

Question 1.[4,4]. a) Find and sketch the largest region of the xy -plane for which the initial value problem

$$\begin{cases} \sqrt{x^2 - 4} \cdot \frac{dy}{dx} = 1 + e^x \ln y \\ y(-3) = 4 \end{cases}$$

has a unique solution.

b) Solve the initial value problem

$$\begin{cases} \frac{dy}{dx} = x - xy - y + 1, & y \neq 1. \\ y(0) = 0. \end{cases}$$

Question 2.[4,4]. a) Find the general solution of the differential equation

$$\frac{dy}{dx} - (y - 2x)^2 = 3.$$

b) Solve the differential equation

$$\frac{x^2 y'}{y^4} - \frac{x}{y^3} = \sin x, \quad x > 0.$$

Question 3.[5]. Initially there were 80 gram of a radioactive material present. After 10 hours the mass decreases by 5%. If the rate of decay is proportional to the amount of the material at time t , then find the half life of this material.

Question 4.[4]. Show that the following differential equation is homogeneous and solve it.

$$(x^2 + y^2)dx + (x^2 - xy)dy = 0.$$

Complete solutions of First Mid-Term
Math. 204. First Sem. 1434-35 H

Question 1.

(a) IVP:
$$\begin{cases} \sqrt{x^2-4} \frac{dy}{dx} = 1 + e^x \ln y \\ y(-3) = 4 \end{cases}$$

$$\frac{dy}{dx} = y' = \frac{1}{\sqrt{x^2-4}} + \frac{e^x}{\sqrt{x^2-4}} \cdot \ln y \quad ; \quad y > 0 \text{ and } |x| > 2$$

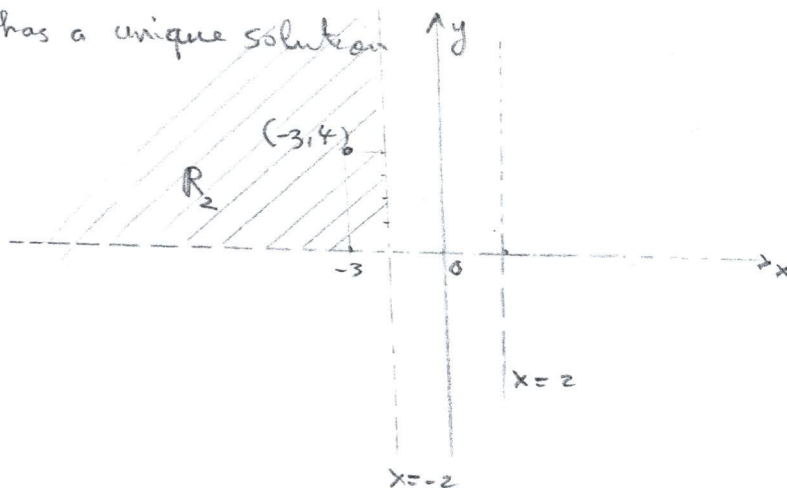
$$y' = f(x, y)$$

$$\frac{\partial f}{\partial y} = \frac{e^x}{\sqrt{x^2-4}} \cdot \frac{1}{y}$$

f and $\frac{\partial f}{\partial y}$ are continuous on $R = \{(x, y) : |x| > 2 \text{ and } y > 0\}$

$$R = \underbrace{\{(x, y) : x > 2 \text{ and } y > 0\}}_{R_1} \cup \underbrace{\{(x, y) : x < -2 \text{ and } y > 0\}}_{R_2}$$

As $(-3, 4) \in R_2$, then R_2 is the largest region for which the IVP has a unique solution



(b)
$$\frac{dy}{dx} = x - xy - y + 1 = x(1-y) + (1-y)$$

$$y' = (x+1)(1-y)$$

$$\frac{dy}{1-y} = (x+1)dx \Rightarrow -\ln|1-y| = \frac{1}{2}(x+1)^2 + C$$

At $x=0, y=0 \Rightarrow -\ln|1-0| = \frac{1}{2} + C = 0$

Hence $C = -\frac{1}{2}$, then the solution of the IVP is

$$\frac{1}{2}(x+1)^2 + \ln|1-y| = \frac{1}{2} \quad \text{or} \quad (x+1)^2 + \ln(1-y)^2 = 1$$

Question 2)

$$a) \quad \frac{dy}{dx} - (y-2x)^2 = 3$$

Let $u = y - 2x$, $y' = u' + 2$, $\Rightarrow u' + 2 - u^2 = 3$

$$u' = u^2 + 1 \Rightarrow \frac{du}{u^2 + 1} = dx \Rightarrow \tan^{-1}(u) = x + C.$$

Then the solution of the D.E. is

$$\boxed{\tan^{-1}(y-2x) - x = C}$$

b) $\frac{x^2 y'}{y^4} - \frac{x}{y^3} = \sin x$; $x > 0$ and $y \neq 0$. is Bernoulli's eq.

then

$$x^2 y' - xy = y^4 \sin x \quad \text{or} \quad y' - \frac{1}{x}y = \frac{\sin x}{x^2} y^4; \quad n=4.$$

hence $y' y^{-4} - \frac{1}{x} y^{-3} = \frac{\sin x}{x^2}$; $x > 0 \Rightarrow y \neq 0$

We put $w = y^{-3} \Rightarrow w' = -3y^{-4}y'$, so $y^{-4}y' = -\frac{w'}{3}$

and $-\frac{w'}{3} - \frac{1}{x}w = \frac{\sin x}{x^2}$

or

$$\boxed{w' + \frac{3}{x}w = -3 \frac{\sin x}{x^2}}$$
 which is a first order linear D.E.

$$\mu(x) = e^{\int \frac{3}{x} dx} = e^{\ln x^3} = x^3$$

then

$$\begin{aligned} w x^3 &= \int -3 \left(\frac{\sin x}{x^2} \right) x^3 dx \\ &= -3 \int x \sin x dx = -3(-x \cos x + \sin x) + C \end{aligned}$$

$$\text{or} \quad y^{-3} x^3 = 3x \cos x - 3 \sin x + C_1 \quad (C_1 = -3C)$$

Hence the solution of the D.E. is given by the family of curves

$$\boxed{x^3 = (3x \cos x - 3 \sin x + C_1) y^3}$$

Question ③

$$\frac{dm}{m} = k dt \Rightarrow \ln m = kt + c, \text{ here } m > 0$$

$$\text{or } m(t) = e^{kt} \cdot e^c$$

$$m(t) = c_1 e^{kt}; \quad c_1 = e^c > 0$$

$$\text{At } t=0, \quad m(0) = 80 \text{ grams} = m_0$$

$$\text{At } t=10 \quad m(10) = 80 - 80\left(\frac{5}{100}\right) = 80 - 4 = 76 \text{ grams.}$$

$$\text{then } m(0) = 80 = c_1, \quad m(t) = 80 e^{kt}$$

$$m(10) = 80 e^{10k} = 76 \quad \text{or} \quad \frac{76}{80} = e^{10k}$$

$$\text{hence } k = \frac{1}{10} \ln\left(\frac{76}{80}\right) \Rightarrow \boxed{k \approx -0.005}, \quad \boxed{m(t) = 80 e^{-0.005t}}$$

$$\text{Now if } m(t) = \frac{m_0}{2} = 40 \text{ grams, then}$$

$$40 = 80 e^{-0.005t} \Rightarrow \frac{1}{2} = e^{-0.005t}$$

$$\text{or } t = \frac{-\ln(2)}{-0.005} \Rightarrow \boxed{t \approx 138.63 \text{ hours}}$$

Question ④

$$(x^2 + y^2) dx + (x^2 - xy) dy = 0 \text{ is homogeneous}$$

We suppose $x > 0$, and put $u = \frac{y}{x}$, $y = xu \Rightarrow dy = x du + u dx$

$$\left[1 + \left(\frac{y}{x}\right)^2\right] dx + \left[1 - \left(\frac{y}{x}\right)\right] dy = 0$$

$$(1 + u^2) dx + (1 - u)(x du + u dx) = 0$$

$$(1 + u^2 + u - u^2) dx + (1 - u)x du = 0$$

$$(1 + u) dx + x(1 - u) du = 0$$

$$\frac{dx}{x} + \left(\frac{1-u}{1+u}\right) du = 0; \quad u \neq -1 \Rightarrow (y \neq -x)$$

$$\frac{dx}{x} + \left(-1 + \frac{2}{u+1}\right) du = 0 \Rightarrow \dots$$

$$\ln x + 2 \ln|u+1| - u = c \quad \text{or}$$

$$\boxed{\ln x + \ln\left(1 + \frac{y}{x}\right)^2 - \frac{y}{x} = c} \text{ is the solution of the D.E.}$$