King Saud University | MATH351 | Exam I, 29/12/1435H
College of Sciences | | Time: 90 Minutes
Mathematics Department | | Full mark: 25

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Q.1[8] Show that the equation $x^3 - 2x - 1 = 0$ has a solution in the interval $[0, 2]$. Then, apply the Bisection Method to compute the second approximation of this solution. Also, find the number of iterations needed to get an approximation accurate to within 10^{-5}.

Q.2[8] Let $g(x) = \frac{e^{-x}}{2}$,
(i) show that g has a unique fixed point α in the interval $[0, 1]$,
(ii) starting with $x_0 = 0$ use the fixed point iterative formula to find the second approximation of α,
(iii) find an upper bound for the error in your approximation.

Q.3[4] Derive Newton's formula for approximating $\sqrt{3}$.

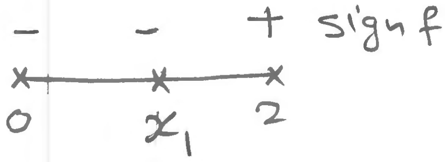
Q.4[5] The equation $x - ln(x+1) = 0$ has a repeated root $\alpha = 0$. Find the multiplicity of this root, then use a suitable iterative method to find the first approximation of this root starting with $x_0 = 0.5$.

Q.1. Let $f(x) = x^3 - 2x - 1$, x in $[0, 2]$.

* (i) f is cont. on $[0,2]$, it is a polynomial.

(ii) $f(0) = -1$ and $f(2) = 3$, hence $f(0) \cdot f(2) < 0$,

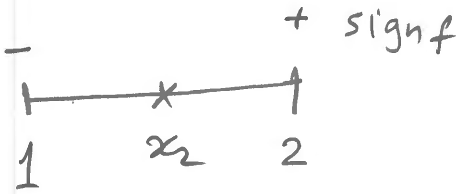
hence the equation $f(x) = 0$ has a solution α in $[0, 2]$.



* Put $a_1 = 0$, $b_1 = 2$,

then $x_1 = \dfrac{a_1 + b_1}{2} = \dfrac{0+2}{2} = 1$

$f(x_1) = f(1) = -2 \implies$ the solution is in $[1, 2]$



Therefore, put $a_2 = 1$, $b_2 = 2$

then $x_2 = \dfrac{a_2 + b_2}{2} = \dfrac{1+2}{2} = 1.5$.

* Since $|\alpha - x_n| \le \dfrac{b-a}{2^n}$, where $a = 0$, $b = 2$,

we need to find n such that

$$|\alpha - x_n| \le \frac{2-0}{2^n} \le 10^{-5}$$

or $\quad 2^{1-n} \le 10^{-5}$

$\implies (1-n)\ln 2 \le (-5)\ln 10$

$\implies \quad n - 1 \ge \dfrac{5 \ln 10}{\ln 2} \approx 16.6$

or $\quad n \ge 17.6$, we may take $n = 18$.

Q.2. (i) $g(x) = \frac{e^{-x}}{2}$ is cont. on $[0,1]$.

Since $g'(x) = -\frac{e^{-x}}{2} < 0$ for all x in $(0,1]$

$\Longrightarrow$ g is decreasing on $[0,1]$. $\Longrightarrow$

$0 < 0.184 \approx g(1) \leq g(x) \leq g(0) = \frac{1}{2} < 1$ for all x in $[0,1]$.

Hence $g(x)$ in $[0,1]$ for all x in $[0,1]$.

Also $\max_{0 \leq x \leq 1} |g'(x)| = \max_{0 \leq x \leq 1} \left| \frac{-e^{-x}}{2} \right| = \frac{1}{2} < 1$

Therefore g has a unique F.P. α in $[0,1]$.

(ii) Since $x_0 = 0$, and $x_{n+1} = g(x_n)$, $n = 0, 1, \ldots$

$\Longrightarrow$ $x_{n+1} = \frac{e^{-x_n}}{2}$, for $n = 0, 1, 2, \ldots$

For $n = 0$, we have: $x_1 = \frac{e^{-x_0}}{2} = \frac{e^{0}}{2} = \frac{1}{2} = 0.5$

" $n = 1$, " " : $x_2 = \frac{e^{-x_1}}{2} = \frac{e^{-0.5}}{2} \approx 0.3$.

(iii) Since $|\alpha - x_n| \leq \frac{k^n}{1-k} |x_1 - x_0|$, for $n = 1, 2, \ldots$

$\Longrightarrow |\alpha - x_2| \leq \frac{(\frac{1}{2})^2}{1 - (\frac{1}{2})} |0.5 - 0| = 0.25.$

Q.3. Let $x = \sqrt{3}$

$\Longrightarrow x^2 = 3$

$\Longrightarrow x^2 - 3 = 0$

Let $f(x) = x^2 - 3 \Longrightarrow f'(x) = 2x$

Hence Newton's formula for $\sqrt{3}$ is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^2 - 3}{2x_n}$$

$$= x_n - \frac{1}{2}\left[x_n - \frac{3}{x_n}\right]$$

$$= \frac{1}{2}\left[x_n + \frac{3}{x_n}\right], \quad n = 0, 1, \ldots$$

Q.4 Let $f(x) = x - \ln(x+1) \Rightarrow f(\alpha) = f(0) = 0$

$\Rightarrow \alpha = 0$ is a root.

Now, $f'(x) = 1 - \frac{1}{x+1} \Rightarrow f'(0) = 0$

and $f''(x) = \frac{1}{(x+1)^2} \Rightarrow f''(0) = \frac{1}{1} = 1 \neq 0$

$\Longrightarrow \alpha = 0$ is a root with multiplicity $m = 2$.

Applying the modified Newton's method we get:

$$x_{n+1} = x_n - \frac{m f(x_n)}{f'(x_n)}, \quad x_0 = 0.5$$

$$\Longrightarrow x_1 = x_0 - \frac{2 f(x_0)}{f'(x_0)} = 0.5 - \frac{2 f(0.5)}{f'(0.5)}$$

$$= 0.5 - \frac{2(0.5 - \ln 1.5)}{1 - \frac{2}{3}} \cong -0.067$$