King Saud University,
College of Sciences
Mathematical Department.

Mid-Term1 /S2/2012
Full Mark:40. Time 1H30mn
20/04/1433

Question 1[8]. Find and sketch the largest region of the $xy-$plane for which the initial value problem

$$\begin{cases} (x^2-9).\frac{dy}{dx} = \ln(1-y^2) \\ y(0) = \frac{1}{2} \end{cases}$$

has a unique solution.

Question 2[7+7]. **a)** Solve the initial value problem

$$\begin{cases} 3xy\frac{dy}{dx} = 4+6y^2, \quad x>0, \quad y \neq 0. \\ y(1) = 1 \end{cases}$$

b) Solve the differential equation

$$\frac{x}{y}\frac{dy}{dx} = 1 - xy, \quad x > 0.$$

Question 3[8]. Find the genral solution of the differential equation

$$x\frac{dy}{dx} - y = xe^{y/x} = 0, \quad x > 0.$$

Question 4[10]. A cake is removed from an oven having a temperature 350 ^0C. Five minutes later, its temperature is 200 ^0C. How long will it take for the cake to cool off to 150 ^0C if it is put in a room with constant temperature 100 ^0C.

Question 5[14]. A thermometer reading 70^0F is placed in an oven pre-heated to a constant temperature. Through a glass window in the oven door, an observer records that the termometer reads 110^0F after $\frac{1}{2}$ minute and 145^0F after 1 minute. How hot is the oven?.

Remark: Answer either question 4 or question 5 (**Do not answer both**)

Solutions:

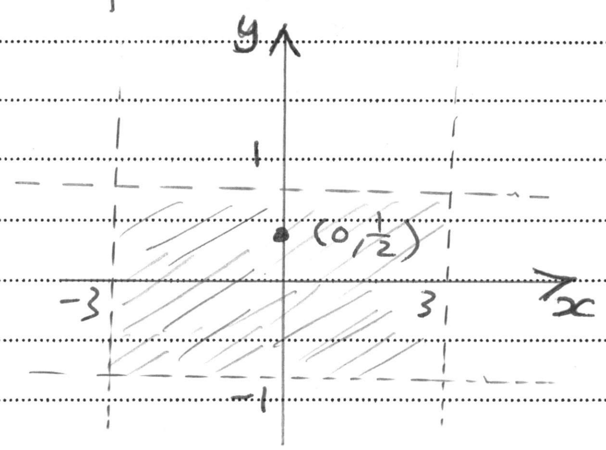
Q1. $f(x,y) = \dfrac{\ln(1-y^2)}{x^2-9}$, $x_0 = 0$, $y_0 = \frac{1}{2}$

$$\frac{\partial f}{\partial y} = \frac{-2y}{(x^2-9)(1-y^2)}$$

f is cont. if $x \neq \pm 3$ and $1-y^2 > 0$,
i.e. $x \neq \pm 3$ and $|y| < 1$,
i.e. $x \neq \pm 3$ and $-1 < y < 1$.

$\frac{\partial f}{\partial y}$ is cont. if $x \neq \pm 3$ and $y \neq \pm 1$.

$\therefore$ f and $\frac{\partial f}{\partial y}$ are cont. if $x \neq \pm 3$ and $-1<y<1$.



$\therefore$ The Largest region containing the point $(x_0, y_0) = (0, \frac{1}{2})$ is

$$R = \{(x,y) \in \mathbb{R}^2 : -3 < x < 3 \text{ and } -1 < y < 1\}.$$

Q.2. (a) The D.E is separable

$$\Rightarrow \frac{3y}{4+6y^2}\,dy = \frac{1}{x}\,dx,$$

$$\Rightarrow \int \frac{3y}{4+6y^2}\,dy = \int \frac{1}{x}\,dx$$

$$\frac{1}{4}\ln(4+6y^2) = \ln(x) + C.$$

Using the condition $y(1) = 1 \Rightarrow C = \frac{\ln 10}{4}$

$\therefore$ the solution is $\frac{1}{4}\ln(4+6y^2) = \ln(x) + \frac{\ln 10}{4}$.

Q.2. (b) Multiplying both sides by $\frac{y}{x}$ implies:

$$\frac{dy}{dx} = \frac{y}{x} - y^2 \quad \text{or} \quad \frac{dy}{dx} - \frac{1}{x} y = -y^2 \quad \text{...(1)}$$

which is Bernoulli's D.E.

Put $u = y^{1-2} = y^{-1} \Longrightarrow \frac{du}{dx} = -y^{-2} \frac{dy}{dx}$

or $\frac{dy}{dx} = -y^2 \frac{du}{dx}$, use this in (1) to get

$-y^2 \frac{du}{dx} - \frac{1}{x} y = -y^2$, Now $\div (-y^2)$

$\Longrightarrow \frac{du}{dx} + \frac{1}{x} y^{-1} = 1$, But $(y^{-1} = u)$

$\Longrightarrow \frac{du}{dx} + \frac{1}{x} u = 1$, (L.D.E in u)

$\therefore P(x) = \frac{1}{x}$ & $f(x) = 1$.

$\therefore$ I.F is $\mu(x) = e^{\int \frac{1}{x} dx} = x$

$\therefore$ the G.S. is $u = \frac{1}{\mu(x)} \left[c + \int \mu(x) f(x) dx \right]$

or $u = \frac{1}{x} \left[c + \int x \, dx \right]$

$\Longrightarrow y^{-1} = \frac{1}{x} \left(c + \frac{x^2}{2} \right)$.

Q.3. Divide both sides by x to get:

$\frac{dy}{dx} - \frac{y}{x} = e^{y/x}$ This is Homog. D.E

Put $y = ux \Longrightarrow \frac{dy}{dx} = u + x \frac{du}{dx}$

$\Longrightarrow u + x \frac{du}{dx} - u = e^{u}$

$\Longrightarrow x \frac{du}{dx} = e^{u}$, (Sep. D.E).

$\Rightarrow e^{-u}\,du = \frac{1}{x}\,dx \Rightarrow \int e^{-u}\,du = \int \frac{1}{x}\,dx$

$\Rightarrow -e^{-u} = \ln x + C \quad , \quad \left(u = \frac{y}{x} \right)$

$\Rightarrow -e^{-\frac{y}{x}} = \ln x + c$

Q.4. $T = 350\,C^{\circ}$ at $t = 0$ min's
$T = 200\,C^{\circ}$ at $t = 5$ min's
$T_m = 100\,C^{\circ}$
} Given.

Find t so that $T = 150\,C^{\circ}$

By Newton's law of cooling:

$$\frac{dT}{dt} = k(T - T_m) \Rightarrow T = T_m + Ce^{kt}$$

$\Rightarrow T = 100 + Ce^{kt}$,

at $t = 0$ we have $350 = 100 + C \Rightarrow C = 250$

$\therefore T = 100 + 250\,e^{kt}$,

at $t = 5$, $T = 200 \Rightarrow 200 = 100 + 250\,e^{5k}$

$\Rightarrow k = \frac{\ln(0.4)}{5} \Rightarrow T = 100 + 250\,e^{\left(\frac{\ln 0.4}{5}\right)t}$

$\therefore$ At $T = 150\,C^{\circ}$ we get:

$$150 = 100 + 250\,e^{\left(\frac{\ln 0.4}{5}\right)t}$$

$$\left(\frac{\ln 0.4}{5}\right)t = \ln(0.2)$$

or $t = \dfrac{5 \ln 0.2}{\ln 0.4}$ min.

Q.5 $\left.\begin{array}{ll} T = 70 F^\circ \text{ at } t = 0 & (1) \\ T = 110 F^\circ \text{ at } t = \frac{1}{2} \text{ min} & (2) \\ T = 145 F^\circ \text{ at } t = 1 \text{ min} & (3) \end{array}\right\}$ Given

$T_m = ?$

By Newton's Law of cooling we get,

$$T(t) = T_m + C e^{kt}$$

$$(1) \Longrightarrow 70 = T_m + C \quad (4)$$

$$(2) \Longrightarrow 110 = T_m + C e^{\frac{1}{2}k} \quad (5)$$

$$(3) \Longrightarrow 145 = T_m + C e^{k} \quad (6)$$

Now, $(5) - (4) \Longrightarrow 40 = C\left(e^{\frac{1}{2}k} - 1\right) \quad \cdots (7)$

$(6) - (4) \Longrightarrow 75 = C\left(e^{k} - 1\right) \quad \cdots (8)$

$$(7) \div (8) \Longrightarrow \frac{\overset{8}{\cancel{40}}}{\underset{15}{\cancel{75}}} = \frac{e^{\frac{1}{2}k} - 1}{e^{k} - 1}$$

$$\Longrightarrow 8e^{k} - 8 = 15 e^{\frac{1}{2}k} - 15$$

$$\Longrightarrow 8e^{k} - 15 e^{\frac{1}{2}k} + 7 = 0 \quad , \left[e^{k} = \left(e^{\frac{1}{2}k}\right)^2\right]$$

$$8\left(e^{\frac{1}{2}k}\right)^2 - 15 e^{\frac{1}{2}k} + 7 = 0$$

$$\left(8e^{\frac{1}{2}k} - 7\right)\left(e^{\frac{1}{2}k} - 1\right) = 0$$

$$\Longrightarrow e^{\frac{1}{2}k} = \frac{7}{8} \quad \text{or} \quad e^{\frac{1}{2}k} = 1$$

$$\Longrightarrow k = 2\ln\left(\frac{7}{8}\right) \quad \text{or} \quad \cancel{k = 0} \qquad (k \neq 0)$$

$$(7) \Longrightarrow c = 40 \Big/ \left(e^{\ln\frac{7}{8}} - 1\right) = -320$$

$$(4) \Longrightarrow T_m = 70 + 320 = 390 \ F^\circ$$