Algebra Topics Search...

Evaluate: $\int_0^{\frac{\pi}{4}} \log(1+\tan(x)) \cdot dx$?

The answer is $\frac{\pi}{2}$, but how?

Algebra

1 Answer

Shwetank Mauria
Nov 14, 2017

$$\int_0^{\frac{\pi}{4}} \log(1+\tan x)dx = \frac{\pi}{8}\log 2$$

Explanation:

As $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Let $I = \int_0^{\frac{\pi}{4}} \log(1+\tan x)dx = \int_0^{\frac{\pi}{4}} \log\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$

= $\int_0^{\frac{\pi}{4}} \log\left(1+\frac{1-\tan x}{1+\tan x}\right)dx$

= $\int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan x}\right)dx$

= $\int_0^{\frac{\pi}{4}} \log 2dx - \int_0^{\frac{\pi}{4}} \log(1+\tan x)dx$

= $\int_0^{\frac{\pi}{4}} \log 2dx - I$

Hence $2I = \int_0^{\frac{\pi}{4}} \log 2dx$

and $I = \log 2 \times \frac{\pi}{4} \times \frac{1}{2} = \frac{\pi}{8}\log 2$

Answer link

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MATHEMATICS

the sum: $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}=\ln(2)$ using Riemann Integral and other methods

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I need to prove the following:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+(-1)^{n+1}\frac{1}{n}+\cdots=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}=\ln(2)$$

Method 1:)

The series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ is an alternating series, thus it is convergent, say to l. Therefore, both s_{2n} and s_n are convergent to the same limit l.

$$=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{2n}=\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2n}\right)-2\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2}\right.$$

$$=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}$$

It is an easy exercise to prove that:

$$\lim_{n\to\infty}s_{2n}=\lim_{n\to\infty}s_n=\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right]=\ln(2)$$

which implies that the given alternating series converges to $l=\ln 2$

However, I am interested to see a proof of this problem using the definition of the Riemann Integral as a sum of infinitely many rectangles of widths tending to zero. I tried to come up with a proof for this, but I couldn't. Can anyone share please?

Also, I am interested to see other methods of solving this problem (other than my method and the Riemann method). If anyone of you is aware of any other methods, please share :)

calculus real-analysis sequences-and-series analysis integration

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edited Jul 5 '12 at 17:46
Michael Hardy
248k 28 245 527

asked Jul 5 '12 at 17:37
C. Lambda
869 9 20

5 asked lot of times math.stackexchange.com/questions/165657/... math.stackexchange.com/a/155212/312 – leonbloy Jul 5 '12 at 17:50

@leonbloy: Oops! I see that I have answered all three. At least the answers were somewhat different. – robjohn ♦ Jul 7 '12 at 14:10

5 Answers

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9

Usually, to use the Riemann integral, alternating terms cause a problem. We want to have the finer partitions converge nicely, but an alternating series does not allow this. So, as far as I can see, we pretty much have to use the ζ trick you employ in your method:

$$\begin{aligned}
&\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}\\
&=\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{2n}\right)-2\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right)\\
&=\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{2n}\\
&=\sum_{k=n+1}^{2n}\frac{n}{k}\frac{1}{n} \qquad (1)
\end{aligned}$$

Then to use (1) as a Riemann sum (with $x=k/n$ and $\mathrm{d}x=1/n$) for

$$\int_1^2\frac{1}{x}\,\mathrm{d}x=\log(2) \qquad (2)$$

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answered Jul 5 '12 at 18:13
robjohn ♦
306k 32 377 732

2 Or $\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{2n}=\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}}$ – N. S. Jul 5 '12 at 18:20

@N.S.: Indeed. That essentially changes variables to shift the integral to

$$\int_0^1 \frac{1}{1+x}\,dx = \log(2)$$

– robjohn ♦ Jul 5 '12 at 18:30

6

Here's another method by the Riemann integral, but not by definition:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = \lim_{m\to\infty}\sum_{n=1}^{m}(-1)^{n+1}\frac{1}{n} =$$

$$\lim_{m\to\infty}\int_0^1 (1-x+\ldots+(-1)^{m-1}x^{m-1})\,dx = \lim_{m\to\infty}\int_0^1 \frac{1-(-x)^m}{1+x}\,dx = \int_0^1 \frac{dx}{1+x} = \ln$$

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answered Jul 5 '12 at 17:49

Andrew

10.4k 1 25 49

Ah, but the second to last step is the trick - why can you bring the limit inside the integral? – Thomas Andrews Jul 5 '12 at 17:51

@ThomasAndrews since the region of integration is $[0,1]$ and the only "bad point" is at $x=-1$, one may apply the Dominated Convergence Theorem. en.wikipedia.org/wiki/Dominated_convergence_theorem – nullUser Jul 5 '12 at 17:55

5 @ThomasAndrews not nessesary to bring, $\lim_{m\to\infty}\int_0^1 \frac{|(-x)^m|}{1+x}\,dx \le \lim_{m\to\infty}\int_0^1 x^m\,dx = 0$. – Andrew Jul 5 '12 at 17:56

1 Yeah, I knew it could be clarified and completed. However, it is always risky when writing out a line like this - the beginner might believe he can "always" do this, where, by "this," I mean, move a limit inside an integral. – Thomas Andrews Jul 5 '12 at 18:03

3

You can use the following geometric series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, |x|<1.$$

Then integrate both sides to get

$$\sum_{n=1}^{\infty}\frac{x^n}{n} = \int_0^x \frac{1}{1-t}dt = -\ln(1-x), |x|<1.$$

Since the LHS converges at $x=-1$, letting $x\to -1$ will give us

$$\sum_{n=1}(-1)^{n+1}\frac{1}{n} = \ln 2.$$

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answered Mar 11 '13 at 19:06

xpaul

29 1

1

As you asked for other methods of solving this problem, here is one using Taylor series.

Now that you know you are looking for the RHS of $\ln(2)$, expand $f(x) = \ln(x)$ into Taylor series around $x = 1$:

$f^{(n)}(x) = (-1)^{n+1}x^{-n}(n-1)!$, for $n > 0, x \neq 0$

so choosing to evaluate at $x = 1$, using $f(1) = \ln(1) = 0$, we get

$f(x) = f(1) + \sum_{n=1}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}$

and using $x = 2$, $\ln 2 = \sum_{n=1}^{\infty}(-1)^{n+1}/n$ as desired.

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answered Jul 5 '12 at 19:40
gt6989b
49.5k 2 29 63

1 While this idea can be used to obtain the result, it is much more subtle than this, and needs more details. Keep in mind that the theory of Taylor series (or you can use the general power series starting with the geometric series and integration) only guarantees that $\ln(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}$ on the interval $(0,2)$. The equality at $x = 2$ needs more effort ;) – N. S. Jul 6 '12 at 14:02

1

However one should note that,

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$$

is not absolutely convergent as

$$\sum_{n=1}^{\infty}|(-1)^{n+1}\frac{1}{n}| = \sum_{n=1}^{\infty}\frac{1}{n}$$

is divergent. (Such a series is said to be conditionally convergent)

Hence there is no fixed value for the series. (Refer to the Riemann Series Theorem) $\ln 2$ is just a value that the series would take for the particular permutation of S_{2n}.

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answered Jul 6 '12 at 13:07
Kasun Fernando
867 5 10

التمرين 1 :

أوجد قيم النهايات التالية

1. $\lim\limits_{n\longrightarrow+\infty} \frac{1}{n}\sum_{k=1}^{n}\frac{k^2}{n^2}$

2. $\lim\limits_{n\to+\infty} \sum_{k=1}^{n}\frac{k}{n^2+k^2}$

3. $\lim\limits_{n\longrightarrow+\infty} \sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k^2}}$

4. $\lim_{n\to+\infty} \sum_{k=1}^{n}\frac{1}{n}\cos\left(\frac{k\pi}{n}\right)$

.2

نعتبر في هذه الدالة $f(x) = \frac{1}{1+x}$ ، $0 \le x \le 1$ وهي قابلة للتكامل على $[0,1]$

وحسب هذا المثال

$$\int_0^1 \frac{dx}{1+x} = \lim_{n\to+\infty} \frac{1}{n}\sum_{0}^{n-1} f\left(\frac{k}{n}\right)$$

$$\ln(2) = \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{n+k} \quad \text{(محسوبة)}$$

(b)

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{k}{n^2+k^2} = \sum_{k=0}^{n-1} \frac{1}{n} \cdot \frac{k/n}{1+\left(\frac{k}{n}\right)^2}$$

نعتبر الدالة $f(x) = \frac{x}{1+x^2}$ المعرفة على $[0,1]$ وحسب المثال

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{k}{n^2+k^2} = \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{n} f\left(\frac{k}{n}\right)$$

$$= \int_0^1 \frac{x\,dx}{x^2+1}$$

$$= \frac{1}{2}\ln 2$$

.4

1. إذا كانت $f \in \mathcal{R}(0,1)$ فأثبت أن

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(k/n) = \int_0^1 f(x)\,dx$$

(1) باستخدام تمرين (1):

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{n} \cos\left(\frac{k\pi}{n}\right) = \lim_{n\to\infty} \sum_{k=1}^{n+1} \frac{1}{n+1} \cos\left(\frac{(k+1)\pi}{n+1}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n+1} \sum^{n} \left[\cos\frac{k\pi}{n} \cos\frac{\pi}{n} - \sin\frac{k\pi}{n} \sin\frac{\pi}{n}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n+1} \sum \cos\left[\frac{(k+1)\pi}{n+1}\right] = \int_0^1 \cos(\pi x)\,dx$$

$$f\left(\frac{k}{n+1}\right)$$

تمارين الباب الثاني- المتسلسلات

د. المنجي بلال

جامعة الملك سعود

19 سبتمبر 2020

تمرين 1 :

ادرس تقارب المتسلسلات التالية

1. $$\sum_{n\geq 1} \frac{\sin n^2}{n^2}$$

2

$$\sum_{n\geq 1} n\sin(\frac{1}{n})$$

3 $$\sum_{n\geq 1}\frac{n^2}{n!}$$

4 $$\sum_{n\geq 1} \frac{1}{(\ln(n+1))^n}$$

5 $$\sum_{n\geq 1} \frac{2^n n!}{n^n}$$

6

$$\sum_{n\geq 1} \frac{3^n n!}{n^n}$$

7 $$\sum_{n\geq 1} \frac{n^n}{2^n}$$

8

$$\sum_{n\geq 1}(\frac{1}{2})^{\sqrt{n}}$$

9

$$\sum_{n\geq 1} \ln \frac{(n^3+1)^2}{(n^2+1)^3}$$

10 $$\sum_{n\geq 1} \frac{e^n n!}{n^n}$$

11 $$\sum_{n\geq 1} \frac{(-1)^n}{n^\alpha + (-1)^n}$$

12

$$\sum_{n\geq 1} \frac{\cos n}{\sqrt{n}+\cos n}$$

تمرين 2 :

لتكن $\sum_{n\geq 0} u_n$ و $\sum_{n\geq 0} v_n$ متسلسلتين متقاربتين ذات حدين موجبين

1. أثبت أنّ المتسلسلتين $\sum_{n\geq 0} u_n^2$ و $\sum_{n\geq 0} \sqrt{u_n v_n}$ متقاربة

لتكن $\sum_{n\geq 0} w_n$ متسلسلة ذات حد موجب و $\lim_{n\longrightarrow +\infty} n w_n = \ell$

2. أثبت أنه إذا كانت المتسلسلة $\sum_{n\geq 0} w_n$ متقاربة، فإنّ $\ell = 0$

تمرين 3 :

1. لتكن f دالة قابلة للتفاضل باتصال (C^1) بحيث يكون التكامل $\int_0^{+\infty} f(t)\,dt$ متقاربا و أن التكامل $\int_0^{+\infty} f'(t)\,dt$ متقاربا مطلقا.
أثبت أنّ المتسلسلة $\sum_{n\geq 0} f(n)$ متقاربة. (يمكن استعمال مفكوك تايلور).

2. ادرس تقارب المتسلسلة التالية $\sum_{n=1}^{+\infty} \frac{\sin(\pi\sqrt{n})}{n}$.

تمرين 4 :

1. أثبت أنّ المتسلسلة $\sum_{n\geq 0} \frac{(-1)^n}{n+1}$ متقاربة.
2. أثبت أنّ $\left|\sum_{k=0}^{n} \frac{(-1)^k}{k+1} - \int_0^1 \frac{dt}{1+t}\right| \leq \frac{1}{n+2}$.
3. استنتج أنّ $\ln 2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$.

تمارين الباب الأول

التمرين 1 :

أوجد قيم النهايات التالية

1. $\lim\limits_{n\longrightarrow+\infty}\frac{1}{n}\sum\limits_{k=1}^{n}\frac{k^2}{n^2}$
2. $\lim\limits_{n\to+\infty}\sum\limits_{k=1}^{n}\frac{k}{n^2+k^2}$
3. $\lim\limits_{n\longrightarrow+\infty}\sum\limits_{k=1}^{n}\frac{1}{\sqrt{n^2+k^2}}$
4. $\lim_{n\to+\infty}\sum\limits_{k=1}^{n}\frac{1}{n}\cos\left(\frac{k\pi}{n}\right)$

التمرين 2 :

أُحسب التكاملات التالية:

1. $\int_1^x \frac{dt}{(t+1)\sqrt{t}}$
2. $\int_0^\pi \frac{dx}{1+\sin x}$
3. $\int_0^\pi \frac{dx}{1+\sin^2 x}$
4. $\int_0^\pi \frac{dx}{3+\cos(2x)}$
5. $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)dx$
6. $\int_1^3 \frac{dx}{x(1+x^4)}$
7. $\int_0^1 \sqrt{1-x^2}dx$

التمرين 3 :

لتكن $f\colon [a,b]\longrightarrow\mathbb{R}$ دالة قابلة للتفاضل باتصال من الدرجة 2، (C^2). أثبت أنّ

$$\int_a^b f(x)dx=\frac{b-a}{2}(f(a)+f(b))+\frac{1}{2}\int_a^b(x-a)(x-b)f''(x)dx.$$

التمرين 4 :

أثبت أنّ التكامل $I_n=\int_1^{+\infty}\frac{dx}{x^{n+1}\sqrt{x-1}}$ متقارب ثم أوجد قيمته باستعمال الإستقراء الرياضي

التمرين 5 :

1. أثبت أنّ التكامل $J=\int_a^b\frac{dx}{\sqrt{(x-a)(b-x)}}$ متقارب.

2. اُحسب التكامل باستعمال التعويض $x = a\cos^2 t + b\sin^2 t$.

التمرين 6 :

ادرس تقارب التكاملات المعتلة التالية.

1. $\int_0^{+\infty} \frac{x\sin x}{(1+x^2)}dx$
2. $\int_0^{+\infty} \frac{\cos x}{(1+x^\alpha)}dx$
3. $\int_0^{+\infty} \frac{\cos(\alpha x)}{1+e^x}dx$
4. $\int_0^{+\infty} \frac{\sin x}{\sqrt{x+\cos x}}dx$
5. $\int_0^{+\infty} \frac{2\tan^{-1}x-\pi}{\sqrt{x}}dx$
6. $\int_0^{+\infty} (\frac{\tan^{-1}x}{x} - \frac{\pi}{2(1+x)})dx$
7. $\int_1^{+\infty} \frac{1}{x}\left(e^{\frac{1}{x}} - \cos\frac{1}{x}\right)dx$
8. $\int_0^1 \frac{dx}{\cos^{-1}x}$

Q1: Discuss whether the series $\sum_{n=1}^{\infty}(\frac{1}{2})^{\sqrt{n}}$ converges or diverges.

Solution: We have the function f(x) $= (\frac{1}{2})^{\sqrt{x}}$ is positive, continuous, and decreasing on [1,∞). Therefore, we can use the integral test to study the convergence of the series.
If we integrate by parts, substituting t=$\sqrt{x}$ yields $dx = 2t\ dt$. So substituting into the integral we get

$$I = \int_1^{\infty}(\frac{1}{2})^{\sqrt{x}}\ dx = \int_1^{\infty}(\frac{1}{2})^t\ 2\ t\ dt$$

.
We compute the integral on the right by parts, taking u= 2t and dv= $(\frac{1}{2})^t\ dt \Rightarrow$ du=2 dt and v= $\frac{-1}{ln(2)}(\frac{1}{2})^t$. Hence,

$$I = -2t\frac{1}{ln(2)}\ (\frac{1}{2})^t\ ]_1^{t\to\infty}\ + \int_1^{\infty}\ 2\frac{1}{ln(2)}(\frac{1}{2})^t dt$$

.
Now the integral $\int_1^{\infty}\ 2\frac{1}{ln(2)}(\frac{1}{2})^t dt = -2(\frac{1}{ln(2)})^2(\frac{1}{2})^t\ ]_1^{t\to\infty} = (\frac{1}{ln(2)})^2$.

We have $\lim_{t\to\infty} t\ (\frac{1}{2})^t = 0$ by L'hopital's rule. This show that the improper integral is convergent. Which means that the series is convergent by the integral test.

Q2:
(i) Show that the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{1+n}$ is convergent.
(ii) Prove that

$$|\sum_{k=0}^{n}\frac{(-1)^k}{k+1} - \int_0^1\frac{dt}{1+t}| \le \frac{1}{n+2}$$

.
(iii) Conclude that $\sum_{n=0}^{\infty}\frac{(-1)^n}{1+n} = ln(2)$.

Solution:
(i) This is an alternating series where $a_n = \frac{1}{1+n}$ is positive decreasing and $a_n \to\ 0$, thus the series is convergent by the alternating series test.

(ii) If we integrate on [0,1]

$$\sum_{k=0}^{n}(-1)^k t^k - \frac{1}{1+t} = \frac{(-t)^{2+n}}{1+t},$$

(because $\sum_{k=0}^{n}(-1)^k t^k$ is geometric series. The summation is $\frac{1-(-t)^{n+1}}{1-(-t)}$). We get

$$\sum_{k=0}^{n}\frac{(-1)^k}{k+1} - \int_0^1\frac{dt}{1+t}dt = \int_0^1\frac{(-t)^{2+n}}{1+t}dt$$

.
However,

$$|\int_0^1\frac{(-t)^{2+n}}{1+t}dt| \le \int_0^1 |\frac{(-t)^{2+n}}{1+t}|\ dt \le \int_0^1 t^{n+2}dt = \frac{1}{n+3} \le \frac{1}{n+2}$$

. Hence, the desired result.
(iii) This follows directly from (ii) if we let n goes to infinity, the quantity inside the absolute value must converge to zero, and since $\int_0^1\frac{dt}{1+t} = ln(2)$. We obtain the result.

تمارين الباب الثاني

التمرين 1 :

ادرس تقارب المتسلسلات التالية

$\sum_{n\geq1}\frac{\sin n^2}{n^2}$; $\sum_{n\geq1}\ln\frac{(n^3+1)^2}{(n^2+1)^3}$;

$\sum_{n\geq1} n\sin(\frac{1}{n})$; $\sum_{n\geq1}\frac{e^n n!}{n^n}$;

$\sum_{n\geq1}\frac{n^2}{n!}$; $\sum_{n\geq1}\frac{(-1)^n}{n^\alpha+(-1)^n}$;

$\sum_{n\geq1}\frac{1}{(\ln(n+1))^n}$; $\sum_{n\geq1}\frac{\cos n}{\sqrt{n}+\cos n}$;

$\sum_{n\geq1}\frac{2^n n!}{n^n}$; $\sum_{n\geq1}\frac{\cos n}{n+\cos n}$;

$\sum_{n\geq1}\frac{3^n n!}{n^n}$; $\sum_{n\geq1}(-1)^n\frac{\sqrt{n}+2}{n}$;

$\sum_{n\geq1}\frac{n^n}{2^n}$;

$\sum_{n\geq1}(\frac{1}{2})^{\sqrt{n}}$; $\sum_{n\geq1}\ln\left(1+\frac{(-1)^n}{2n+1}\right)$.

التمرين 2 :

لتكن $\sum_{n\geq0}u_n$ و $\sum_{n\geq0}v_n$ متسلسلتين متقاربتين ذات حدين موجبين

1. أثبت أنّ المتسلسلتين $\sum_{n\geq0}u_n^2$ و $\sum_{n\geq0}\sqrt{u_nv_n}$ متقاربتين

لتكن $\sum_{n\geq0}w_n$ متسلسلة ذات حد موجب و $\lim_{n\longrightarrow+\infty} nw_n=\ell$

2. أثبت أنه إذا كانت المتسلسلة $\sum_{n\geq0}w_n$ متقاربة، فإنّ $\ell=0$

التمرين 3 :

1. أثبت أنّ المتسلسلة $\sum_{n\geq 0} \frac{(-1)^n}{n+1}$ متقاربة.

2. أثبت أنّ $\left|\sum_{k=0}^{n} \frac{(-1)^k}{k+1} - \int_0^1 \frac{dt}{1+t}\right| \leq \frac{1}{n+2}$.

3. استنتج أنّ $\ln 2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$.

$$\sum_{n\geq 2} \frac{(-1)^n}{n^\alpha + (-1)^n} \quad , \quad \begin{array}{l} \alpha \leq 1/2 \\ \alpha > 1/2 \end{array}$$

$$\sum_{n\geq 2} \frac{(-1)^n}{n^\alpha + (-1)^n} - \frac{(-1)^n}{n^\alpha + (-1)^n} = - \sum_{n\geq 2} \frac{1}{n^\alpha (n^\alpha + (-1)^n)} \geq 0 \quad , \quad v_n = \frac{1}{n^{2\alpha}}$$

$$\frac{\frac{1}{n^\alpha (n^\alpha + (-1)^n)}}{\frac{1}{n^{2\alpha}}}$$

$$= \frac{n^{2\alpha}}{n^\alpha (n^\alpha + (-1)^n)} = \frac{1}{1 + \frac{(-1)^n}{n^\alpha}} \longrightarrow 1$$

ابحث التقارب والتقارب المنتظم للمتسلسلة $\sum_1^\infty f_n$ في الحالات التالية

ⓒ $f_n(x) = \dfrac{x^n}{x^n+1}$

نثبت أن $\sum_1^\infty f_n(x)$ متقاربة لما $|x|<1$ وذلك باستخدام اختبار النسبة

كلية / إدارة .... الموضوع:

$$\lim_{n\to\infty} \frac{\frac{|x|^{n+1}}{|x^{n+1}+1|}}{\frac{|x|^n}{|x^n+1|}} = \lim_{n\to\infty} \frac{|x|\,|x^n+1|}{|x^{n+1}+1|} = |x| < 1$$

وعندما $x=1$ فإن $S_n = f_1(1)+\dots+f_n(1) = \frac{n}{2} \to \infty$

أي أن المتسلسلة $\sum_1^\infty f_n(1)$ متباعدة

وعندما $x=-1$ فإن $f_n(-1)$ غير معرف عندما تكون n فردي

وعندما $|x|>1$ فإن $\lim_{n\to\infty} \frac{x^n}{1+x^n} = \lim_{n\to\infty} \frac{1}{(\frac{1}{x})^n+1} = 1$

أي أن $\lim_{n\to\infty} f_n(x) \neq 0$ لما $|x|>1$ $\Leftarrow$ $\sum_1^\infty f_n(x)$ متباعدة

نأخذ الآن $0<a<1$ ومن أجل كل x تحقق $|x|\le a$ لدينا:

$$|1+x^n| \ge 1-|x|^n \ge 1-a^n$$

$$\left|\frac{x^n}{1+x^n}\right| \le \frac{a^n}{1-a^n} \quad \forall\ |x|<a$$

وبما أن $\sum_1^\infty \frac{a^n}{1-a^n}$ متقاربة (اختبار النسبة) فإن $\sum_1^\infty \frac{x^n}{1+x^n}$ متقاربة

بانتظام على $[-a,a]$ حيث $0<a<1$

1. ادرس التقارب البسيط و التقارب المنتظم لمتسلسلة الدوال التالية:

$$\sum_{n=0}^{+\infty} \frac{(-1)^n x}{(1+x^2)^n}$$ ، حيث $x \in \mathbb{R}$.

2. أوجد مجموع المتسلسلة $\sum_{n=0}^{+\infty} \frac{(-1)^n x}{(1+x^2)^n}$ لكل $x \in \mathbb{R}$.

نظرية 3.11 (اختبار دريشليه Dirichlet للتقارب المنتظم) لتكن (u_n) و (v_n) متتاليتين من الدوال الحقيقية على $D \subset \mathbb{R}$ وافرض أنهما تحققان الشروط التالية:

(i) متتالية مجاميع (u_n) الجزئية، (U_n)، محدودة بانتظام على D، أي يوجد K بحيث

$$|U_n(x)| = \left|\sum_{k=1}^{n} u_k(x)\right| \leq K \quad \forall x \in D, \ \forall n \in \mathbb{N}.$$

(ii) المتتالية (v_n) متناقصة على D، أي أن

$$v_{n+1}(x) \leq v_n(x) \quad \forall x \in D, \ \forall n \in \mathbb{N}.$$

(iii) $v_n \xrightarrow{u} 0$.

عندئذ تكون المتسلسلة $\sum u_n v_n$ متقاربة بانتظام على D.

السؤال 3 :

أوجد متسلسلة القوى للدالة $f(x) = \dfrac{1}{1 - x + x^2}$ و أوجد شعاع تقاربها.

السؤال 4 :

لتكن $f_n(x) = (-1)^n \dfrac{x^n}{n(n-1)}$، لكل $n \geq 2$.

1. أوجد فترة التقارب للمتسلسلة التالية $\sum_{n\geq 2} (-1)^n \dfrac{x^n}{n(n-1)}$ و ادرس تقارب المتسلسلة عند رؤوس فترة التقارب.

2. ادرس تقارب المتسلسلة $\sum_{n\geq 2} f_n'(x)$ و المتسلسلة $\sum_{n\geq 2} f_n''(x)$.

3. استنتج مجموع المتسلسلة $\sum_{n\geq 2} f_n(x)$.

السؤال 2 :

1. أثبت أن $\lim_{n\to+\infty} nx^n = 0$ ، لكل $x \in (-1,1)$.

2. ادرس تقارب المتسلسلتين التاليتين $\sum_{n\geq 1} \frac{x^n}{1+x^{2n}}$ و $\sum_{n\geq 1} \frac{x^n}{1+x^n}$، حيث $x \in [0,+\infty)$.

3. لتكن $(a_n)_n$ متتالية من الأعداد الحقيقية حيث $a_0 = 1$, $a_1 = -2$ و $a_{n-1} + 2a_n + a_{n+1} = 0$ لكل $n \geq 1$.
أوجد a_n لكل $n \geq 2$.

السؤال 4 :

1. ادرس التقارب البسيط للمتسلسلة $\sum_{n\geq 1}(-1)^n \ln\left(1+\frac{x}{n}\right)$ على الفترة $[0,+\infty)$.