## Digital Filters

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Digital Filtering: Realization

Digital Filtering:

\[ y(n) = \sum_{i=0}^{M} b_i x(n - i) - \sum_{j=1}^{N} a_j y(n - j). \]

Matlab Implementation:
3-tap (2\textsuperscript{nd} order) IIR filter

\[
\begin{align*}
\gg B &= [0 \ 1]; A = [1 \ 0 \ -0.5]; \\
\gg x &= [1 \ 0.5 \ 0.25 \ 0.125]; \\
\gg y &= \text{filter}(B, A, x) \\
\end{align*}
\]

\[ y = \\
0 \quad 1.0000 \quad 0.5000 \quad 0.7500 \]
Adder, Multiplier & Delay

Three components of Filters

(a) Adder, (b) multiplier, (c) delay.
Digital Filters: Response (Impulse, Step, Frequency)

Input signal \( \otimes \) impulse res. = output sig.

Convolution = weighted sum of input samples.

Filter Kernel

Finite Impulse Response (FIR) filters

Recursion = input sample + previous outputs

Impulse response of recursive filter

Exponentially decaying sinusoids

Infinitely long

Infinite Impulse Response (IIR) filters
FIR (Finite Impulse Response) Filter

\[ y(n) = b_0 x(n) + b_1 x(n-1) + \ldots + b_M x(n-M) \]

\[ y(n) = \sum_{j=0}^{M} b_j x(n-j) \]

Convolution

![Diagram of FIR filter](image)
IIR (Infinite Impulse Response) Filter

First-order IIR filter.

\[ y(n) + a_1 y(n-1) = b_0 x(n) + b_1 x(n-1) \]
\[ \Rightarrow y(n) = b_0 x(n) + b_1 x(n-1) - a_1 y(n-1) \]
\[ \Rightarrow y(n) = b_0 x(n) + D \{ b_1 x(n) - a_1 y(n) \} \]
IIR (Infinite Impulse Response) Filter

Second-order IIR filter.

\[ y(n) + a_1 y(n-1) + a_2 y(n-2) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) \]
Differential Equation:
\[ y(n) = b_0 x(n) + b_1 x(n - 1) + \cdots + b_M x(n - M) - a_1 y(n - 1) - \cdots - a_N y(n - N). \]

z- Transform:
\[ Y(z) = b_0 X(z) + b_1 X(z)z^{-1} + \cdots + b_M X(z)z^{-M} - a_1 Y(z)z^{-1} - \cdots - a_N Y(z)z^{-N} \]

Transfer Function:
\[ H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} \]
**Example: Transfer Function**

**Given:**
\[ y(n) = x(n) - x(n - 2) - 1.3y(n - 1) - 0.36y(n - 2) \]

**z- Transform:**
\[ Y(z) = X(z) - X(z)z^{-2} - 1.3Y(z)z^{-1} - 0.36Y(z)z^{-2} \]

**Rearrange:**
\[ Y(z)(1 + 1.3z^{-1} + 0.36z^{-2}) = (1 - z^{-2})X(z) \]

**Transfer Function:**
\[ H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-2}}{1 + 1.3z^{-1} + 0.36z^{-2}} \]

**Given:**
\[ H(z) = \frac{z^2 - 1}{z^2 + 1.3z + 0.36} \]

**Rearrange:**
\[ H(z) = \frac{(z^2 - 1)/z^2}{(z^2 + 1.3z + 0.36)/z^2} = \frac{1 - z^{-2}}{1 + 1.3z^{-1} + 0.36z^{-2}} \]

**Differential Equation:**
\[ y(n) = x(n) - x(n - 2) - 1.3y(n - 1) - 0.36y(n - 2) \]
Pole – Zero from Transfer Function

\[
H(z) = \frac{z^{-1} - 0.5z^{-2}}{1 + 1.2z^{-1} + 0.45z^{-2}}
\]

\[
H(z) = \frac{(z^{-1} - 0.5z^{-2})z^2}{(1 + 1.2z^{-1} + 0.45z^{-2})z^2} = \frac{z - 0.5}{z^2 + 1.2z + 0.45} \quad \frac{(z - 0.5)}{(z + 0.6 - j0.3)(z + 0.6 + j0.3)}
\]

The system is stable.

The zeros do not affect system stability.
System Stability

Depends on poles’ location
Example: System Stability

\[ H(z) = \frac{z^2 + z + 0.5}{(z - 1)^2(z + 1)(z - 0.6)} \]

Zeros are \( z = -0.5 \pm j0.5 \).

Poles: \( z = 1, \ |z| = 1 \); \( z = 1, \ |z| = 1 \); \( z = -1, \ |z| = 1 \); \( z = 0.6, \ |z| = 0.6 < 1 \).

Since the outermost pole is multiple order (2\textsuperscript{nd} order) at \( z = 1 \) and is on the unit circle, the system is unstable.
Digital Filter: Frequency Response

\[ H(z)|_{z=e^{j\omega T}} = H(e^{j\omega T}) = |H(e^{j\omega T})| \angle H(e^{j\omega T}) \]

Putting \( \Omega = \omega T \)

\[ H(e^{j\Omega}) = H(z)|_{z=e^{j\Omega}} = |H(e^{j\Omega})| \angle H(e^{j\Omega}) \]

Example: Given \( y(n) = 0.5x(n) + 0.5x(n - 1) \) Sampling rate = 8k Hz

Transfer function: \( H(z) = \frac{Y(z)}{X(z)} = 0.5 + 0.5z^{-1} \).

Frequency response: \( H(e^{j\Omega}) = 0.5 + 0.5e^{-j\Omega} \)

\[ = 0.5 + 0.5\cos(\Omega) - j0.5\sin(\Omega) \]

\[ |H(e^{j\Omega})| = \sqrt{(0.5 + 0.5\cos(\Omega))^2 + (0.5\sin(\Omega))^2} \quad \text{and} \quad \angle H(e^{j\Omega}) = \tan^{-1}\left( \frac{-0.5\sin(\Omega)}{0.5 + 0.5\cos(\Omega)} \right) \]

Low Pass Filter (LPF)  Band Pass Filter (BPF)

Matlab: Frequency Response

\[ \text{[h, w] = freqz(B, A, N)} \]
Impulse Response of FIR Filters

Frequency response of ideal LPF:

Impulse response of ideal LPF:

After truncating $2M+1$ major components:

$$H(z) = h(M)z^M + \cdots + h(1)z + h(0) + h(1)z^{-1} + \cdots + h(M)z^{-M}$$

Making causal,

$$H(z) = b_0 + b_1z^{-1} + \cdots + b_{2M}(2M)z^{-2M}$$

Where,

$$b_n = h(n-M) \text{ for } n = 0, 1, \ldots, 2M.$$
Ideal Low Pass Filter

Impulse Response:

\[ h(n) = \begin{cases} 
\frac{\Omega_c}{\pi} & n = 0 \\
\frac{\sin (\Omega_c n)}{n\pi} & \text{for } n \neq 0 \\
-M \leq n \leq M 
\end{cases} \]

Example:

3-tap FIR LPF with cutoff freq. = 800 Hz and sampling rate = 8k Hz.

\[ \Omega_c = 2\pi f_c T_s = 2\pi \times \frac{800}{8000} = 0.2\pi \text{ radians} \]

2\(M + 1 = 3 \]

\[ h(0) = \frac{\Omega_c}{\pi} \quad \text{for } n = 0 \]

\[ h(0) = \frac{0.2\pi}{\pi} = 0.2 \]

\[ h(n) = \frac{\sin (\Omega_c n)}{n\pi} = \frac{\sin (0.2\pi n)}{n\pi}, \quad \text{for } n \neq 1 \]

\[ h(1) = \frac{\sin [0.2\pi \times 1]}{1 \times \pi} = 0.1871 \]

Using symmetry:

\[ h(-1) = h(1) = 0.1871 \]
Ideal Low Pass Filter – contd.

Delaying $h(n)$ by $M = 1$ sample,

\[ b_0 = h(0 - 1) = h( - 1) = 0.1871 \]
\[ b_1 = h(1 - 1) = h(0) = 0.2 \]
\[ b_2 = h(2 - 1) = h(1) = 0.1871 \]

Transfer function $H(z) = 0.1871 + 0.2z^{-1} + 0.1871z^{-2}$

Differential Eq: \[ y(n) = 0.1871x(n) + 0.2x(n - 1) + 0.1871x(n - 2) \]

Frequency response \[ H(e^{j\Omega}) = 0.1871 + 0.2e^{-j\Omega} + 0.1871e^{-j2\Omega} \]
\[ = e^{-j\Omega}(0.1871e^{j\Omega} + 0.2 + 0.1871e^{-j\Omega}) \]
\[ = e^{-j\Omega}(0.2 + 0.3742 \cos(\Omega)) \]

Magnitude: \[ |H(e^{j\Omega})| = |0.2 + 0.3472 \cos(\Omega)| \]

Phase: \[ \angle H(e^{j\Omega}) = \begin{cases} -\Omega & \text{if } 0.2 + 0.3472 \cos(\Omega) > 0 \\ -\Omega + \pi & \text{if } 0.2 + 0.3472 \cos(\Omega) < 0 \end{cases} \]

\[ e^{jx} + e^{-jx} = 2 \cos(x) \]

Complete Plot!
Linear Phase

If filter has linear phase property, the output will simply be a delayed version of input.

Let, 17-tap FIR filter with linear phase property.
Nonlinear Phase

Input: 
\[ x(n) = x_1(n) + x_2(n) = \sin(0.05\pi n)u(n) - \frac{1}{3}\sin(0.15\pi n)u(n) \]

Linear phase filter output: 
\[ y_1(n) = \sin[0.05\pi (n-8)] - \frac{1}{3}\sin[0.15\pi (n-8)] \]

90 d phase delay filter output: 
\[ y_2(n) = \sin(0.05\pi n - \pi/2) - \frac{1}{3}\sin(0.15\pi n - \pi/2) \]

Distorted!
Linear Phase: Zero Placement

• A single zero can be either at $z = 1$ or $z = -1$. (B or D)

• Real zeros not on the unit circle always occur in pairs with $r$ and $r^{-1}$. (C)

• If the zero is complex, its conjugate is also zero. (E) [on the unit circle]

• Complex zeros not on the unit circle always occur in quadruples with $r$ and $r^{-1}$. (A)
Example: FIR Filtering With Window Method

Problem:
Design a 5-tap FIR band reject filter with a lower cutoff frequency of 2,000 Hz, an upper cutoff frequency of 2,400 Hz, and a sampling rate of 8,000 Hz using the Hamming window method.

Solution:

\[
\begin{align*}
\Omega_L &= 2\pi f_L T = 2\pi \times 2000/8000 = 0.5\pi \text{ radians} \\
\Omega_H &= 2\pi f_H T = 2\pi \times 2400/8000 = 0.6\pi \text{ radians} \\
2M + 1 &= 5 \\
h(n) &= \begin{cases} \\
\frac{\pi - \Omega_H + \Omega_L}{\pi} & n = 0 \\
\frac{\sin((\Omega_H n)/n\pi)}{-\sin((\Omega_L n)/n\pi)} + \frac{\sin((\Omega_H n)/n\pi)}{\sin((\Omega_L n)/n\pi)} & n \neq 0 \\
-2 \leq n \leq 2.
\end{cases} \\
h(0) &= \frac{\pi - \Omega_H + \Omega_L}{\pi} = \frac{\pi - 0.6\pi + 0.5\pi}{\pi} = 0.9 \\
h(1) &= \frac{\sin(0.5\pi \times 1)}{1 \times \pi} - \frac{\sin(0.6\pi \times 1)}{1 \times \pi} = 0.01558 \\
h(2) &= \frac{\sin(0.5\pi \times 2)}{2 \times \pi} - \frac{\sin(0.6\pi \times 2)}{2 \times \pi} = 0.09355 \\
M &= 2
\end{align*}
\]
Hamming window function

\[ w_{ham}(0) = 0.54 + 0.46 \cos \left( \frac{0 \times \pi}{2} \right) = 1.0 \]
\[ w_{ham}(1) = 0.54 + 0.46 \cos \left( \frac{1 \times \pi}{2} \right) = 0.54 \]
\[ w_{ham}(2) = 0.54 + 0.46 \cos \left( \frac{2 \times \pi}{2} \right) = 0.08 \]

Symmetry

\[ w_{ham}(-1) = w_{ham}(1) = 0.54 \]
\[ w_{ham}(-2) = w_{ham}(2) = 0.08 \]

Windowed impulse response

\[ h_w(0) = h(0)w_{ham}(0) = 0.9 \times 1 = 0.9 \]
\[ h_w(1) = h(1)w_{ham}(1) = 0.01558 \times 0.54 = 0.00841 \]
\[ h_w(2) = h(2)w_{ham}(2) = 0.09355 \times 0.08 = 0.00748 \]
\[ h_w(-1) = h(-1)w_{ham}(-1) = 0.00841 \]
\[ h_w(-2) = h(-2)w_{ham}(-2) = 0.00748 \]

By delaying \( h_w(n) \) by \( M = 2 \) samples,

\[ b_0 = b_4 = 0.00748, \quad b_1 = b_3 = 0.00841, \quad \text{and} \quad b_2 = 0.9 \]

\[ H(z) = 0.00748 + 0.00841z^{-1} + 0.9z^{-2} + 0.00841z^{-3} + 0.00748z^{-4} \]
## FIR Filter Length Estimation

<table>
<thead>
<tr>
<th>Window Type</th>
<th>Window Function $w(n)$, $-M \leq n \leq M$</th>
<th>Window Length, $N$</th>
<th>Passband Ripple (dB)</th>
<th>Stopband Attenuation (dB)</th>
</tr>
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<tr>
<td>Rectangular</td>
<td>$1$</td>
<td>$N = 0.9/\Delta f$</td>
<td>$0.7416$</td>
<td>$21$</td>
</tr>
<tr>
<td>Hanning</td>
<td>$0.5 + 0.5 \cos\left(\frac{n\pi}{M}\right)$</td>
<td>$N = 3.1/\Delta f$</td>
<td>$0.0546$</td>
<td>$44$</td>
</tr>
<tr>
<td>Hamming</td>
<td>$0.54 + 0.46 \cos\left(\frac{n\pi}{M}\right)$</td>
<td>$N = 3.3/\Delta f$</td>
<td>$0.0194$</td>
<td>$53$</td>
</tr>
<tr>
<td>Blackman</td>
<td>$0.42 + 0.5 \cos\left(\frac{n\pi}{M}\right) + 0.08 \cos\left(\frac{2n\pi}{M}\right)$</td>
<td>$N = 5.5/\Delta f$</td>
<td>$0.0017$</td>
<td>$74$</td>
</tr>
</tbody>
</table>

**Diagram:**

- $\delta_p$: Passband ripple
- $\delta_s$: Stopband attenuation

\[ \Delta f = \frac{|f_{\text{stop}} - f_{\text{pass}}|}{f_s} \]

\[ f_c = \frac{(f_{\text{pass}} + f_{\text{stop}})}{2} \]

\[ \delta_p \text{ dB} = 20 \cdot \log_{10} (1 + \delta_p) \]

\[ \delta_s \text{ dB} = -20 \log_{10} (\delta_s) \]
Example: FIR Filter Length Estimation

Problem:
Design a BPF with

<table>
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<th>Lower stopband = 0–500 Hz</th>
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<td>Passband = 1,600–2,300 Hz</td>
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<tr>
<td>Upper stopband = 3,500–4,000 Hz</td>
</tr>
<tr>
<td>Stopband attenuation = 50 dB</td>
</tr>
<tr>
<td>Passband ripple = 0.05 dB</td>
</tr>
<tr>
<td>Sampling rate = 8,000 Hz</td>
</tr>
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Solution:

\[ \Delta f_1 = \frac{|1600 - 500|}{8000} = 0.1375 \]
\[ \Delta f_2 = \frac{|3500 - 2300|}{8000} = 0.15 \]

\[ N_1 = \frac{3.3}{0.1375} = 24 \]
\[ N_2 = \frac{3.3}{0.15} = 22 \]

Choose nearest higher odd \( N = 25 \)

Cutoff frequencies:
\[ f_1 = \frac{(1600 + 500)}{2} = 1050 \text{ Hz} \]
\[ f_2 = \frac{(3500 + 2300)}{2} = 2900 \text{ Hz} \]

Normalized
\[ \Omega_L = \frac{1050 \times 2\pi}{8000} = 0.2625\pi \text{ radians} \]
\[ \Omega_H = \frac{2900 \times 2\pi}{8000} = 0.725\pi \text{ radians} \]

Now design the filter with hint from slide 14.
Application: Noise Reduction

Input waveform: sinusoid + broadband noise

Spectrum:

Want to remove this noise

Specification: LPF

Pass band frequency [0 – 800 Hz]  Pass band ripple < 0.02 dB
Stop band frequency [1000 – 4000 Hz]  Stop band attenuation = 50 dB
Application: Noise Reduction — contd.

133- tap FIR filter, so a delay of 66

However, noise reduction in real world scenario is not so easy!
Frequency Sampling Design Method

Simple to design

Filter length = $2M + 1$

Magnitude response in the range $[0 \sim \pi]$

Calculate FIR filter coefficients:

$$h(n) = \frac{1}{2M + 1} \left\{ H_0 + 2 \sum_{k=1}^{M} H_k \cos \left( \frac{2\pi k(n - M)}{2M + 1} \right) \right\}$$

for $n = 0, 1, \ldots, M$.

Use the symmetry:

$$h(n) = h(2M - n) \text{ for } n = M + 1, \ldots, 2M.$$
Example: Frequency Sampling Design Method

Problem: Design a linear phase lowpass FIR filter with 7 taps and a cutoff frequency of $\Omega_c = 0.3\pi$ radian using the frequency sampling method.

Solution:

\[ N = 2M + 1 = 7 \quad \Rightarrow \quad M = 3 \]

\[ \Omega_k = \frac{2\pi}{7} k \text{ radians}, \quad k = 0, 1, 2, 3. \]

For $\Omega_0 = 0$ radians, $H_0 = 1.0$

For $\Omega_1 = \frac{2}{7}\pi$ radians, $H_1 = 1.0$

For $\Omega_2 = \frac{4}{7}\pi$ radians, $H_2 = 0.0$

For $\Omega_3 = \frac{6}{7}\pi$ radians, $H_3 = 0.0$.

By symmetry:

\[ h(4) = h(2) = 0.32100 \]

\[ h(5) = h(1) = 0.07928 \]

\[ h(6) = h(0) = -0.11456. \]

\[ h(n) = \frac{1}{7}\left\{ 1 + 2 \sum_{k=1}^{3} H_k \cos \left[ 2\pi k(n-3)/7 \right] \right\}, \quad n = 0, 1, \ldots, 3. \]

\[ = \frac{1}{7}\left\{ 1 + 2 \cos \left[ 2\pi(n-3)/7 \right] \right\} \]

\[ h(0) = \frac{1}{7}\{1 + 2\cos(-6\pi/7)} = -0.11456 \]

\[ h(1) = \frac{1}{7}\{1 + 2\cos(-4\pi/7)} = 0.07928 \]

\[ h(2) = \frac{1}{7}\{1 + 2\cos(-2\pi/7)} = 0.32100 \]

\[ h(3) = \frac{1}{7}\{1 + 2\cos(-0 \times \pi/7)} = 0.42857. \]
Coefficient Quantization Effect

Filter coefficients are usually truncated or rounded off for the application.

Transfer function with infinite precision:

\[ H(z) = \sum_{n=0}^{K} b_n z^{-n} = b_0 + b_1 z^{-1} + \ldots + b_K z^{-K} \]

Transfer function with quantized precision:

\[ H^q(z) = \sum_{n=0}^{K} b^q_n z^{-n} = b^q_0 + b^q_1 z^{-1} + \ldots + b^q_K z^{-K} \]

Error of the magnitude frequency response:

\[ |H(e^{j\Omega}) - H^q(e^{j\Omega})| = \sum_{n=0}^{K} |(b_n - b^q_n)e^{-jn\Omega}| \]

\[ < \sum_{n=0}^{K} |b_n - b^q_n| < (K + 1) \cdot 2^{-B-1} \]

Example

25 - tap FIR filter; 7 bits used for fraction

Let infinite precision coeff. = 0.00759455135346

0.00759455135346 \times 2^7 = 0.9721 = 1 (rounded up to the integer)

Quantized coeff. = 1 / 2^7 = 0.0078125

Error is bounded by < 25 / 256 = 0.0977
Consider a signal that is the sum of two real exponentials:

\[ x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]. \]

The \( z \)-transform is then

\[
X(z) = \sum_{n=-\infty}^{\infty} \left\{ \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n] \right\} z^{-n}
\]

\[
= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n] z^{-n} + \sum_{n=-\infty}^{\infty} \left(-\frac{1}{3}\right)^n u[n] z^{-n}
\]

\[
= \sum_{n=0}^{\infty} \left(\frac{1}{2}z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(-\frac{1}{3}z^{-1}\right)^n
\]

\[
= \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}} = \frac{2 \left(1 - \frac{1}{12}z^{-1}\right)}{\left(1 - \frac{1}{2}z^{-1}\right) \left(1 + \frac{1}{3}z^{-1}\right)}
\]

\[
= \frac{2z \left(z - \frac{1}{12}\right)}{\left(z - \frac{1}{2}\right) \left(z + \frac{1}{3}\right)}.
\]

The \( z \)-transform of \( a^n u(n) = \frac{z}{z - a} = \frac{1}{1 - az^{-1}} \)

### z - plane diagram
Complementary Example - II

**Given:**

\[ y[n] = \left( \frac{1}{2} \right)^n x[n] \]

\[ y[n] = \left( \frac{1}{2} \right)^n x[n] \Rightarrow Y(z) = X(2z) = \frac{4z^2 + 1}{2z - \frac{1}{2}} \]

zeros \( \pm \frac{1}{2} j \)

poles \( \frac{1}{4}, \infty \)

**Given:**

\[ X(z) = \frac{z^2 + 1}{z - \frac{1}{2}} \]
IIR Filter Design: Bilinear Transformation Method

1. Transformation with frequency warping
2. Transformation by lowpass prototype filter
3. Bilinear transformation

Digital filter specifications

Analog filter specifications

Analog filter transfer function

Digital filter transfer function and frequency response verification

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Bilinear Transformation Method

For LPF and HPF: \[ \omega_a = \frac{2}{T} \tan \left( \frac{\omega_a T}{2} \right) \]

For BPF and BRF: \[ \omega_{al} = \frac{2}{T} \tan \left( \frac{\omega_{al} T}{2} \right), \quad \omega_{ah} = \frac{2}{T} \tan \left( \frac{\omega_{ah} T}{2} \right) \]
\[ \omega_0 = \sqrt{\omega_{al} \omega_{ah}}, \quad W = \omega_{ah} - \omega_{al} \]

From LPF to LPF: \[ H(s) = H_P(s) \bigg|_{s = \frac{s}{\omega_a}} \]
From LPF to HPF: \[ H(s) = H_P(s) \bigg|_{s = \frac{s}{\omega_a}} \]
From LPF to BPF: \[ H(s) = H_P(s) \bigg|_{s = \frac{s^2 + \omega_0^2}{s W}} \]
From LPF to BRF: \[ H(s) = H_P(s) \bigg|_{s = \frac{sW}{s^2 + \omega_0^2}} \]

Obtained Transfer Function: \[ H(z) = H(s) \bigg|_{s = \frac{2z - 1}{T(z + 1)}} \]
Example 1: Bilinear Transformation Method

Problem:
Design a first-order digital highpass Chebyshev filter with a cutoff frequency of 3 kHz and 1 dB ripple on passband using a sampling frequency of 8,000 Hz.

Solution:

\[
\omega_d = 2\pi f = 2\pi(3000) = 6000\pi \text{ rad/sec, and } T = 1/f_s = 1/8000 \text{ sec.}
\]

\[
\omega_a = \frac{2}{T} \tan\left(\frac{\omega_d T}{2}\right) = 16000 \times \tan\left(\frac{6000\pi/8000}{2}\right) = 3.8627 \times 10^4 \text{ rad/sec.}
\]

First-order LP Chebyshev filter prototype:

\[
H_P(s) = \frac{1.9652}{s + 1.9625}
\]

Applying transformation LPF to HPF:

\[
H(s) = H_P(s)\bigg|_{\frac{\omega_a}{s}} = \frac{1.9652}{\frac{o_a}{s} + 1.9652} = \frac{1.9652s}{1.9652s + 3.8627 \times 10^4}
\]

Applying BLT:

\[
H(z) = \left. \frac{s}{s + 1.9656 \times 10^4} \right|_{s=16000(z-1)/(z+1)} = \frac{0.4487 - 0.4487z^{-1}}{1 + 0.1025z^{-1}}
\]
Example 2: Bilinear Transformation Method

Problem:
Design a second-order digital bandpass Butterworth filter with the following specifications:
- an upper cutoff frequency of 2.6 kHz and
- a lower cutoff frequency of 2.4 kHz,
- a sampling frequency of 8,000 Hz.

Solution:
\[ \omega_h = 2\pi f_h = 2\pi(2600) = 5200\pi \text{ rad/sec} \]
\[ \omega_l = 2\pi f_l = 2\pi(2400) = 4800\pi \text{ rad/sec, and } T = 1/f_s = 1/8000 \text{ sec}. \]
\[ \omega_{ah} = \frac{2}{T} \tan \left( \frac{\omega_h T}{2} \right) = 16000 \times \tan \left( \frac{5200\pi/8000}{2} \right) = 2.6110 \times 10^4 \text{ rad/sec} \]
\[ \omega_{al} = 16000 \times \tan \left( \frac{\omega_l T}{2} \right) = 16000 \times \tan (0.3\pi) = 2.2022 \times 10^4 \text{ rad/sec} \]
\[ W = \omega_{ah} - \omega_{al} = 26110 - 22022 = 4088 \text{ rad/sec} \]
\[ \omega_0^2 = \omega_{ah} \times \omega_{al} = 5.7499 \times 10^8 \]

A first-order LPF prototype will produce second-order BPF prototype.
Example 2: Bilinear Transformation Method

Contd.

1st order LPF prototype:

\[ H_P(s) = \frac{1}{s + 1} \]

Applying transformation LPF to BPF:

\[ H(s) = H_P(s) \bigg|_{s^2 + \omega_0^2} = \frac{W_s}{s^2 + W_s + \omega_0^2} = \frac{4088s}{s^2 + 4088s + 5.7499 \times 10^8} \]

Applying BLT:

\[ H(z) = \frac{4088s}{s^2 + 4088s + 5.7499 \times 10^8}_{s=16000(z-1)/(z+1)} = \]

\[ H(z) = \frac{0.0730 - 0.0730z^{-2}}{1 + 0.7117z^{-1} + 0.8541z^{-2}} \]
Pole Zero Placement Method

Second-Order BPF Design

\( r \): controls bandwidth
\( \theta \): controls central frequency

Location of poles & zeros:
controls magnitude

Location of pole:
determines stability

Number of zero:
determines phase linearity

\[ r \approx 1 - \left( \frac{BW_{3dB}}{f_s} \right) \times \pi \]

\[ \theta = \left( \frac{f_0}{f_s} \right) \times 360^\circ \]

\[ K = \frac{(1 - r) \sqrt{1 - 2r \cos 2\theta + r^2}}{2|\sin \theta|} \]
Pole Zero Placement Method

Second-Order BRF Design

\[ r \approx 1 - \left( \frac{BW_{3dB}}{f_s} \right) \times \pi \]

\[ \theta = \left( \frac{f_0}{f_s} \right) \times 360^0 \]

\[ H(z) = \frac{K(z - e^{j\theta})(z + e^{-j\theta})}{(z - re^{j\theta})(z - re^{-j\theta})} = \frac{K(z^2 - 2z\cos \theta + 1)}{(z^2 - 2rz\cos \theta + r^2)} \]

\[ K = \frac{(1 - 2r\cos \theta + r^2)}{(2 - 2\cos \theta)} \]

Example

Sampling rate = 8,000 Hz
3 dB bandwidth: \( BW = 100 \text{ Hz} \)
Narrow passband centered at \( f_0 = 1,500 \text{ Hz} \).

\[ r \approx 1 - \left( \frac{100/8000}{1} \right) \times \pi = 0.9607 \]

\[ \theta = \left( \frac{1500}{8000} \right) \times 360^0 = 67.5^0 \]

\[ H(z) = \frac{0.9620(z^2 - 2z\cos 67.5^0 + 1)}{(z^2 - 2 \times 0.9607z\cos 67.5^0 + 0.9607^2)} \]

\[ = \frac{0.9620 - 0.7363z^{-1} + 0.9620z^{-2}}{1 - 0.7353z^{-1} + 0.9229} \]
Pole Zero Placement Method

First-Order LPF Design

When $f_c < f_s/4$, $\alpha \approx 1 - 2 \times (f_c/f_s) \times \pi$.

$$H(z) = \frac{K(z + 1)}{(z - \alpha)}$$

$$K = \frac{(1 - \alpha)}{2}$$

Example

Sampling rate = 8,000 Hz
3 dB cutoff frequency: $f_c = 100$ Hz

$100 \text{ Hz} < f_s/4 = 2,000 \text{ Hz}$

$\alpha \approx 1 - 2 \times (100/8000) \times \pi = 0.9215$

When $f_c > f_s/4$, $\alpha \approx -(1 - \pi + 2 \times (f_c/f_s) \times \pi)$.

$$H(z) = \frac{0.03925(z + 1)}{(z - 0.9215)} = \frac{0.03925 + 0.03925z^{-1}}{1 - 0.9215z^{-1}}$$

$K = \frac{(1 - 0.9215)}{2} = 0.03925$
Pole Zero Placement Method

First-Order HPF Design

When $f_c < f_s/4$, $\alpha \approx 1 - 2 \times (f_c/f_s) \times \pi$

When $f_c > f_s/4$, $\alpha \approx -(1 - \pi + 2 \times (f_c/f_s) \times \pi)$

$$H(z) = \frac{K(z - 1)}{(z - \alpha)}$$

$$K = \frac{(1 + \alpha)}{2}$$

Practice examples.
Application: 60 - Hz Hum Eliminator

Hum noise: created by poor power supply or electromagnetic interference and characterized by a frequency of 60 Hz and its harmonics.

Hum eliminator

Frequency response of Hum eliminator

Corrupted by hum & harmonics
ECG + Hum $\rightarrow$ makes difficult to analyze.

Heart beat /min $= \frac{60000}{T}$
Heart Beat Detection Using ECG Pulse

1. Input ECG signal
   \[\xrightarrow{60\text{-Hz eliminator}}\]
   Enhanced ECG signal
   \[\xrightarrow{\text{Bandpass filtering passband: 0.25 Hz to 40 Hz}}\]
   Signal for heart rate detection

To filter muscle noise \(\approx 40\) Hz

Practice example