King Saud University,
College of Sciences
Mathematical Department.

Mid-Term1 /S2/2013
Full Mark:50. Time 1H30mn
2/5/1434

Question 1.[10,6] **a)** Find and sketch the largest region of the $xy-$plane for which the initial value problem

$$\begin{cases} (y^2+2y+1)\frac{dy}{dx} - \ln(4-x^2) = 0 \\ y(-1) = 1. \end{cases}$$

has a unique solution.

b) By using a suitable substitution, find the general solution of the differential equation

$$\frac{dy}{dx} - (x+y)^2 = 0.$$

Question 2.[8] Find the equation of a curve having a slope given by

$$\frac{dy}{dx} = xy^2 - y/x, \quad x > 0$$

and passing through the point (1.1).

Question 3[8,6]**.** Solve the following two differential equations

$$1)\ (\cos x.\ln x - y\tan x)dx - dy = 0, \quad x > 0,$$

$$2)\ (x + \sqrt{xy})dy - ydx = 0, \quad x > 0.$$

Question 4[12]

A cup of tea at temperature 80^0C is left in a room of temperature 20^0C. One minute later the temperature of tea is 70^0C. What will be its temperature after 3 minutes. At what time the tea cools down to 50^0C.

Solutions:

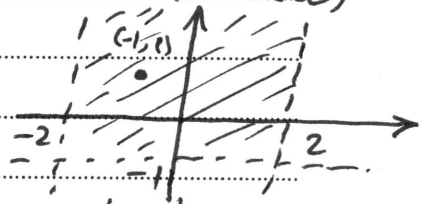
Q(1)(a) $f(x,y) = \dfrac{\ln(4-x^2)}{(y+1)^2}$, is continuous

on $R_1 = \{(x,y) \in \mathbb{R}^2 ; \ 4-x^2>0, \ y+1 \neq 0\}$

$= \{(x,y) \in \mathbb{R}^2 : |x|<2, \ y \neq -1\}$

$\dfrac{\partial f}{\partial y} = \dfrac{-2\ln(4-x^2)}{(y+1)^3}$, is continuous

on the same region R_1.



Since the point $(-1,1)$ lies in the region

$R = \{(x,y) \in \mathbb{R}^2 : |x|<2, \ y>-1\}$,

the given I.V.P has a unique solution at any point in the region R.

(b) Let $u = x+y \Longrightarrow \dfrac{du}{dx} = 1 + \dfrac{dy}{dx}$

or $\dfrac{dy}{dx} = \dfrac{du}{dx} - 1$, using the values of $x+y$ and $\dfrac{dy}{dx}$ in the D. Eq. implies

$\dfrac{du}{dx} - 1 = u^2 \Longrightarrow \dfrac{du}{1+u^2} = dx$

$\Longrightarrow \tan^{-1} u = x + c \Longrightarrow \tan^{-1}(x+y) = x + c$.

Q.(2) The D. Eq. can be written as:

$y' + \dfrac{y}{x} = x y^2$, $x>0$

which is a Bernoulli's D. Eq.

Let $u = y^{1-2} = y^{-1} \Longrightarrow \frac{du}{dx} = -y^{-2}\frac{dy}{dx}$.

Dividing the D. Eq. by y^2 imply:

$$y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = x$$

$$\Longrightarrow -\frac{du}{dx} + \frac{1}{x}u = x$$

$$\Longrightarrow \frac{du}{dx} - \frac{1}{x}u = -x$$, which is L. D. Eq. of 1^{st} order.

$\therefore$ I. F. is $\mu(x) = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$.

$$\therefore \quad u = \frac{1}{\mu(x)}\left[C + \int \mu(x).(-x)\,dx\right]$$

$$= x\left[C + \int - dx\right]$$

$$\Longrightarrow y^{-1} = x\left[C - x\right].$$

Since $y(1) = 1 \Longrightarrow C = 2$

$\therefore$ the curve is given by:

$$\frac{1}{y} = 2x - x^2.$$

Q.(3) (1) The D. Eq. can be written as

$$\frac{dy}{dx} = \cos x \ln x - y\tan x \quad , x>0$$

or

$$\frac{dy}{dx} + y\tan x = \cos x \ln x$$. Linear D. Eq.

$$\therefore \mu(x) = e^{\int \tan x\,dx} = e^{\ln(\sec x)} = \sec x = \frac{1}{\cos x}.$$

$\therefore\ y = \frac{1}{\mu(x)}\left[C + \int \mu(x) \cdot \cos x \ln x \, dx\right]$

$= \cos x \left[C + \int \ln x \, dx\right]$

$= \cos x \left[C + x\ln x - x\right]$

(2) $(x + \sqrt{xy})\,dy - y\,dx = 0, \quad x > 0.$

This is homogeneous D. Eq.

Let $x = Vy \Longrightarrow dx = V\,dy + y\,dV$

$\Longrightarrow (\cancel{Vy} + y\sqrt{V})\,dy - y(\cancel{V\,dy} + y\,dV) = 0$

$\Longrightarrow y\sqrt{V}\,dy = y^2\,dV$

or $\frac{dy}{y} = \frac{dV}{\sqrt{V}} \Longrightarrow \ln y = 2\sqrt{V} + C,$

or $\ln y = 2\sqrt{\frac{x}{y}} + C.$

Q.(4) $\frac{dT}{dt} = k(T - T_m)$

$T(0) = 80, \quad T(1) = 70, \quad T_m = 20.$

$\Longrightarrow T(t) = 20 + Ce^{kt}$

$T(0) = 80 \Longrightarrow C = 60 \Longrightarrow T(t) = 20 + 60e^{kt}$

$T(1) = 70 \Longrightarrow 70 = 20 + 60e^{k} \Longrightarrow k = -0.18$

Hence, $T(t) = 20 + 60e^{-0.18t}$

Now, at $t=3$ we have

$$T(3) = 20 + 60\, e^{(-0.18)(3)} = 54.96\ C^{\circ}$$

At $T = 50\ C^{\circ}$, we have

$$50 = 20 + 60\, e^{-0.18t} \Longrightarrow t = 3.85 \text{ minutes}$$