

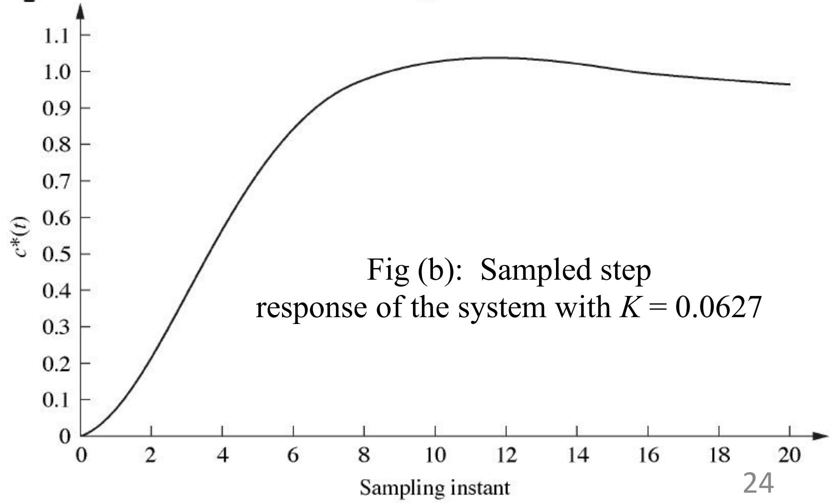


Fig (b): Sampled step response of the system with *K* = 0.0627

Note: Valid only at integer values of sampling instant

10 Cascade Compensation via the s-Plane

- Rather than designing directly in the z-domain, we can design on the s-plane, using S-plane analysis, and then convert the continuous compensator to a digital compensator using the ***bilinear transformation***.

- A bilinear transformation that yields a digital transfer function whose output response at the sampling instants is approximately the same as the equivalent analog transfer function is called the ***Tustin transformation***.

- ***Tustin transformation*** is used to transform the continuous compensator, $G_c(s)$, to the digital compensator, $G_c(z)$, by:

Tustin transformation

$$s = \frac{2(z-1)}{T(z+1)}$$

Inverse Tustin transformation

$$z = \frac{-\left(s+\frac{2}{T}\right)}{\left(s-\frac{2}{T}\right)} = \frac{1+\frac{T}{2}s}{1-\frac{T}{2}s}$$

- As the sampling interval, *T*, gets smaller (higher sampling rate), the designed digital compensator's output yields a closer match to the analog compensator.

Example (Digital Cascade Compensator Design)

For the digital control system of Figure(a), where the plant $G_p(s)$ is given, design a digital lead compensator, $G_c(z)$, as shown in Figure (b), so that the system will operate with 20% overshoot and a settling time of 1.1 seconds. Create your design in the s-domain and transform the compensator to the z-domain (Sampling period T=0.01 second).

$$G_p(s) = \frac{1}{s(s+6)(s+10)}$$

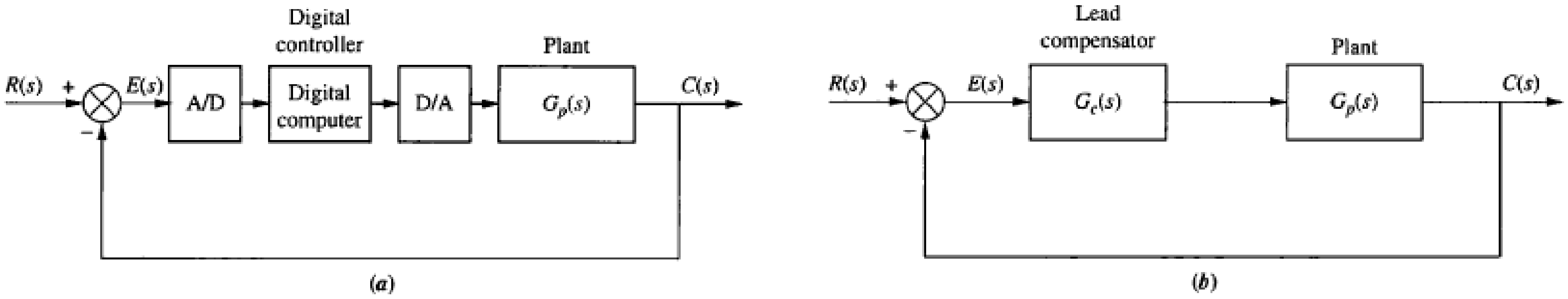


FIGURE a. Digital control system showing the digital computer performing compensation; b. continuous system used for design;

SOLUTION

Using Figure(b), design a lead compensator using the techniques described previously. The design was created as part of an Example, where we found that the lead compensator was (see previous chapters)

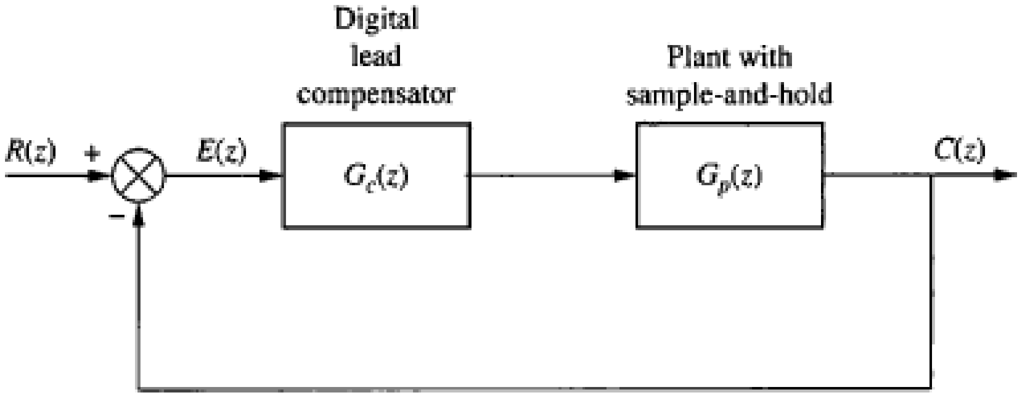
$$G_c(s) = \frac{1977(s+6)}{(s+29.1)}$$

- We have the analog compensator transfer function $G_c(s) = \frac{1977(s+6)}{(s+29.1)}$

- Using **_Tustin transformation_** with T = 0.01 $s = \frac{2(z-1)}{T(z+1)}$ Yields to the digital compensator TF $G_c(z) = \frac{1778z - 1674}{z - 0.746}$

- The z-transform of the plant and zero-order hold, with T = 0.01 second, is $G_p(z) = \frac{(1.602 \times 10^{-7}z^2) + (6.156 \times 10^{-7}z) + (1.478 \times 10^{-7})}{z^3 - 2.847z^2 + 2.699z - 0.8521}$

- The transformed digital system



- The time response in Figure (T = 0.01 s) shows that the compensated closed-loop system meets the transient response requirements. The figure also shows the response for a compensator designed with sampling times at the extremes of Astrom and Wittenmark's guideline.

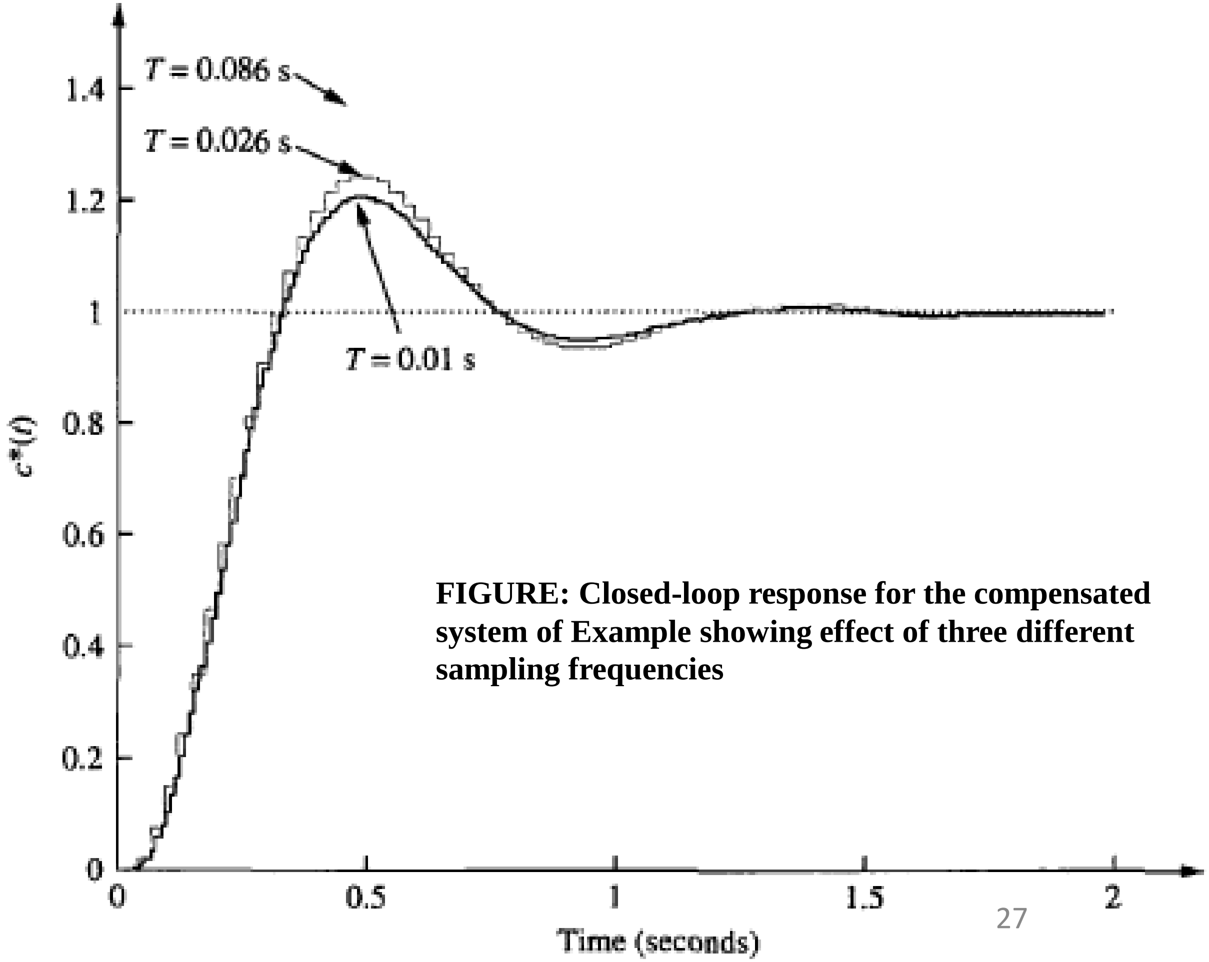


FIGURE: Closed-loop response for the compensated system of Example showing effect of three different sampling frequencies