Kingdom of Saudi Arabia
Ministry of Education
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Department of mathematics

المملكة العربية السعودية
وزارة التعليم
جامعة الملك سعود
كلية العلوم
قسم الرياضيات

MIDTERM 1 EXAM

SEMESTER:	FIRST TERM	YEAR	2017/2016
	COURSE:	ACTU 466	
DATE:	15/03/2017	DURATION:	1H 30 MNS

	رقم الشعبة:		إسـم الطالب(ة):
	توقيع الطالب(ة):		الرقم الجامعي للطالب(ة):

INSTRUCTIONS

1) Please check that your exam contains **07 pages** total (including the first page!!), **08** questions.
2) **Answer all questions.**
3) No books, No notes and no phones are allowed.
4) A standard no programmable calculator is allowed.
5) Table for most used distributions is included.
6) Z-table is included.

Question	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	
Total score	**2**	**2**	**4**	**4**	**3**	**2**	**2**	**6**	
Score									

1) (2 marks) Consider the density function:

$$f(x) = \begin{cases} 4x^3 & if \quad 0 < x < 1 \\ 0 & if\ not \end{cases}$$

Compute the median and the 0.23 quantile.

$*$ $p = F(x) = x^4 \Rightarrow x_p = F^{-1}(p) = \sqrt[4]{p}$.

$\rightarrow$ median $= \sqrt[4]{0.5} =$

$\rightarrow$ 0.23 quantile $= \sqrt[4]{0.23} =$

2) (2 marks) Consider the following mass function:

x	0	1	2	3	4	5	6
$f(x)$	0.1	0.15	0.15	0.2	0.2	0.1	0.1
$F(x)$	0.1	0.25	0.40	0.60	0.80	0.90	1

Compute the median and the 80th percentile.

$*$ median $= 3$ since $P(X < 3) = 0.40 \le 0.50$
and $P(X \le) = 0.60 \geq 0.50$

$*$ $P_{80} = 4 \rightarrow$ $P(X < 4) = 0.60 \le 0.80$
$P(X \le 4) = 0.80 \geq 0.80$

3) (4 marks) Let $X \sim Exponential(\delta)$ and define $Y = e^X$.

a) Compute the cdf and the pdf of Y.

b) Determine whether the distribution of Y is light-tailed or heavy-tailed, using

- the method of moments.
- the hazard rate function method.

a) $F(y) = P(e^X \leq y) \leq F_X(\ln y) = 1 - e^{-\lambda \ln(y)}$

$= 1 - y^{-\lambda} \; ; \; y \geq 1.$

$f(y) = \lambda y^{-\lambda - 1} \; ; \; y \geq 1.$

b) $* \quad E(Y^m) = E e^{mX} < \infty$ for all $m \geq 0 \Rightarrow$

$* \quad h(y) = -\frac{S'(y)}{S(y)} = -\frac{-\lambda y^{-\lambda-1}}{y^{-\lambda}} = \frac{\lambda}{y} \searrow .$

$\rightarrow$

4) (4 marks) Let $X \sim N(0,1)$ with density function ϕ. f

a) Show that the tail-value-at-risk of X at 100p%, is given by:

$$TVaR_p(X) = \frac{1}{1-p} \phi(VaR_p(X)).$$ f

b) Compute $VaR_p(X)$ and $TVaR_p(X)$ for $p = 0.95$.

$e^{-x^2/2}$ $\sqrt{2\pi}$

a) $Tvar_p(X) = \frac{1}{1-p} \int_{\pi_p}^{\infty} x\,\phi(x)\,dx = \frac{1}{1-p} \left(-\phi(x)\right)\Big|_{\pi_p}^{\infty}$

$= \frac{1}{1-p} \phi(\pi_p) .$

b) $p = 0.95$ $\quad Var_p(X) = 1.645$

$TVaR_p(X) = \frac{1}{0.05} \phi(1.645) =$

5) (3 marks) Let X have a cdf $F(x) = 1 - (1+x)^{-\alpha}$ where $x > 0$ and $\alpha > 0$. Determine the pdf and the cdf of $Y = cX$ with $c > 0$.

- $F(y) = F_X(y/c) = 1 - (1 + \frac{y}{c})^{-\alpha}$.
- $f(y) = \frac{\alpha}{c} (1 + y/c)^{-\alpha-1}$.

6) (2 marks) Let $X \sim Uniform(0, a)$. Find the pdf of $Y = X^{\frac{1}{\tau}}$ with $\tau > 0$.

- $F(y) = F_X(y^{\tau})$, $0 \le y \le a^{1/\tau}$.
- $f(y) = \tau y^{\tau-1} f_X(y^{\tau})$ $0 \le y \le a^{1/\tau}$.

$= \tau y^{\tau-1}$;

7) (2 marks) Show that the binomial distribution with parameters m and p, belongs to the linear exponential family with respect to the parameter $\theta = p$.

$$f(x) = \binom{m}{x} p^{x} (1-p)^{m-x}$$

$$= \binom{m}{x} \left(\frac{p}{1-p}\right)^{x} (1-p)^{m}$$

$$= \frac{\binom{m}{x}}{(1-p)^{-m}} e^{\ln\left(\frac{p}{1-p}\right) x}$$

$p(x)$ $q(p)$ $r(p)$

8) (6 marks) Let $N \sim Poisson(\theta)$ and let N^T be the zero-truncated random variable associated to N. Compute:

a) $p^T(k)$ for $k = 0$ and $k \geq 1$.

b) the mean of N^T.

c) the mgf of N^T.

a) $p^T(0) = 0, \quad p^T(k) = \frac{1}{1-p(0)} p(k)$

$$= \frac{1}{1-e^{-\lambda}} e^{-\lambda} \frac{\lambda^k}{k!}, \quad k \geq 1.$$

b) $E(N^T) = \sum_{k=0}^{\infty} k p^T(k) = \sum_{k=1}^{\infty} k p^T(k)$

$$= c \sum_{k=1}^{\infty} k p(k) = c \sum_{k=0}^{\infty} k p(k)$$

$$= c \, E(N) = \frac{\lambda}{1-e^{-\lambda}}.$$

c) $m_{N^T}(t) = E e^{tN^T} = \sum_{k=0}^{\infty} e^{tk} p^T(k)$

$$= c \sum_{k=1}^{\infty} e^{tk} p(k) = c \left(\sum_{k=0}^{\infty} e^{kt} p(k) - p(0) \right)$$

$$= c \left(m_N(t) - p(0) \right)$$

$$= \frac{1}{1-e^{-\lambda}} \left(e^{\lambda(e^t-1)} - e^{-\lambda} \right).$$