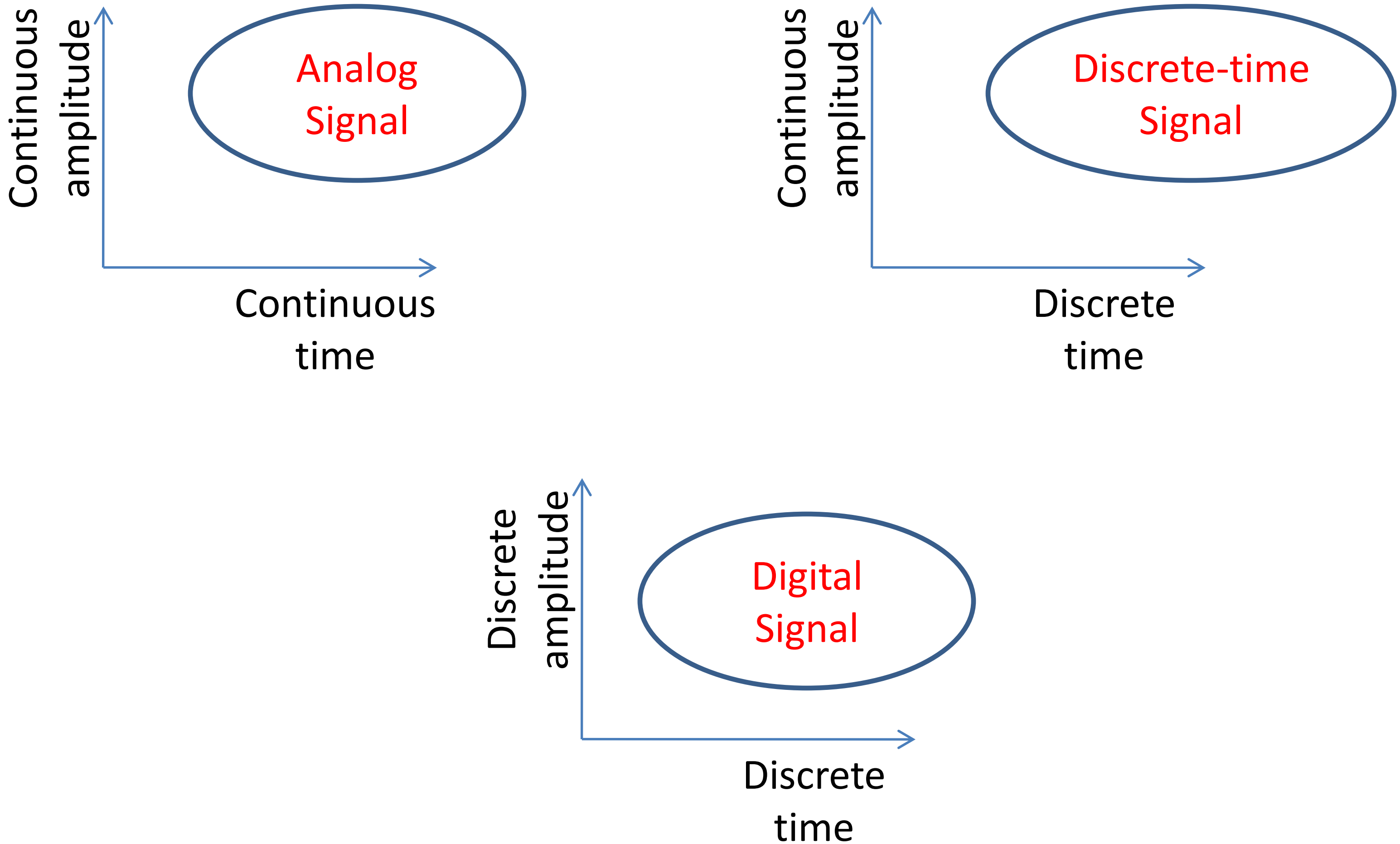
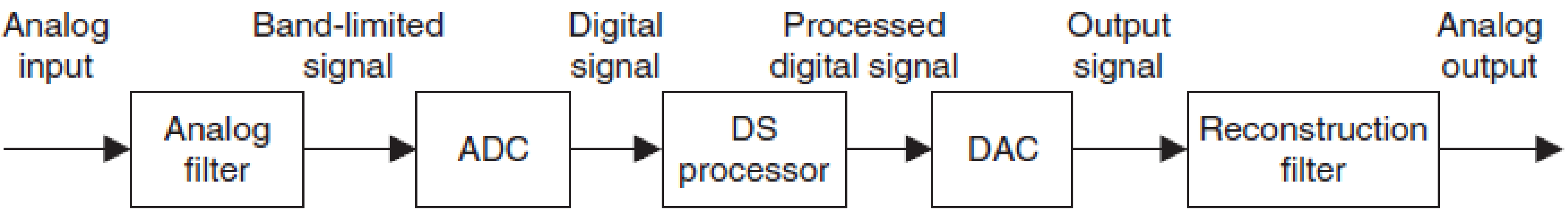
Digital Signal



DSP (Digital Signal Processing)

A digital signal processing scheme



To avoid aliasing for sampling

Analog to Digital Converter

Computer / microprocessor / micro controller/ etc.

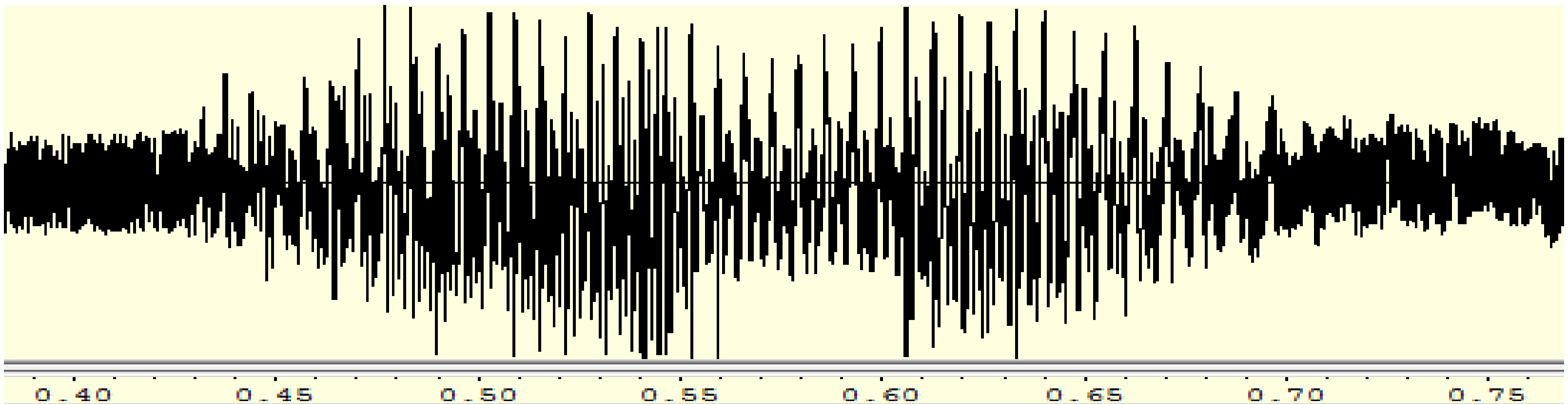
Digital to Analog Converter

To avoid aliasing for sampling

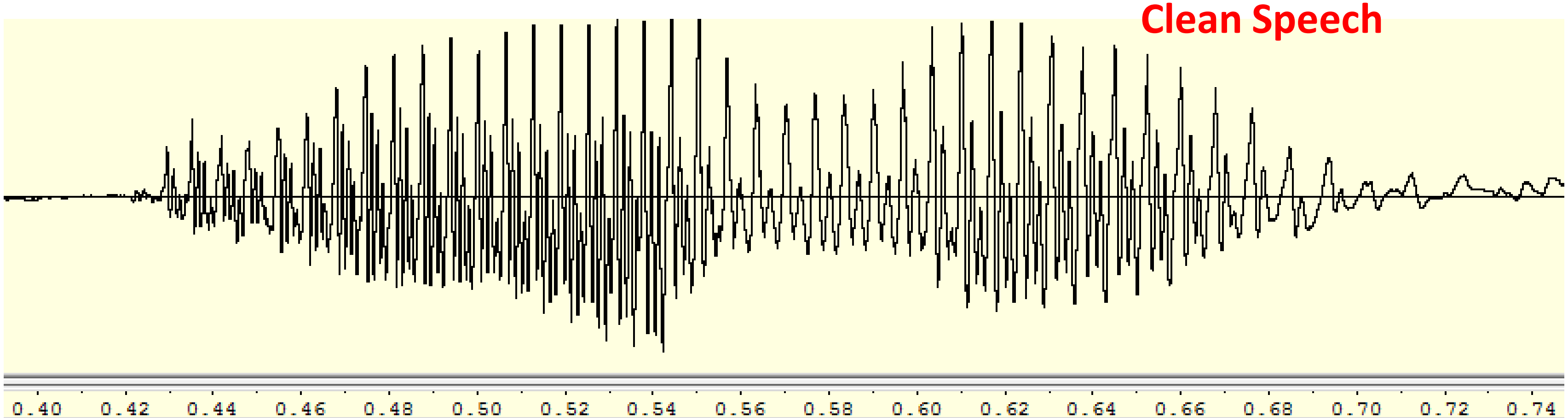
Some Applications of DSP

- Noise removal from speech.

Noisy Speech



Clean Speech



Some Applications of DSP

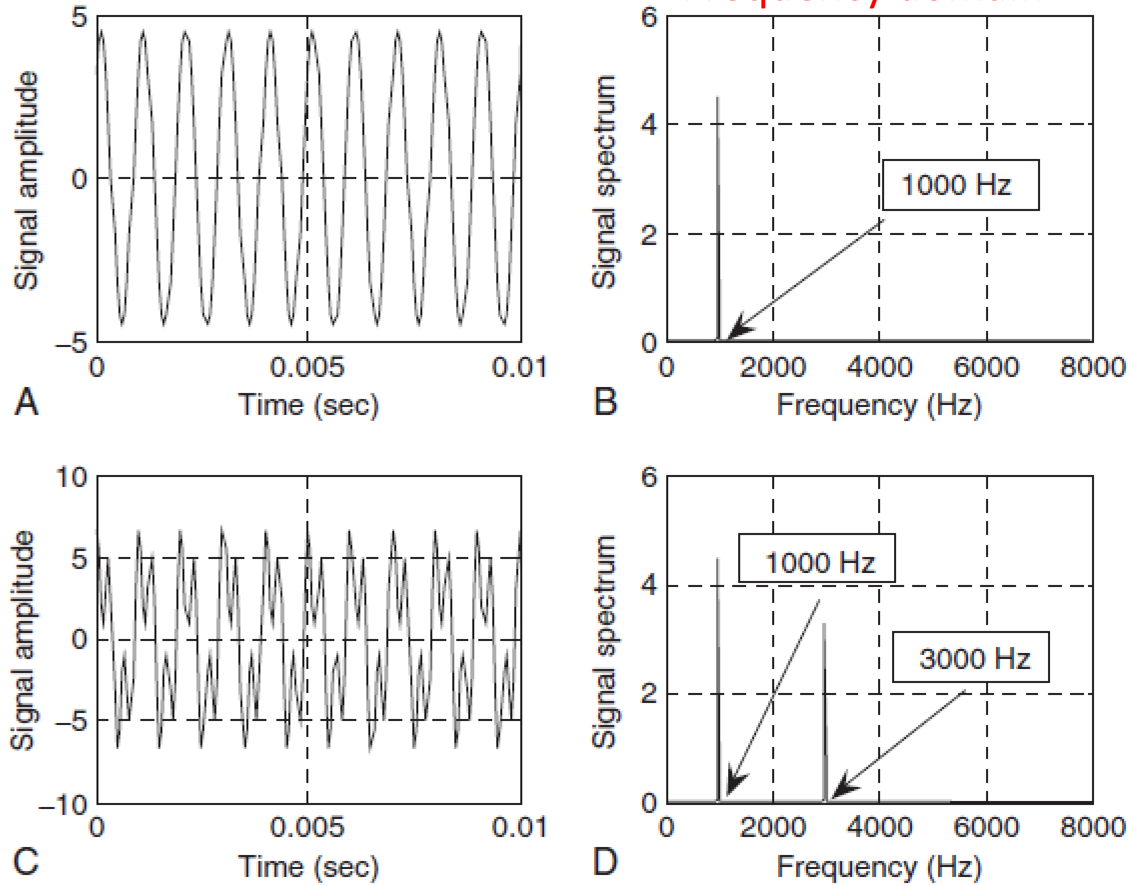
- **Signal spectral analysis.**

Single tone: 1000 Hz

Double tone: 1000 Hz and 3000 Hz

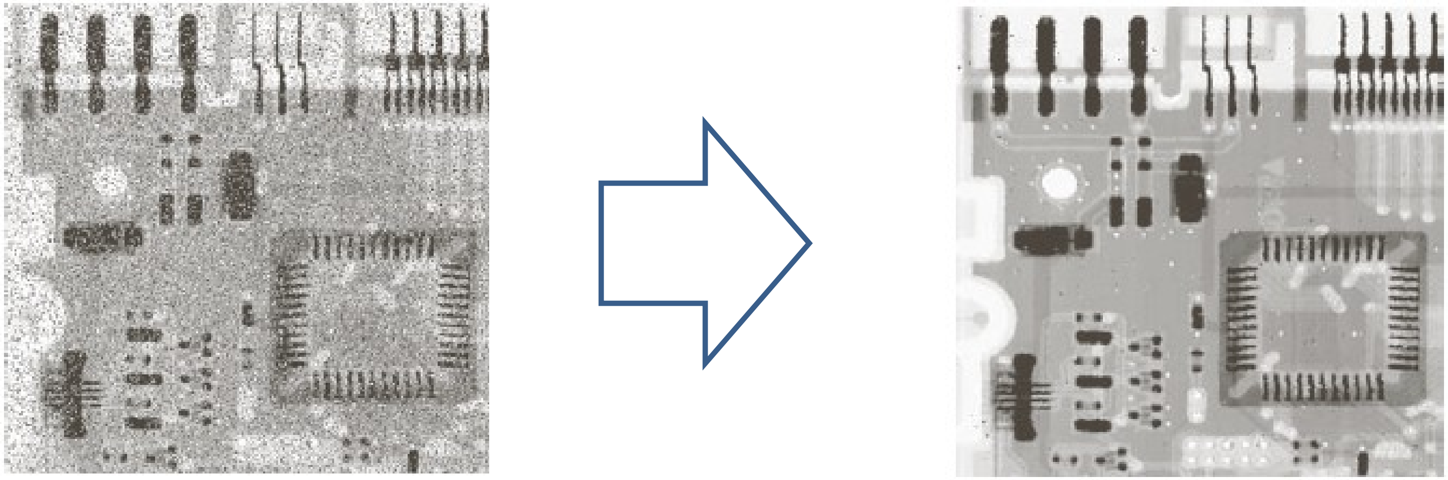
Time domain

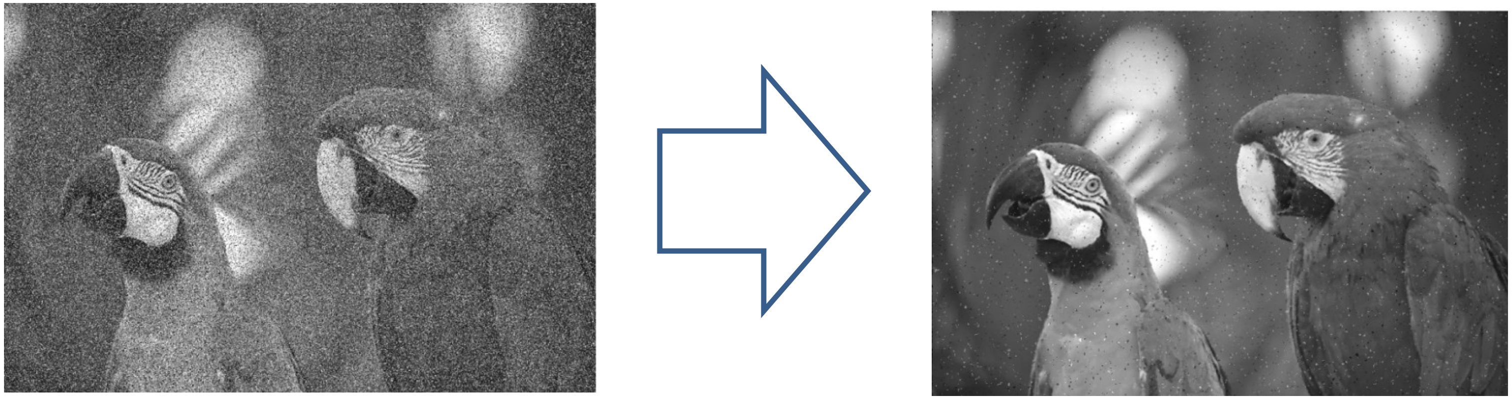
Frequency domain



Some Applications of DSP

- **Noise removal from image.**





Some Applications of DSP

- **Image enhancement.**

Summary Applications of DSP

Digital speech and audio:

- Speech recognition
- Speaker recognition
- Speech synthesis
- Speech enhancement
- Speech coding

Digital Image Processing:

- Image enhancement
- Image recognition
- Medical imaging
- Image forensics
- Image coding

Multimedia:

- Internet audio, video, phones
- Image / video compression
- Text-to-voice & voice-to-text
- Movie indexing

....

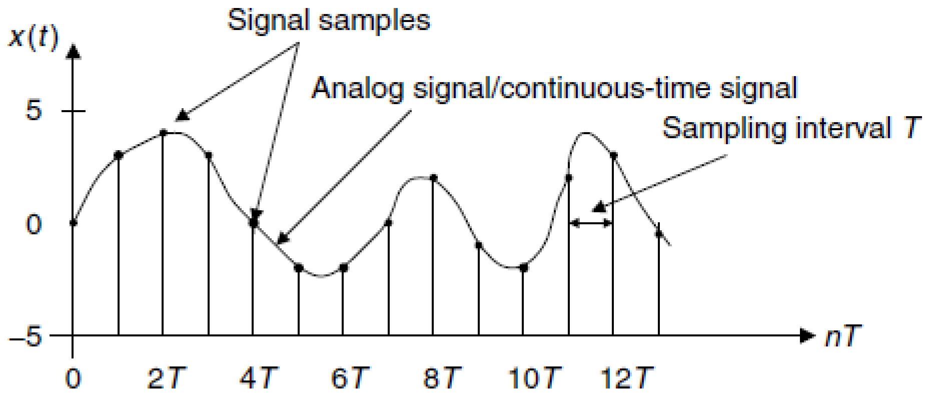
Sampling

For a given sampling interval T, which is defined as the time span between two sample points, the sampling rate is given by

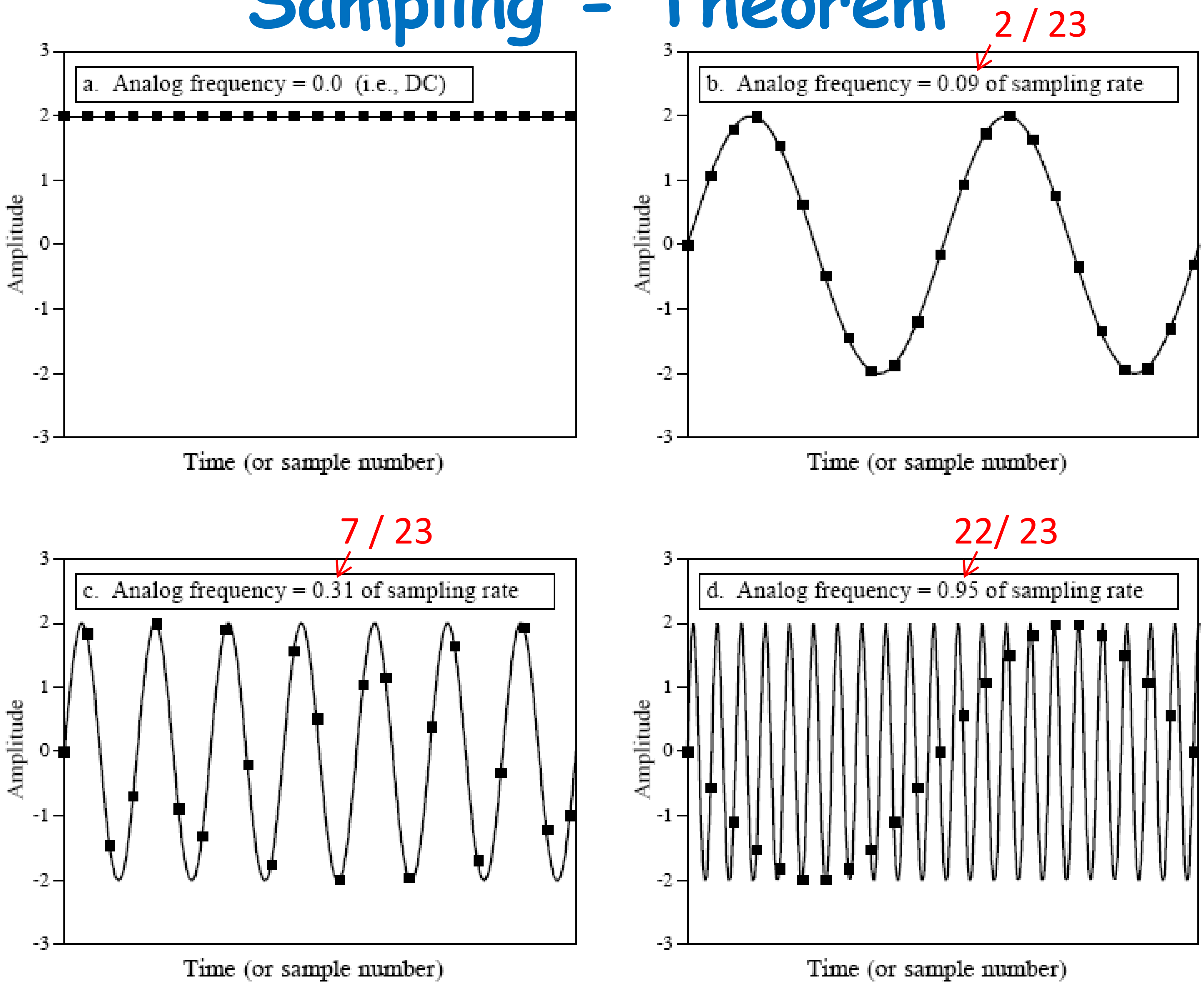
$$f_s = \frac{1}{T}$$

samples per second (Hz).

For example, if a sampling period is T = 125 microseconds, the sampling rate is determined as fs =1/125 μs or 8,000 samples per second (Hz).



Sampling - Theorem



Sampling - Theorem

The sampling theorem guarantees that an analog signal can be in theory perfectly recovered as long as the sampling rate is at least twice as large as the highest-frequency component of the analog signal to be sampled.

The condition is:

$$f_s \geq 2f_{\max},$$

where $f_{\max}$ is the maximum-frequency component of the analog signal to be sampled.

For example, to sample a speech signal containing frequencies up to 4 kHz, the minimum sampling rate is chosen to be at least 8 kHz, or 8,000 samples per second.

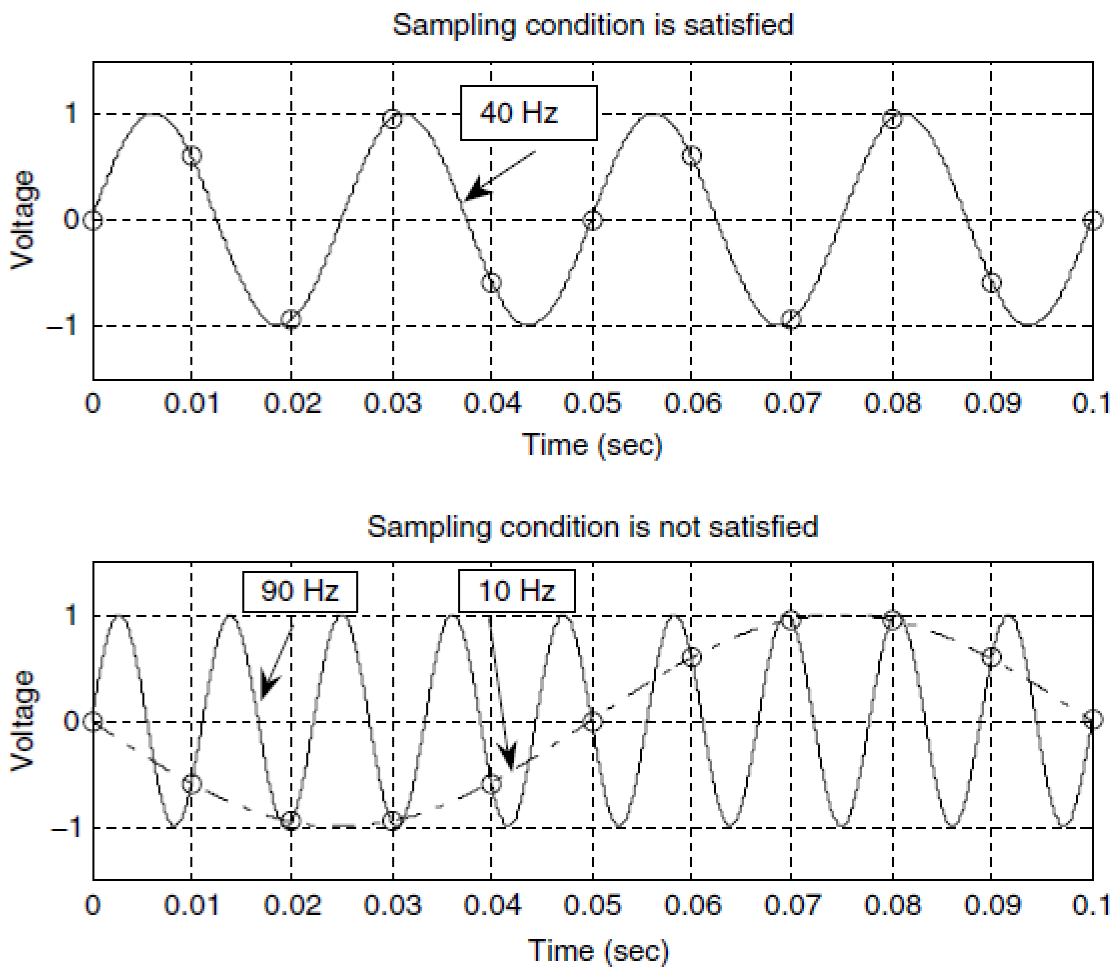
Sampling - Theorem

Sampling interval T= 0.01 s
Sampling rate f_s= 100 Hz
Sinusoid freq. = 4 cycles / 0.1
= 40 Hz

$$2f_{\max} = 80 \text{ Hz} < f_s.$$

Sampling condition is satisfied, so reconstruction from digital to analog is possible.

Do this by yourself! →



Sampling Process

In frequency domain:

$$X_s(f) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

X_s(f): Sampled spectrum
X(f): Original spectrum
X(f±nf_s): Replica spectrum

$$\Rightarrow \quad X_s(f) = \cdots + \frac{1}{T}X(f + f_s) + \frac{1}{T}X(f) + \frac{1}{T}X(f - f_s) + \cdots$$

Sampling Process

Original spectrum

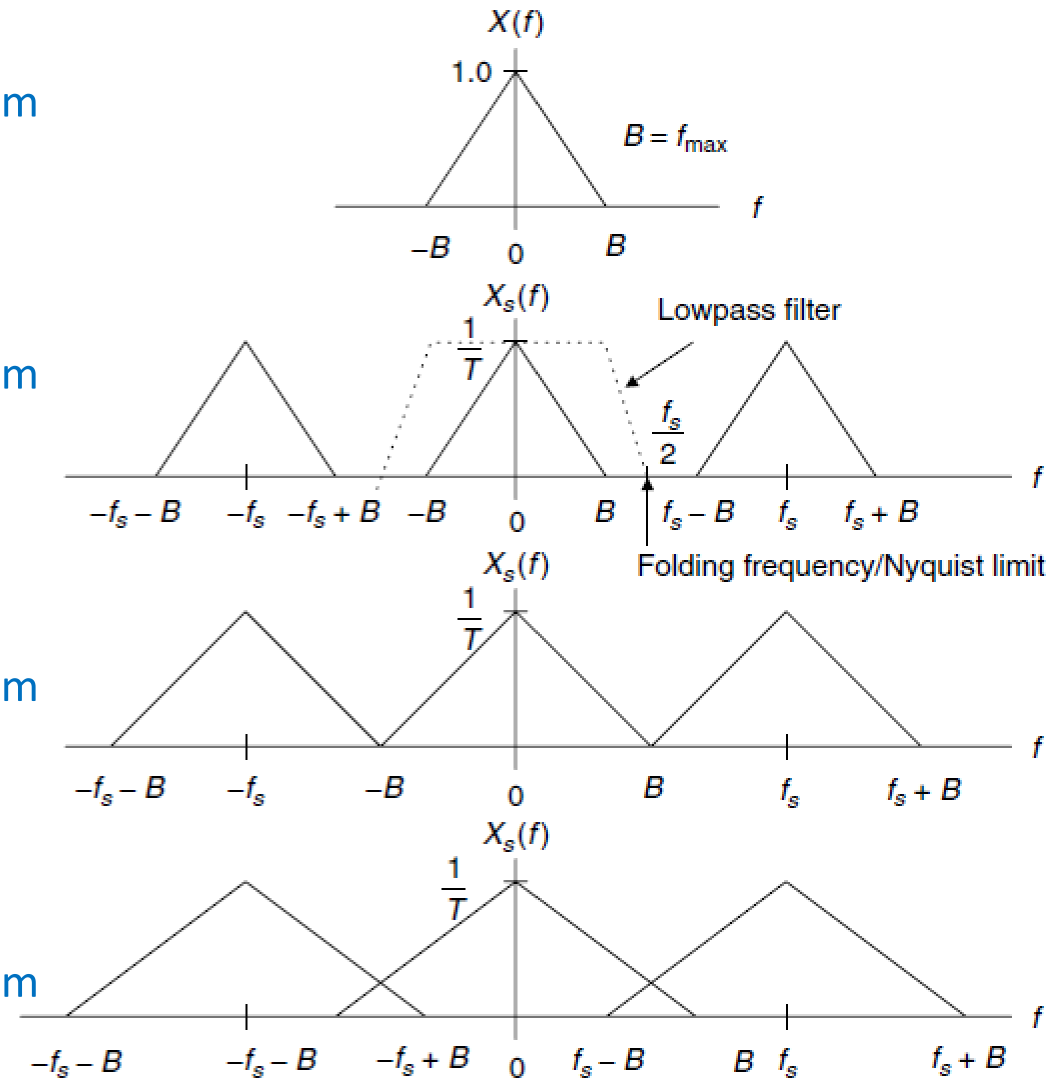
Original spectrum plus its replicas

Original spectrum plus its replicas

Minimum requirement for Reconstruction

Original spectrum plus its replicas

Reconstruction not possible



Shannon Sampling Theorem

For a uniformly sampled DSP system, an analog signal can be perfectly recovered as long as the sampling rate is at least twice as large as the highest-frequency component of the analog signal to be sampled.

$$f_s - f_{\max} \geq f_{\max} \quad \Rightarrow \quad f_s \geq 2 f_{\max}$$

The minimum sampling rate is called the Nyquist rate
Half of the sampling frequency is called the folding frequency.

Problem: Find the Nyquist rate for the following signal.

$$x(t) = 3\cos 50\pi t + 10\sin 300\pi t - \cos 100\pi t$$

Solution:

The maximum frequency present is 150 Hz = f_{max}.

Therefore Nyquist rate = 2× f_{max} = 300 Hz.

Example 1

Problem:

Suppose that an analog signal is given as

$$x(t) = 5\cos(2\pi \cdot 1000t), \text{ for } t \geq 0$$

and is sampled at the rate of 8,000 Hz.

a. Sketch the spectrum for the original signal.

b. Sketch the spectrum for the sampled signal from 0 to 20 kHz.

Solution:

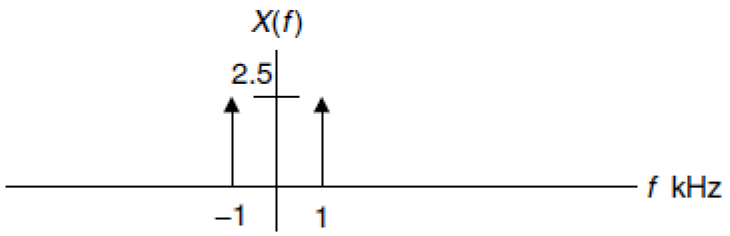
Using Euler's identity,

$$5\cos(2\pi \times 1000t) = 5 \cdot \left(\frac{e^{j2\pi\times 1000t} + e^{-j2\pi\times 1000t}}{2}\right) = 2.5e^{j2\pi\times 1000t} + 2.5e^{-j2\pi\times 1000t}$$

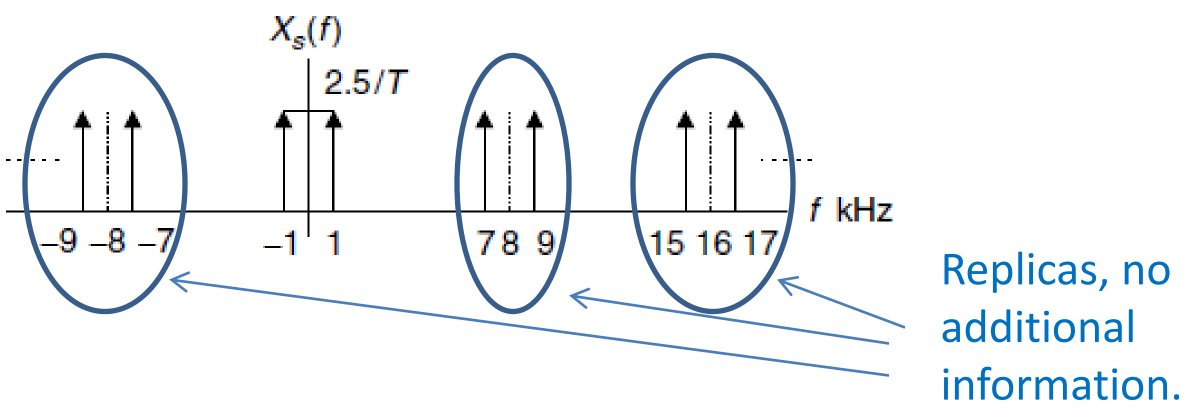
Hence, the Fourier series coefficients are: $c_1 = 2.5$, and $c_{-1} = 2.5$.

Example 1 – contd.

a.

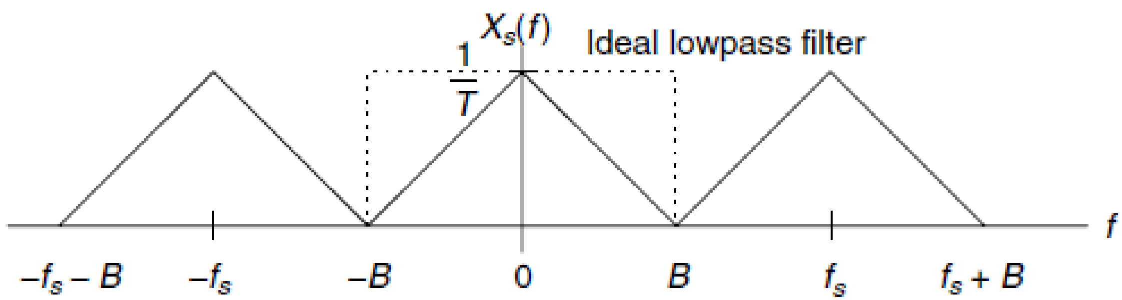


b. After the analog signal is sampled at the rate of 8,000 Hz, the sampled signal spectrum and its replicas centered at the frequencies $\pm nf_s$, each with the scaled amplitude being 2.5/T

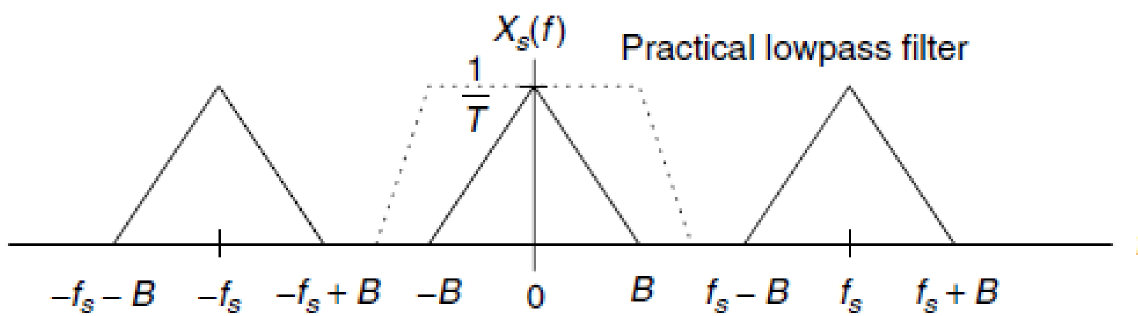


Signal Reconstruction

Case 1: $f_s = 2f_{max}$

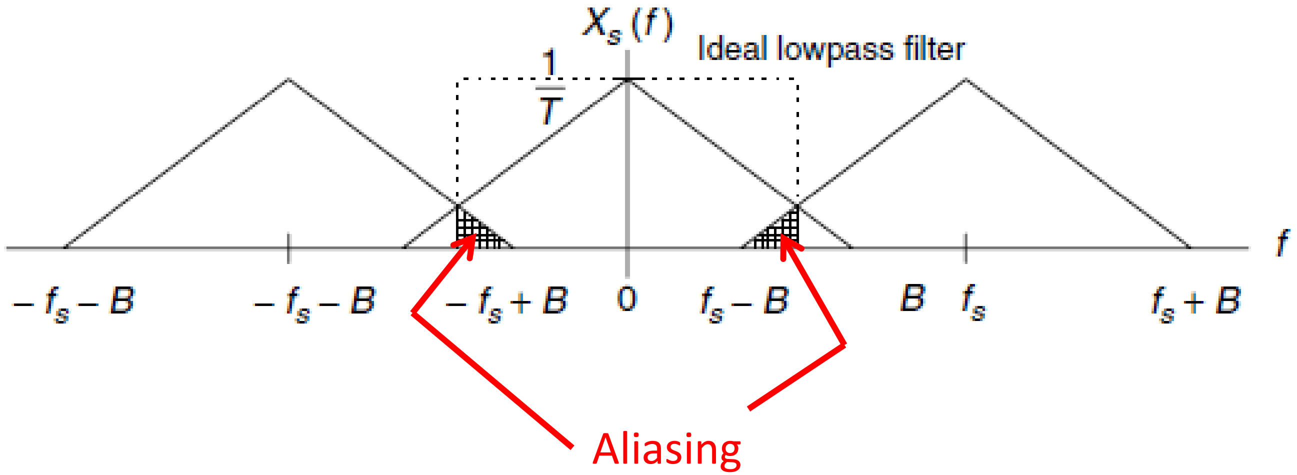


Case 2: $f_s > 2f_{max}$



Signal Reconstruction

Case 3: $f_s < 2f_{max}$



Perfect reconstruction is not possible, even if we use ideal low pass filter.

Example 2

Problem:

Assuming that an analog signal is given by

$$x(t) = 5\cos(2\pi \cdot 2000t) + 3\cos(2\pi \cdot 3000t), \text{ for } t \geq 0$$

and it is sampled at the rate of 8,000 Hz,

a. Sketch the spectrum of the sampled signal up to 20 kHz.

b. Sketch the recovered analog signal spectrum if an ideal lowpass filter with a cutoff frequency of 4 kHz is used to filter the sampled signal ($y(n) = x(n)$ in this case) to recover the original signal.

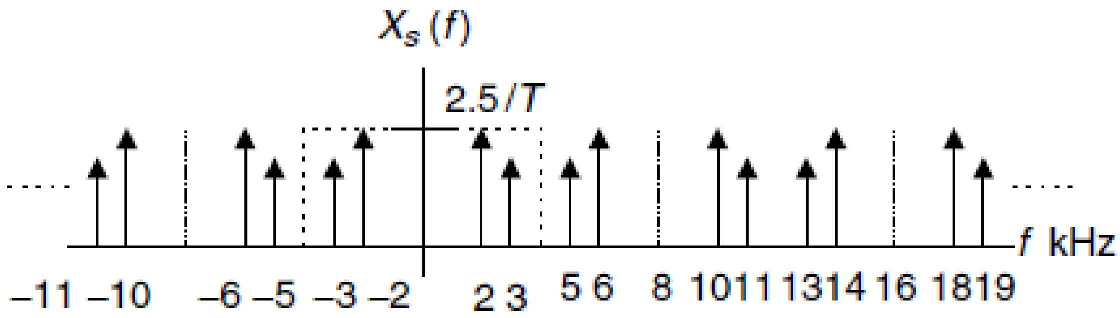
Solution:

Using the Euler's identity:

$$x(t) = \frac{3}{2}e^{-j2\pi \cdot 3000t} + \frac{5}{2}e^{-j2\pi \cdot 2000t} + \frac{5}{2}e^{j2\pi \cdot 2000t} + \frac{3}{2}e^{j2\pi \cdot 3000t}$$

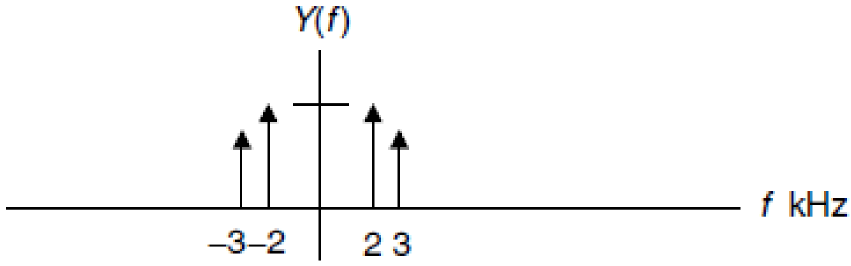
Example 2 – contd.

a.



b.

The Shannon sampling theory condition is satisfied.



Example 3

Problem:

Given an analog signal

$$x(t) = 5\cos(2\pi \times 2000t) + 1\cos(2\pi \times 5000t), \text{ for } t \geq 0,$$

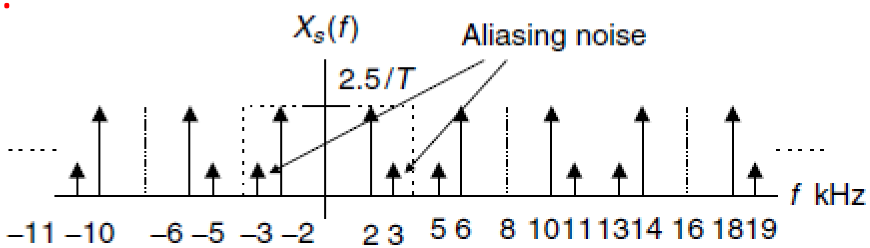
which is sampled at a rate of 8,000 Hz,

a. Sketch the spectrum of the sampled signal up to 20 kHz.

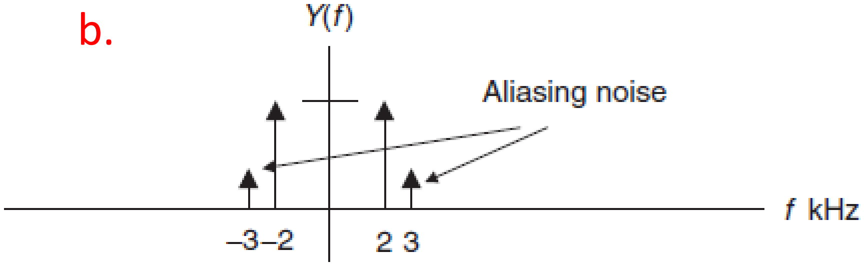
b. Sketch the recovered analog signal spectrum if an ideal lowpass filter with a cutoff frequency of 4 kHz is used to recover the original signal ($y(n) = x(n)$ in this case).

Solution:

a.



b.



Example 4

Problem: Consider the analog signal $x(t) = 3\cos 2000\pi t + 5\sin 6000\pi t + 10\cos 12000\pi t$

(a) What is the Nyquist rate for this signal?
(b) What is the discrete-time signal after sampling it at F_s = 5000 samples/s?
(c) What is the analog signal y(t) that we reconstruct from the samples if we use ideal interpolation?

Solution: (a) $F_1 = 1KHz,\quad F_2 = 3KHz,\quad F_3 = 6KHz$ Max.

Nyquist rate = 2×6K Hz = 12K Hz.

(b)

$$x(n) = x(nT) = x\left(\frac{n}{F_s}\right)$$

$$= 3\cos 2\pi\left(\frac{1}{5}\right)n + 5\sin 2\pi\left(\frac{3}{5}\right)n + 10\cos 2\pi\left(\frac{6}{5}\right)n$$

$$= 3\cos 2\pi\left(\frac{1}{5}\right)n + 5\sin 2\pi\left(1-\frac{2}{5}\right)n + 10\cos 2\pi\left(1+\frac{1}{5}\right)n$$

$$= 3\cos 2\pi\left(\frac{1}{5}\right)n + 5\sin 2\pi\left(-\frac{2}{5}\right)n + 10\cos 2\pi\left(\frac{1}{5}\right)n$$

$$= 13\cos 2\pi\left(\frac{1}{5}\right)n - 5\sin 2\pi\left(\frac{2}{5}\right)n$$

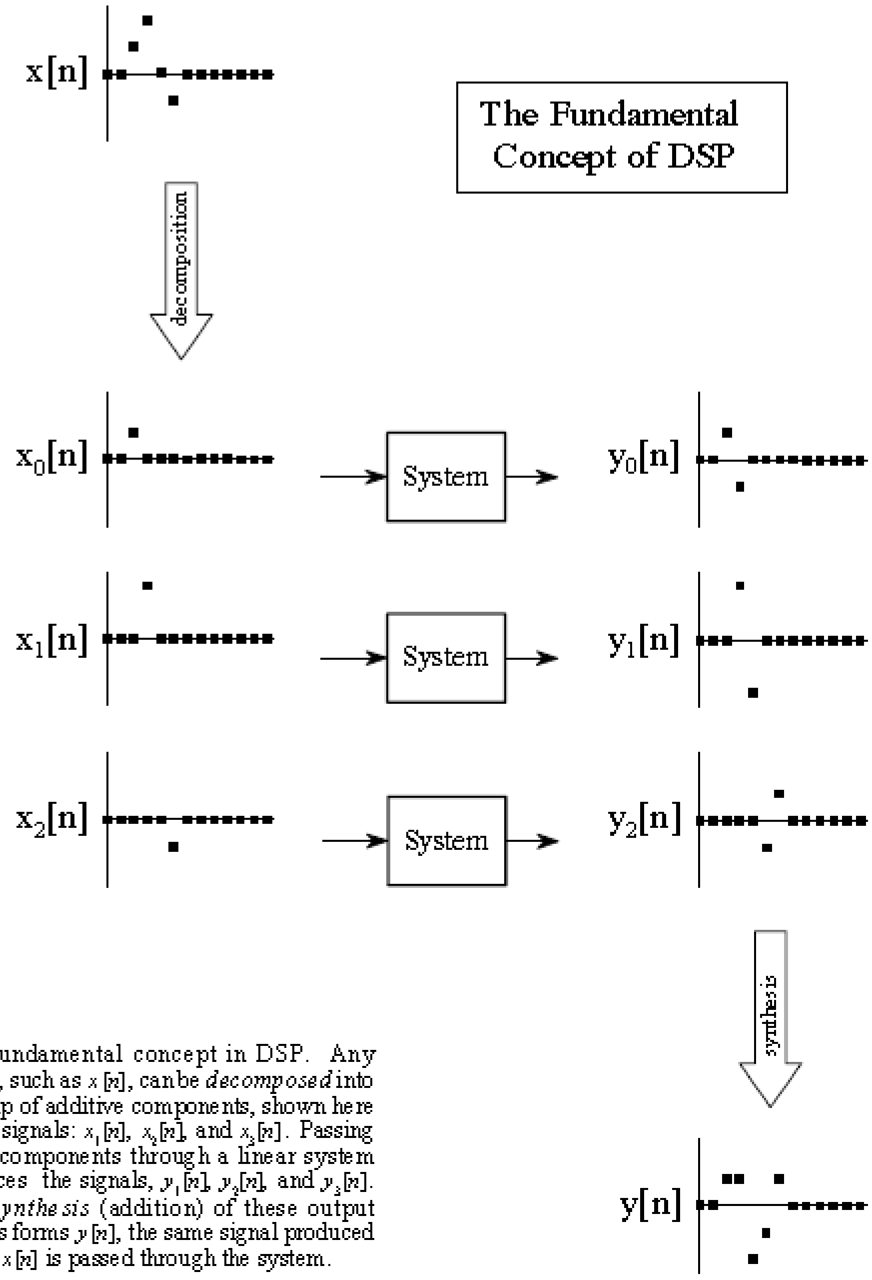
(c)

$$y(t) = 13\cos 2000\pi t - 5\sin 4000\pi t$$

Alias

Decomposition and Synthesis

Any signal can be decomposed into additive components, and the component signals can be added (synthesis) to produce the original signal.



The fundamental concept in DSP. Any signal, such as $x[n]$, can be *decomposed* into a group of additive components, shown here by the signals: $x_1[n]$, $x_2[n]$, and $x_3[n]$. Passing these components through a linear system produces the signals, $y_1[n]$, $y_2[n]$, and $y_3[n]$. The *synthesis* (addition) of these output signals forms $y[n]$, the same signal produced when $x[n]$ is passed through the system.

Decomposition_1

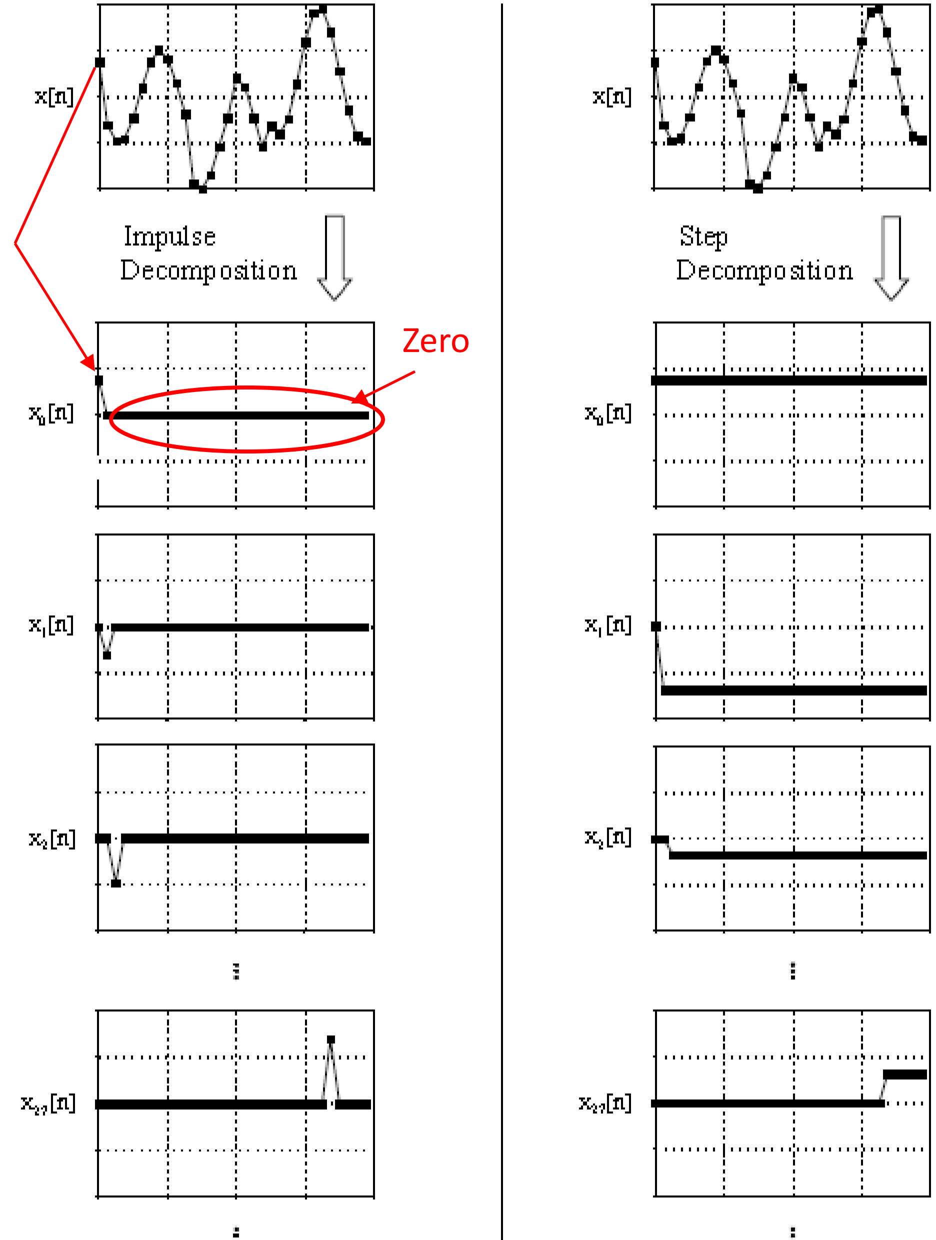
Impulse decomposition:

N samples signal is decomposed into N component signals each containing N samples.

A component signal contains one point from the original signal, with the remainder of the values being zero.

Step decomposition:

k-th component signal, $x_k[n]$, contains zeros for points through 0 to k-1, while the remaining points have a value equal to $x[k] - x[k-1]$.



Impulse decomposition

Step decomposition

Decomposition_2

Even/Odd decomposition:

$$x_e[n] = \frac{x[n] + x[N-n]}{2}$$

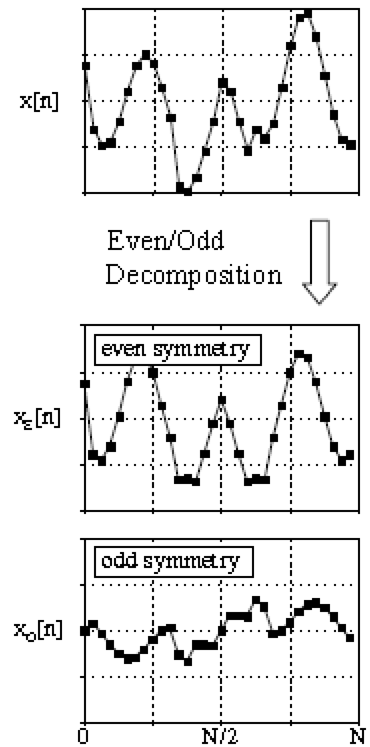
$$x_o[n] = \frac{x[n] - x[N-n]}{2}$$

Even symmetry:

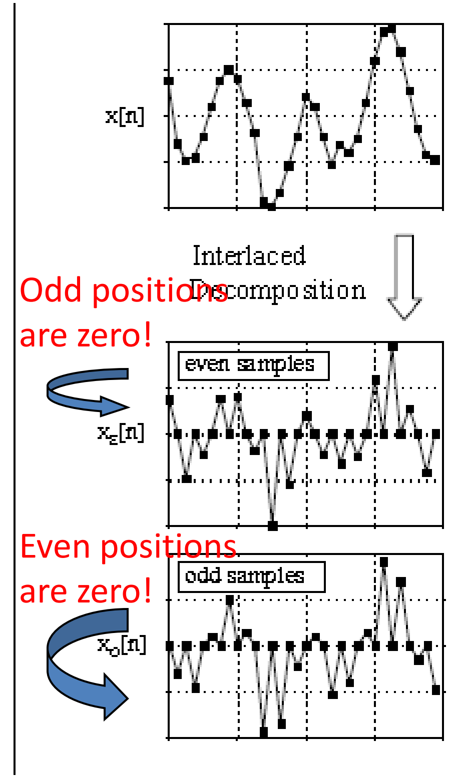
Mirror image around x[N/2].
x[0] and x[N/2] are equal to original value.
Others like: x[N/2+1] = x[N/2-1].

Odd symmetry:

Negative mirror image around x[N/2].
x[0] and x[N/2] are equal to zero.
Others like: x[N/2+1] = -x[N/2-1].



Example of even/odd decomposition. An N point signal is broken into two N point signals, one with even symmetry, and the other with odd symmetry.

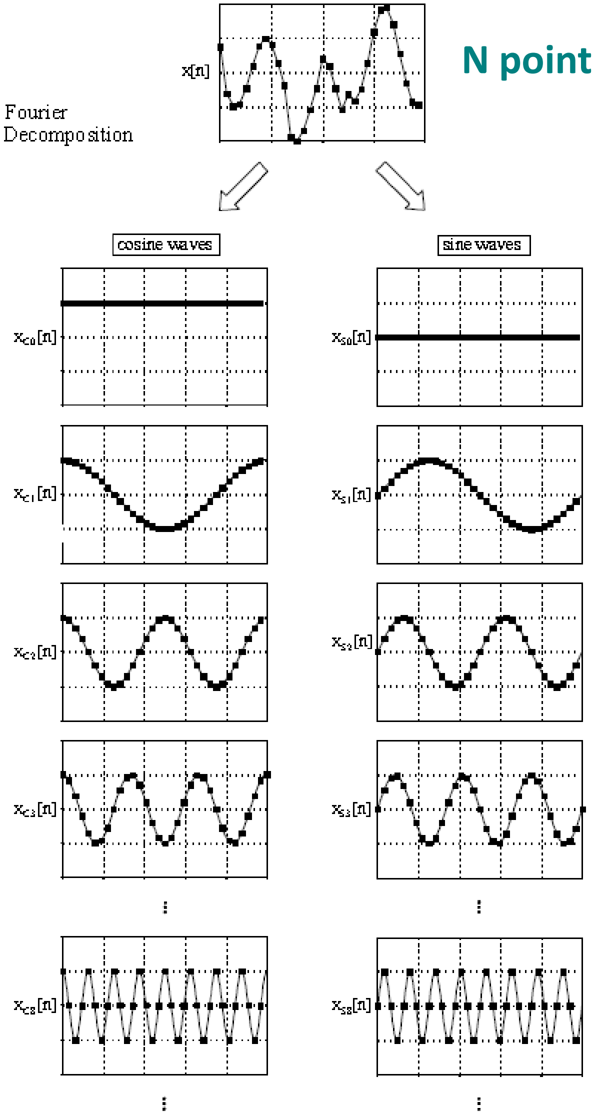


Example of interlaced decomposition. An N point signal is broken into two N point signals, one with the odd samples set to zero, the other with the even samples set to zero.

You can check the result by using the following formula:

$$x[n] = x_e[n] + x_o[n]$$

Fourier Decomposition



N point signal

Zero!

(N/2)+1 cosine signals, each with N points

(N/2)+1 sine signals, each with N points

So, total N significant signals!

Zero!

FIGURE 5-16
Illustration of Fourier decomposition. An *N* point signal is decomposed into *N*+2 signals, each having *N* points. Half of these signals are cosine waves, and half are sine waves. The frequencies of the sinusoids are fixed; only the amplitudes can change.

Example 5

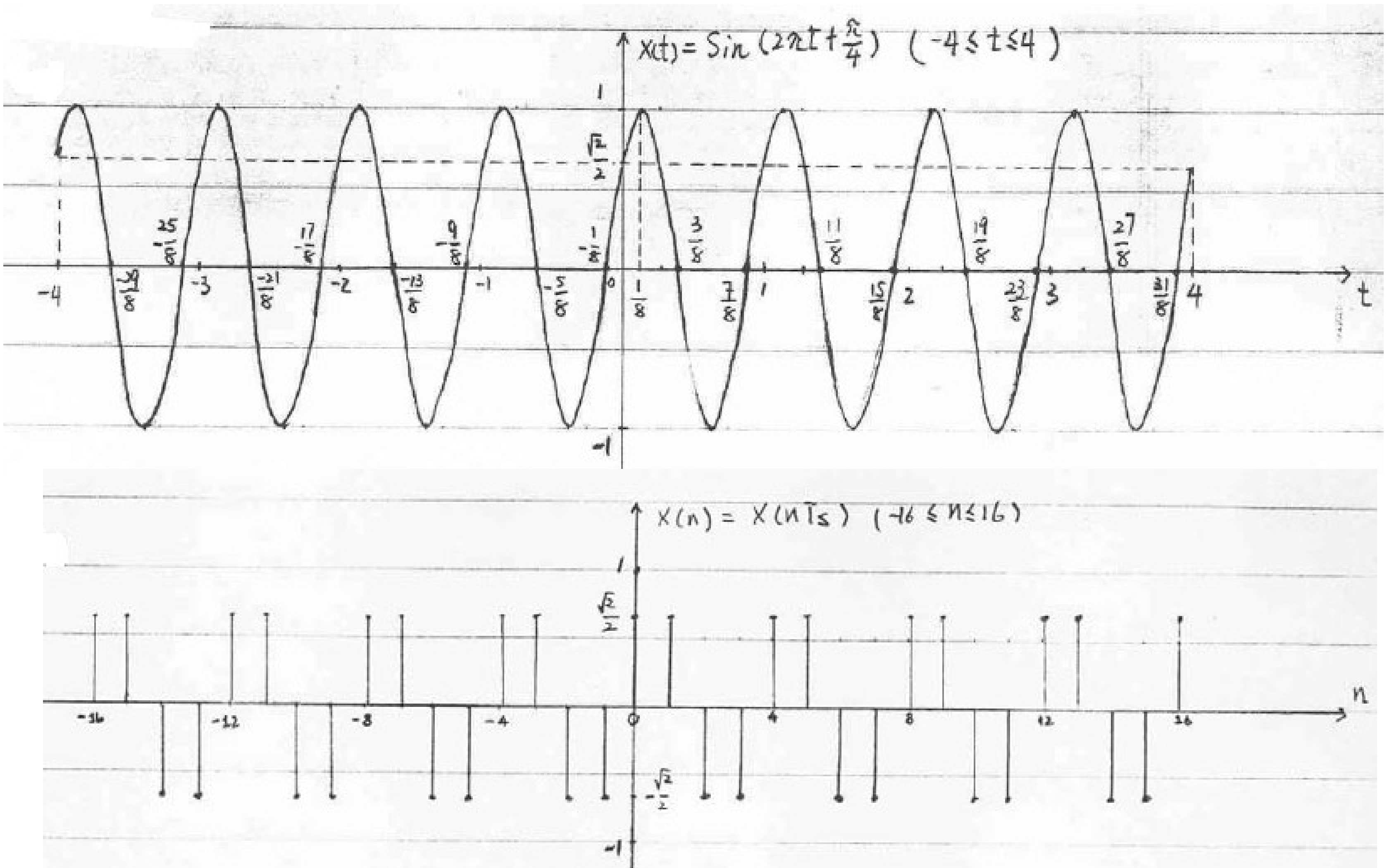
Problem: A signal is defined as x[n]={2, 3, 5, -2, -3, -5, -2, 2}; Find the result of even / odd decomposition. Index starts from 0.

Answer: Even symmetry: x_e = {2, 2.5, 1.5, -3.5, -3, -3.5, 1.5, 2.5}

Odd symmetry: x_o = {0, 0.5, 3.5, 1.5, 0, -1.5, -3.5, -0.5}

Sinusoid Drawing

$$x(t) = \sin\left(2\pi t + \frac{\pi}{4}\right), \quad -4 \le t \le 4$$



Mean and Standard Deviation

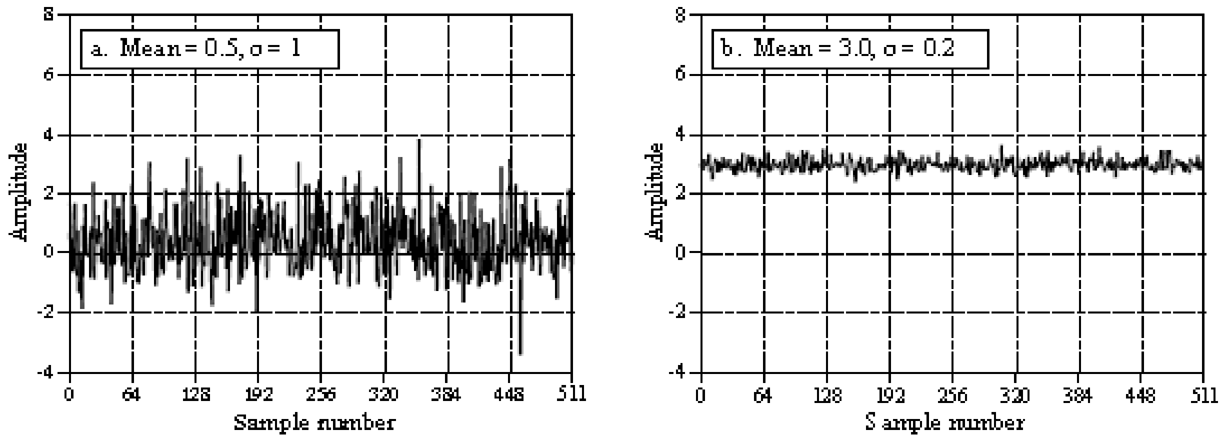
Mean: $$\mu = \frac{1}{N}\sum_{i=0}^{N-1} x_i$$

Standard Deviation: $$\sigma = \sqrt{\frac{1}{N-1}\sum_{i=0}^{N-1}(x_i - \mu)^2}$$

To reduce statistical noise for small number of samples

Mean = DC Value

S.D. = How the signal fluctuates around Mean (AC)



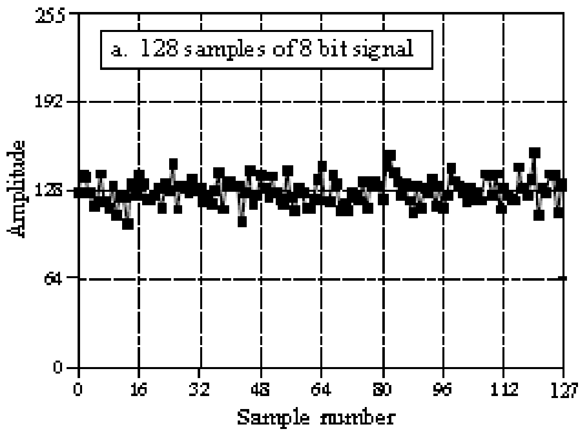
Examples of two digitized signals with different means and standard deviations.

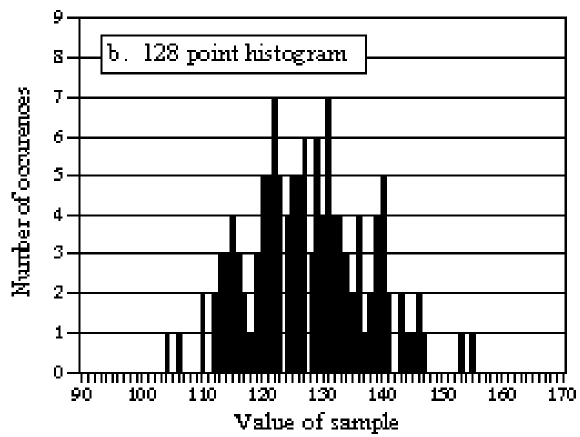
Histogram

More sample produces smoother histogram

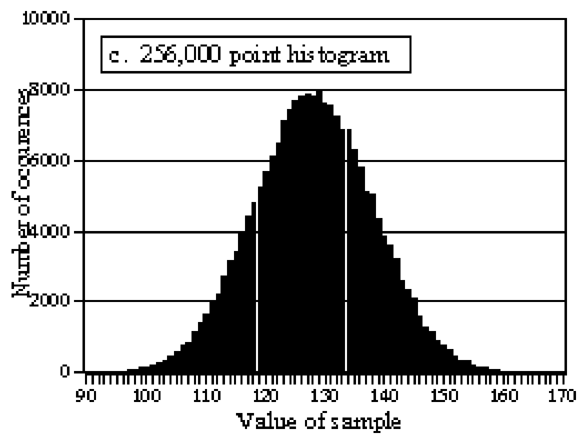
$$N = \sum_{i=0}^{M-1} H_i$$

M = Number of points in the histogram





Examples of histograms. Figure (a) shows 128 samples from a very long signal, with each sample being an integer between 0 and 255. Figures (b) and (c) show histograms using 128 and 256,000 samples from the signal, respectively. As shown, the histogram is smoother when more samples are used.



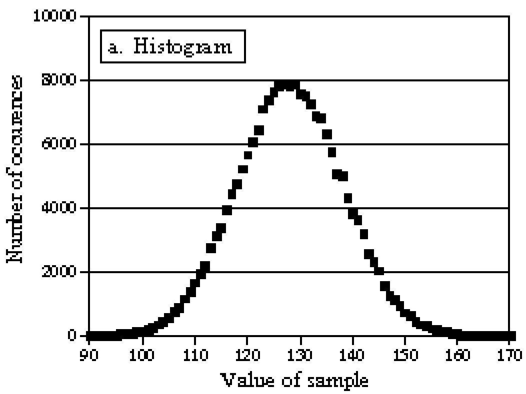
Mean and S.D. using histogram:

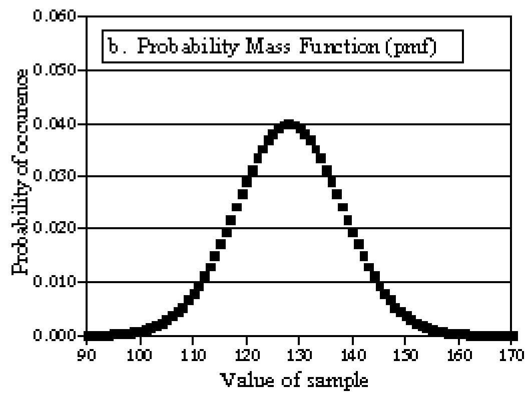
$$\mu = \frac{1}{N} \sum_{i=0}^{M-1} i H_i$$

$$\sigma = \sqrt{\frac{1}{N-1} \sum_{i=0}^{M-1} (i-\mu)^2 H_i}$$

Histogram, pmf, pdf

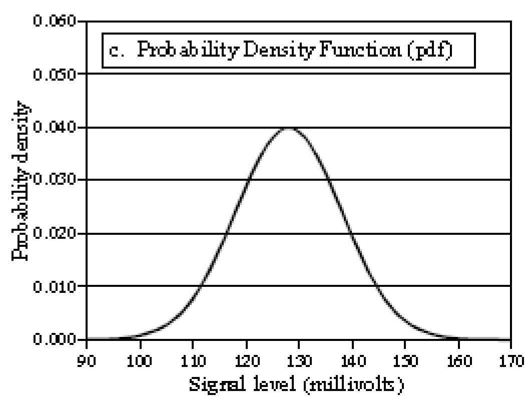
Look at the values along y-axis





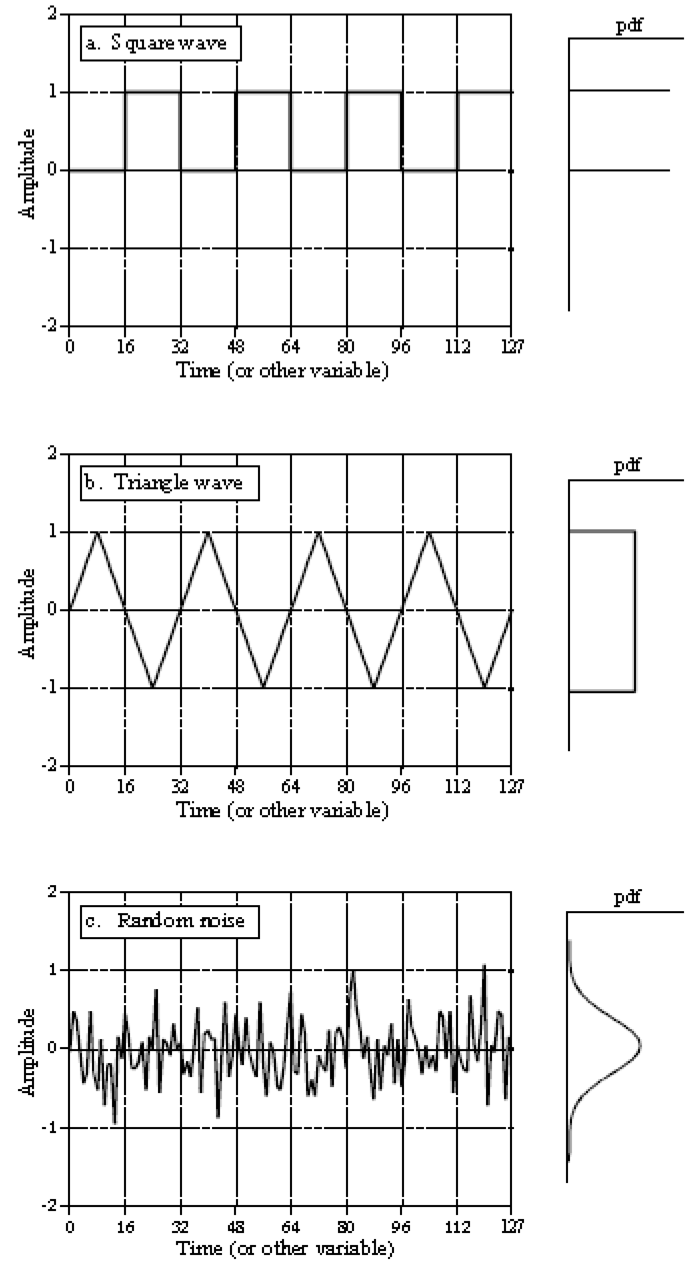
Discrete

The relationship between (a) the histogram, (b) the probability mass function (pmf), and (c) the probability density function (pdf). The histogram is calculated from a finite number of samples. The pmf describes the probabilities of the underlying process. The pdf is similar to the pmf, but is used with continuous rather than discrete signals. Even though the vertical axis of (b) and (c) have the same values (0 to 0.06), this is only a coincidence of this example. The amplitude of these three curves is determined by: (a) the sum of the values in the histogram being equal to the number of samples in the signal; (b) the sum of the values in the pmf being equal to one, and (c) the area under the pdf curve being equal to one.



Continuous

Waveforms and pdfs



Three common waveforms and their probability density functions. As in these examples, the pdf graph is often rotated one-quarter turn and placed at the side of the signal it describes. The pdf of a square wave, shown in (a), consists of two infinitesimally narrow spikes, corresponding to the signal only having two possible values. The pdf of the triangle wave, (b), has a constant value over a range, and is often called a *uniform* distribution. The pdf of random noise, as in (c), is the most interesting of all, a bell shaped curve known as a *Gaussian*.

Look at the pdf of random noise.

It is Gaussian!

Histogram Bins

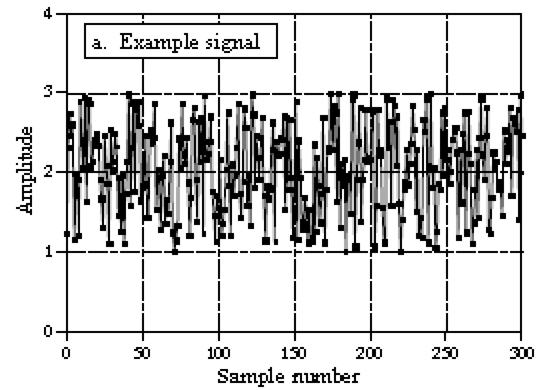
The range 0-4 is divided by 600 in (b) and by 8 in (c).

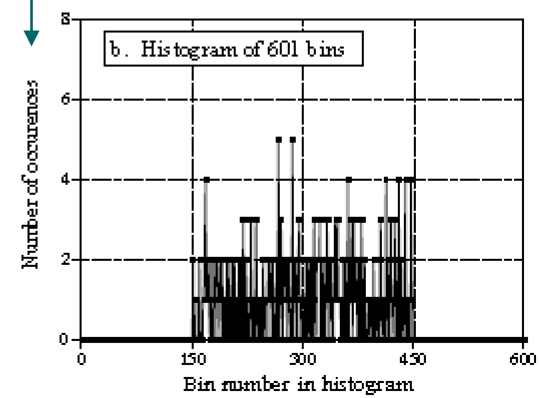
Poor resolution in vertical axis: (b).

Poor resolution in horizontal axis: (c).

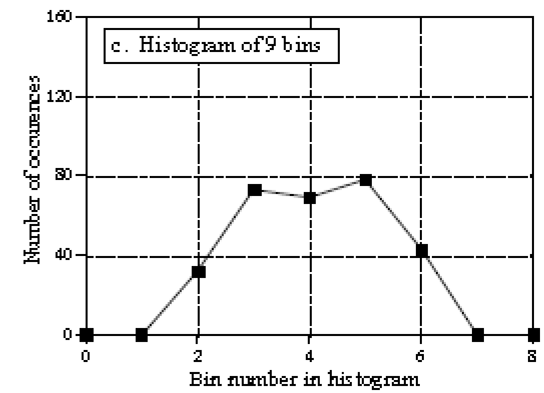
Using more samples makes better resolution.

Look at the vertical axis





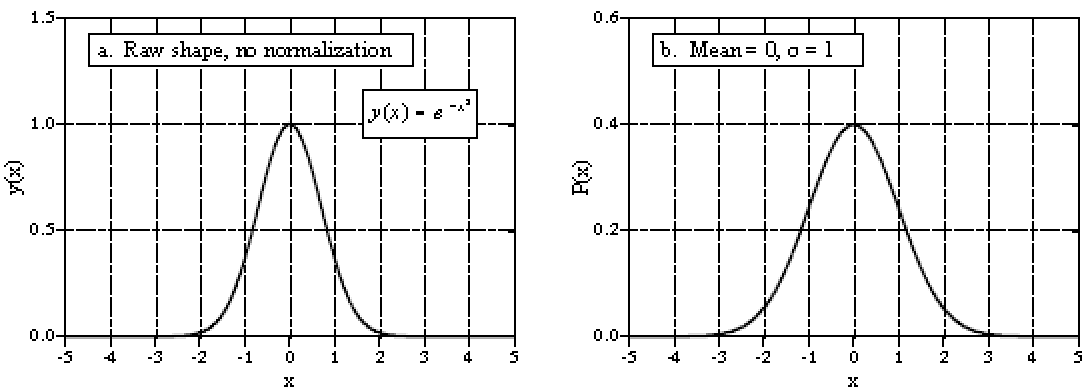
Example of binned histograms. As shown in (a), the signal used in this example is 300 samples long, with each sample a floating point number uniformly distributed between 1 and 3. Figures (b) and (c) show binned histograms of this signal, using 601 and 9 bins, respectively. As shown, a large number of bins results in poor resolution along the *vertical axis*, while a small number of bins provides poor resolution along the *horizontal axis*. Using more samples makes the resolution better in both directions.



Normal Distribution (Gaussian)

Basic shape: $y(x) = e^{-x^2}$

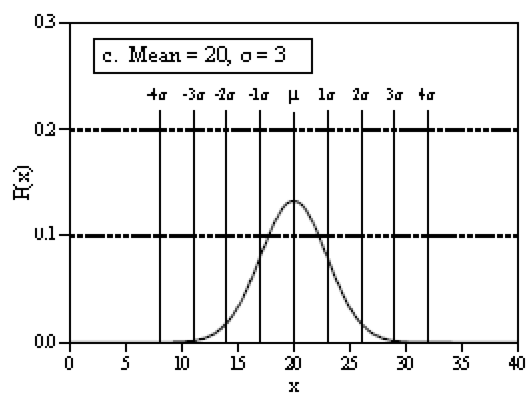
Equation for normal distribution: $P(x) = \dfrac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$



μ: Mean

σ: Standard Deviation

The normalization term is to make the area under curve = 1



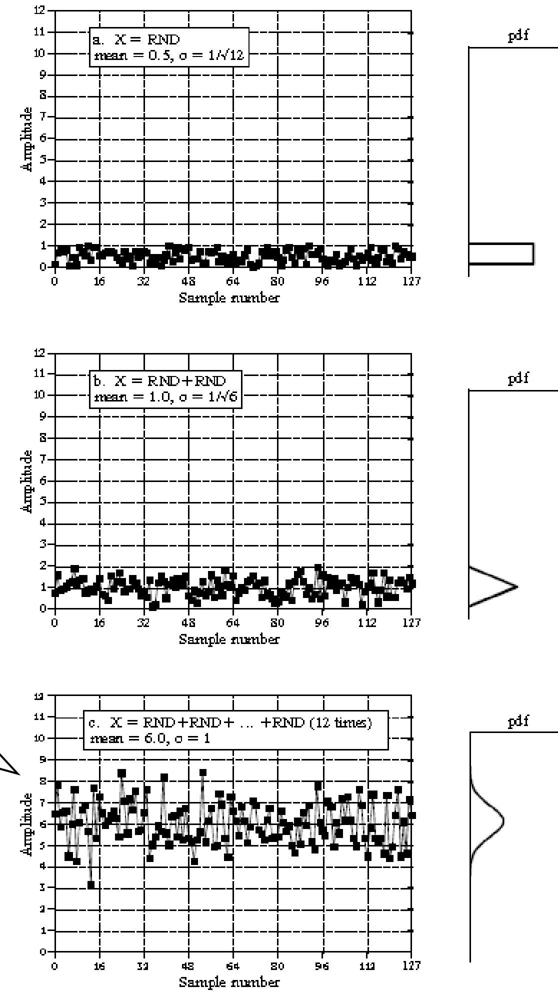
Examples of Gaussian curves. Figure (a) shows the shape of the raw curve without normalization or the addition of adjustable parameters. In (b) and (c), the complete Gaussian curve is shown for various means and standard deviations.

Digital Noise Generation

Uniform distribution: **one** RND function (0~1)

Two times RND functions and add

Twelve times RND functions and add



Gaussian

For each sample:

$$\mu' + \sigma' \times (\sum_{12} RND_{0\sim1} - 6)$$

μ' : desired mean, σ' : desired S.D.

Central Limit Theorem:

A sum of random numbers becomes normally distributed as more and more of the random numbers are added together.

Converting a uniform distribution to a Gaussian distribution. Figure (a) shows a signal where each sample is generated by a random number generator. As indicated by the pdf, the value of each sample is uniformly distributed between zero and one. Each sample in (b) is formed by adding *two* values from the random number generator. In (c), each sample is created by adding *twelve* values from the random number generator. The pdf of (c) is very nearly Gaussian, with a mean of *six*, and a standard deviation of *one*.

Figure Acknowledgement

Most of the figures are taken from the following books:

Li Tan, *Digital Signal Processing, Fundamentals and Applications*, Elsevier, 2008.

Steven W. Smith, *Digital Signal Processing: A Practical Guide for Engineers and Scientists*, Newnes, Elsevier, 2003.