Exercise 1. (2+2+1=6 marks) Suppose $\Lambda \sim \text{Exponential}(\beta)$ and $X_{|\Lambda=\lambda} \sim Poisson(\lambda)$.

a) Compute the mass function of X.

b) Deduce that X has a geometric distribution.

c) Compute the mean of X.

a) $$P(X=n) = \int_0^{\infty} P(X=n \mid \Lambda=\lambda)\, f_\Lambda(\lambda)\, d\lambda$$

$$= \int_0^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \beta e^{-\beta\lambda}\, d\lambda$$

$$= \frac{\beta}{n!} \int_0^{\infty} \lambda^n e^{-(1+\beta)\lambda}\, d\lambda$$

$(\Gamma(n+1) = n!)$

$$= \frac{\beta}{\cancel{n!}} \frac{\cancel{\Gamma(n+1)}}{(1+\beta)^{n+1}} = \frac{\beta}{(1+\beta)^{n+1}}.$$

b) $$P(X=n) = \frac{\beta}{1+\beta}\left(\frac{1}{1+\beta}\right)^n.$$

So $X \sim \text{Geometric}\left(\frac{\beta}{1+\beta}\right)$; $p = \frac{\beta}{1+\beta}$.

c) $$E(X) = \frac{1-p}{p} = \frac{\frac{1}{1+\beta}}{\frac{\beta}{1+\beta}} = \frac{1}{\beta}.$$

or $$E(X) = E\,E(X|\Lambda) = E(\Lambda) = \frac{1}{\beta}.$$

Exercise 2. (2+2+2+2+2=10 marks)

You are given:

(i) The annual size X of claims for a policyholder follows an exponential distribution with mean $1/\lambda$.

(ii) The prior distribution of Λ is $Gamma(5,2)$.

An insured is selected at random and observed to have a claim size of 5 during Year 1 and a claim size of 3 during Year 2.

a) Find the model distribution.

b) Find the joint distribution of (X_1, X_2) and Λ.

c) Find the marginal distribution of (X_1, X_2).

d) Find the posterior distribution of Λ.

e) Find the posterior mean of the claim size in Year 3.

a) $\underline{X} = (X_1, X_2)$.

$$f_{\underline{X}|\Lambda}(\underline{x}|\lambda) = f_{X_1|\Lambda}(x_1|\lambda) \cdot f_{X_2|\Lambda}(x_2|\lambda) = \lambda e^{-\lambda x_1} \cdot \lambda e^{-\lambda x_2}$$

$$= \lambda^2 e^{-8\lambda}.$$

b) $$f_{\underline{X},\Lambda}(\underline{x},\lambda) = \lambda^2 e^{-8\lambda} \frac{2^5}{\Gamma(5)} \lambda^4 e^{-2\lambda}$$

$$= \frac{2^5}{4!} \lambda^6 e^{-10\lambda}$$

c) $$f_{\underline{X}}(\underline{x}) = \int_0^\infty \frac{2^5}{4!} \lambda^6 e^{-10\lambda} \, d\lambda = \frac{2^5}{4!} \frac{6!}{10^7} =$$

d) $$\pi_\Lambda(\lambda) = \frac{10^7}{6!} \lambda^6 e^{-10\lambda} \sim Gamma(7, 10).$$

e) $$E(X_3 | \underline{X} = \underline{x}) = \int_0^\infty E(X_3 | \Lambda = \lambda) \pi_\Lambda(\lambda) \, d\lambda.$$

$$= \int_0^\infty \frac{1}{\lambda} \frac{10^7}{6!} \lambda^6 e^{-10\lambda} \, d\lambda.$$

$$= \int_0^\infty \frac{10^7}{6!} \lambda^5 e^{-10\lambda} \, d\lambda.$$

$$= \frac{10^7}{6!} \cdot \frac{5!}{10^6} = \frac{10}{6}.$$

Exercise 3. (2+2+2+2+2=10 marks)

The model for an annual total claim is given as follows:

(i) The number of claims N follows a negative binomial distribution with parameters $r = 2$ and $p = 0.4$.

(ii) Claim severity Y has the following distribution:

Claim Size	Probability
1	0.3
10	0.5
100	0.2

(iii) The number of claims is independent of the severity of claims.

We suppose that aggregate (total) losses are within 10% of expected aggregate (total) losses with 95% probability.

a) Compute the mean and variance of Y.
b) Compute the mean and variance of N.
c) Compute the mean and variance of the annual total claim $X = Y_1 + \cdots + Y_N$.
d) Determine the standard of full credibility, measured in terms of the number of observations.
e) Compute the credibility factor based on 56 observations.

a) $E(Y) = (1)(0.3) + (10)(0.5) + (100)(0.2) = 0.3 + 5 + 20 = 25.3$

$E(Y^2) = 1(0.3) + 100(0.5) + 10000(0.2) = 0.3 + 50 + 2000 = 2050.3$

$var(Y) = 2050.3 - (25.3)^2 = 1410.21$

b) $E(N) = \frac{2 \times 0.6}{0.4} = 3$, $var(N) = \frac{3}{0.4} = \frac{30}{4} = 7.5$.

c) $E(X) = E(N)\,E(Y) = 3 \times 25.3 = 75.9$

$var(X) = var(Y)\,E(N) + var(N)\,(E(Y))^2$

$= (1410.21) \times 3 + (7.5)(25.3)^2 = 9031.305$

d) $r = 0.1$, $p = 0.95 \rightarrow y_p = 1.96 \rightarrow \lambda_0 = \left(\frac{1.96}{0.1}\right)^2 = 384.16$

$n_F = \lambda_0 \frac{var(S)}{(E(S))^2} = 602.25$

e) $z = \min\left(1, \sqrt{\frac{560}{602.25}}\right) = 0.96$.