Some Remarks on *Derivatives Markets* (third edition, 2013)

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1. The parameter δ in the Black-Scholes formula

The Black-Scholes option-pricing formula is given in Chapter 12 of McDonald without proof. A rather unrealistic assumption on dividend payments is needed to derive formula (12.1). The assumption is given in the first sentence of the last paragraph on page 129: dividends are paid continuously at a rate that is proportional to the stock price. More precisely, for each share of the stock, the amount of dividends paid between time t and time $t+dt$ is assumed to be $S(t)\delta dt$, where $S(t)$ denotes the price of one share of the stock at time t, $t \geq 0$. (Note that the book also writes $S(t)$ as S_t. The symbol S in formula (12.1) is the value of the stock price at time 0, while S in other formulas, such as those in Appendix 12.B, may mean the value of the stock price at time t.) The *dividend yield*, δ, is a nonnegative constant.

The equation that defines the stock price process is (20.1), which can be rewritten as

$$dS(t) + S(t)\delta dt = S(t)\alpha dt + S(t)\sigma dZ(t).$$

The total return on a stock has two components: price change and dividend income. The two terms on the left-hand side of this equation are the two components of the total return on the stock between time t and time $t+dt$.

It is pointed out on page 130 that, if all dividends are re-invested immediately, then one share of the stock at time 0 will grow to $e^{\delta T}$ shares at time T, $T \geq 0$. Here is an alternative derivation. Let $n(t)$ denote the number of shares of the stock held at time t under the reinvestment policy. Thus, $n(0) = 1$. Instead of receiving dividends, the investor receives additional shares of the stock:

$$n(t)S(t)\delta dt = [n(t+dt) - n(t)]S(t).$$

Cancelling $S(t)$ yields

$$n(t)\delta dt = dn(t).$$

Hence, for $T \geq 0$,

$$\int_0^T \delta dt = \int_0^T \frac{dn(t)}{n(t)},$$

or

$$\delta T = \ln[n(T)] - \ln[n(0)] = \ln[n(T)],$$

which means

$$n(T) = e^{\delta T}.$$

Thus, if we want one share of the stock at time T, we can buy $e^{-\delta T}$ shares at time 0 and reinvest all dividends between time 0 and time T. This gives a meaning to the quantity $Se^{-\delta T}$ in formula (12.1). More generally, if we buy $e^{-\delta(T-t)}$ shares at time t, $t < T$, and reinvest all dividends between time t and time T, we get one share of the stock at time T. Hence,

$$e^{-\delta(T-t)}S(t) = F_{t,T}^P(S), \tag{1}$$

the time-t *prepaid forward price* for delivery of one share of the stock at time T. With $t = 0$, this is formula (5.4) on page 130.

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2. Variations of the Black-Scholes formula

2.1 Formula (12.5)

It is stated at the end of the first paragraph on page 354 that formula (12.5) "is interesting because the dividend yield and the interest rate do not appear explicitly; they are implicitly incorporated into the prepaid forward prices." Formula (12.5) can be used to price options on a stock that pays discrete dividends. In this case, the stock price process, $\{S(t)\}$, cannot be a geometric Brownian motion, because there must be a downward jump in the stock price immediately after each dividend is paid. In particular, the logarithm of the stock price cannot be a stochastic process with a constant standard deviation per unit time. So, what is the σ in formula (12.5)? It turns out that formula (12.5) follows from the assumption that the stochastic process of the prepaid forward price for delivery of one share of the stock at time T,

$$\{\, F_{t,T}^{P}(S)\,;\, 0 \le t \le T\},$$

is a geometric Brownian motion, with σ being the standard deviation per unit time of its natural logarithm.

If the stock pays dividends continuously at a rate proportional to its price, then formula (1) holds. In this case, the prepaid forward price process, $\{\, F_{t,T}^{P}(S)\,\}$, is a geometric Brownian motion if and only if the stock price process, $\{S(t)\}$, is a geometric Brownian motion; both stochastic processes have the same parameter σ. Formula (12.1) is a consequence of (12.5), but the converse is not true because (12.1) is not applicable for pricing options on stocks with discrete dividends.

2.2 Exchange options

It is pointed out in Section 14.6 that ordinary calls and puts are special cases of exchange options. As hinted at in the footnote on page 354, formula (12.5) can be generalized to price European exchange options. For $j = 1, 2$, let $S_j(t)$ denote the price of asset j at time t, $t \ge 0$. Consider a European exchange option whose payoff at time T is

$$\max(S_1(T) - S_2(T), 0). \tag{2}$$

If $\{\ln[F_{t,T}^{P}(S_1)]\,;\ 0 \le t \le T\}$ and $\{\ln[F_{t,T}^{P}(S_2)]\,;\ 0 \le t \le T\}$ are a pair of correlated Brownian motions, then it can be shown that the time-t price of the European exchange option is

$$F_{t,T}^{P}(S_1)\, N\!\left(\frac{\ln[F_{t,T}^{P}(S_1)/F_{t,T}^{P}(S_2)]}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t} \right)$$

$$- F_{t,T}^{P}(S_2)\, N\!\left(\frac{\ln[F_{t,T}^{P}(S_1)/F_{t,T}^{P}(S_2)]}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t} \right), \quad 0 \le t < T. \tag{3}$$

Here, $\mathrm{Var}(\ln[F_{t,T}^{P}(S_1)/F_{t,T}^{P}(S_2)]) = \sigma^2 t,\ 0 \le t \le T.$

To emphasize the simplicity of formula (3), let us write $v = \sigma\sqrt{T-t}$, and $F_j = F_{t,T}^{P}(S_j)$, for $j = 1, 2$. Then, (3) becomes

$$F_1 \times N\left(\frac{\ln[F_1/F_2]}{v} + \frac{v}{2}\right) \; - \; F_2 \times N\left(\frac{\ln[F_1/F_2]}{v} - \frac{v}{2}\right), \tag{4}$$

which is not a difficult formula to remember.

To see that formula (14.16) follows from formula (3), we note the assumptions for (14.16): for $j = 1, 2$, $\{S_j(t)\}$ is a geometric Brownian motion with volatility σ_j and dividends of amount $S_j(t)\delta_j dt$ being paid between time t and time $t+dt$; the correlation coefficient between the continuously compounded returns, $\ln[S_1(t)/S_1(0)]$ and $\ln[S_2(t)/S_2(0)]$, is ρ. Thus, similar to (1),

$$F_{t,T}^P(S_j) \; = \; e^{-\delta_j(T-t)} S_j(t), \qquad j = 1, 2, \tag{5}$$

and

$$
\begin{aligned}
\sigma^2 t \; &= \; \mathrm{Var}(\ln[F_{t,T}^P(S_1)/F_{t,T}^P(S_2)]) \\
&= \; \mathrm{Var}(\ln[S_1(t)/S_2(t)]) \qquad\qquad \text{because of (5)} \\
&= \; \mathrm{Var}(\ln[S_1(t)] - \ln[S_2(t)]) \\
&= \; \mathrm{Var}(\ln[S_1(t)]) + \mathrm{Var}(\ln[S_2(t)]) - 2\mathrm{Cov}(\ln[S_1(t)], \ln[S_2(t)]) \\
&= \; \sigma_1^2 t + \sigma_2^2 t - 2\rho\,\sigma_1\sigma_2\, t,
\end{aligned}
$$

which is equivalent to (14.17) on page 424.

2.3 All-or-nothing options

The exchange option payoff, given by (2), can decomposed as the difference of two all-or-nothing option payoffs,

$$\max(S_1(T) - S_2(T), 0) \; = \; S_1(T) \times I[S_1(T) > S_2(T)] - S_2(T) \times I[S_1(T) > S_2(T)].$$

Here, $I[.]$ denotes the indicator function, i.e., $I[A] = 1$ if the event A happens, and $I[A] = 0$ if A does not happen. (All-or-nothing options are discussed in Section 23.1.) It is not a surprise that the time-t price of the first all-or-nothing option payoff is the first term in (3), and the time-t price of the second payoff is the second term in (3).

It turns out that it is sufficient to know only one of these two formulas. Suppose that we know that the time-t price of the time-T payoff

$$S_1(T) \times I[S_1(T) > S_2(T)] \tag{6}$$

is

$$F_{t,T}^P(S_1) \; N\left(\frac{\ln[F_{t,T}^P(S_1)/F_{t,T}^P(S_2)]}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}\right).$$

Then, by symmetry, the time-t price of the time-T payoff

$$S_2(T) \times I[S_1(T) > S_2(T)] \tag{7}$$

is

$$F_{t,T}^P(S_2) \; N\left(\frac{\ln[F_{t,T}^P(S_2)/F_{t,T}^P(S_1)]}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}\right).$$

Because (7) can be rewritten as

$$S_2(T) \; - \; S_2(T) \times I[S_2(T) \geq S_1(T)],$$

its time-t price is

$$F_{t,T}^P(S_2) - F_{t,T}^P(S_2)N\left(\frac{\ln[F_{t,T}^P(S_2)/F_{t,T}^P(S_1)]}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}\right)$$

$$= F_{t,T}^P(S_2)\left[1 - N\left(\frac{\ln[F_{t,T}^P(S_2)/F_{t,T}^P(S_1)]}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}\right)\right]$$

$$= F_{t,T}^P(S_2)\ N\left(\frac{\ln[F_{t,T}^P(S_1)/F_{t,T}^P(S_2)]}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}\right),$$

which is indeed the second term in (3).

2.4 In terms of forward prices
Because

$$\frac{F_{t,T}^P(S_1)}{F_{t,T}^P(S_2)} = \frac{F_{t,T}(S_1)}{F_{t,T}(S_2)},$$

the formulas in the last two subsections can be expressed in terms of forward prices. For example, the time-t forward price for time-T delivery of (2) is

$$F_{t,T}(S_1)N\left(\frac{\ln[F_{t,T}(S_1)/F_{t,T}(S_2)]}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}\right)$$

$$- F_{t,T}(S_2)N\left(\frac{\ln[F_{t,T}(S_1)/F_{t,T}(S_2)]}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}\right), \qquad 0 \le t < T. \quad (8)$$

Note that if F_j in (4) denotes the forward price $F_{t,T}(S_j)$, instead of the prepaid forward price $F_{t,T}^P(S_j)$, $j = 1,\ 2$, then formula (4) is (8). In other words, if the F_1 and F_2 in (4) are forward prices, then (4) gives the forward price for (2); if F_1 and F_2 are prepaid forward prices, then (4) gives the prepaid forward price (i.e., price) for (2).

2.5 Black's formula for pricing options on zero-coupon bonds
With the exchange option formula (3), one can derive formula (25.54), which is Black's formula for pricing a European call option on a zero-coupon bond. For $t \le T$, consider

$$S_1(t) \;=\; P_t(t, T+s)$$

and

$$S_2(t) \;=\; K \times P_t(t, T).$$

Then,

$$F_{t,T}^P(S_1) \;=\; P_t(t, T+s)$$

and

$$F_{t,T}^P(S_2) \;=\; K \times P_t(t, T).$$

We make the assumption

$$\mathrm{Var}(\ln[\, P_t\,(t,\,T+s)/\,P_t\,(t,\,T)]) \;=\; \sigma^2 t, \qquad\qquad 0 \le t \le T.$$

By (3), the time-t price of the European call option with time-T payoff

$$\max[0,\ P_T\,(T,\,T+s) - K] \tag{25.52}$$

is

$$
P_t\,(t,\,T+s)\,N\!\left(\frac{\ln[P_t(t,T+s)/[KP_t(t,T)]]}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}\right)
$$
$$
-\ KP_t\,(t,\,T)\,N\!\left(\frac{\ln[P_t(t,T+s)/[KP_t(t,T)]]}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}\right), \tag{9}
$$

which is (25.54).

Following (25.53), we let $F_{t,T}\,[P(T+s)]$ denote the time-t forward price for time-T delivery of a zero-coupon bond that pays 1 at time $T+s$. Then (9) can be rewritten as

$$
P_t\,(t,\,T)\big[\, F_{t,T}\,[P(T+s)]\,N\!\left(\frac{\ln[F_{t,T}[P(T+s)]/K]}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}\right)
$$
$$
-\ KN\!\left(\frac{\ln[F_{t,T}[P(T+s)]/K]}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}\right)\big], \tag{10}
$$

which is formula (25.55).

We can also obtain (10) using (8) with $F_{t,T}\,(S_1) = F_{t,T}\,[P(T+s)]$ and $F_{t,T}\,(S_2) = K$. Since formula (8) gives the forward price, multiplication with $P_t\,(t,\,T)$ yields (10).

We note that there are five other expressions for the forward price $F_{t,T}\,[P(T+s)]$, namely,

$$F_{t,T,T+s}\,,\ P_t\,(T,\,T+s),\ P_t\,(t,\,T+s)/\,P_t\,(t,\,T),\ P(t,\,T+s)/P(t,\,T),\ F_{t,T}\,[P(T,\,T+s)].$$

For the first four expressions, see the second and third paragraphs on page 752. The fifth expression is used in the second edition of McDonald; it has appeared in Question 14 of the May 2009 MFE examination.

Also, both K_1 in footnote 8 on page 422 should be K_2.

3. The parameters δ and σ in the binomial model

The quantities δ and σ also appear in Chapters 10 and 11, both of which relate to binomial models. On page 295, δ is called the continuous dividend yield. On page 302, σ is called the annualized standard deviation of the continuously compounded returns on the stock. Because a binomial model is a discrete model, it seems strange that these "continuous-time" concepts are involved. A motivation for incorporating δ and σ in a binomial model is touched upon in Section 11.3. By letting the length of each time period, h, tend to zero (and the number of periods, n, tend to infinity), we can obtain a geometric Brownian motion model for stock price movements with the dividend yield δ and volatility σ. Note that McDonald has suggested three pairs of formulas for

$$u \;=\; e^{\alpha(h)+\sigma\sqrt{h}}$$

and

$$d \;=\; e^{\alpha(h)-\sigma\sqrt{h}}.$$

In (10.9), $\alpha(h) = (r - \delta)h$. In (11.13), $\alpha(h) = 0$, which means $u = 1/d$. In (11.14),

$\alpha(h) = (r - \delta - \frac{1}{2}\sigma^2)h$. In the limit as $h \to 0$, option prices derived in these three models are the same.

The usual model in McDonald is $\alpha(h) = (r - \delta)h$. A binomial tree so constructed is called a "forward tree" (page 303). In this case, the risk-neutral probabilities are

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(r-\delta)h} - e^{(r-\delta)h - \sigma\sqrt{h}}}{e^{(r-\delta)h + \sigma\sqrt{h}} - e^{(r-\delta)h - \sigma\sqrt{h}}} = \frac{1 - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}} = \frac{1}{1 + e^{\sigma\sqrt{h}}},$$

and

$$1 - p^* = \frac{e^{\sigma\sqrt{h}}}{1 + e^{\sigma\sqrt{h}}} = \frac{1}{1 + e^{-\sigma\sqrt{h}}}.$$

Because $\sigma > 0$, we have $p^* < \frac{1}{2} < 1 - p^*$, which may be viewed as a built-in bias in the model.

4. Greeks

Greeks are partial derivatives of an option price. The definitions given on page 356 are numerical approximations. We need to be careful about the units in which changes are measured. For example, it is stated on page 356 that "[t]heta (θ) measures the change in the option price when there is a decrease in the time to maturity of 1 day." However, the mathematical definition for theta is the partial derivative of the option price with respect to t. In the Black-Scholes option-pricing formula, the variable t is (usually) in years. Thus it is stated on page 379 that "[t]o obtain a per-day theta, divide by 365." If we measure theta per trading day instead of per calendar day, we divide by 252.

For call and put options, delta, rho and psi are very easy to derive. Start with the Black-Scholes option pricing formula (12.1) or (12.4). Differentiate with respect to S, r or δ, while pretending d_1 and d_2 are constant. (You can check these six partial derivatives with the formulas in Appendix 12.B.) To see this, denote formula (4) as $V(F_1, F_2)$. The partial derivative $\dfrac{\partial}{\partial F_1}V(F_1, F_2)$ is

$$N\left(\frac{\ln[F_1 / F_2]}{v} + \frac{v}{2}\right) + F_1 N'\left(\frac{\ln[F_1 / F_2]}{v} + \frac{v}{2}\right)\frac{1}{vF_1} - F_2 N'\left(\frac{\ln[F_1 / F_2]}{v} - \frac{v}{2}\right)\frac{1}{vF_1},$$

the last two terms of which cancel to zero because of the identity

$$e^{-(x+y)^2/2} = e^{-(x-y)^2/2} \times e^{-2xy}.$$

(The identity above also explains the last formula in footnote 16 on page 379.) Hence,

$$\frac{\partial}{\partial F_1}V(F_1, F_2) = N\left(\frac{\ln[F_1 / F_2]}{v} + \frac{v}{2}\right). \tag{11}$$

Similarly,

$$\frac{\partial}{\partial F_2}V(F_1, F_2) = -N\left(\frac{\ln[F_1 / F_2]}{v} - \frac{v}{2}\right). \tag{12}$$

With $F_j = e^{-\delta_j(T-t)}S_j$, it follows from the chain rule that

$$\frac{\partial}{\partial S_j}[\ ] = \frac{\partial}{\partial F_j}[\ ] \times \frac{\partial F_j}{\partial S_j} = \frac{\partial}{\partial F_j}[\ ] \times e^{-\delta_j(T-t)}$$

and

$$\frac{\partial}{\partial \delta_j}[\ ] = \frac{\partial}{\partial F_j}[\ ] \times \frac{\partial F_j}{\partial \delta_j} = \frac{\partial}{\partial F_j}[\ ] \times S_j e^{-\delta_j(T-t)} \times -(T-t).$$

Hence, for call and put options, delta, rho and psi can be derived by pretending that d_1 and d_2 are constant.

Applying (11) and (12) to (4) yields

$$V(F_1, F_2) = F_1 \times \frac{\partial}{\partial F_1} V(F_1, F_2) + F_2 \times \frac{\partial}{\partial F_2} V(F_1, F_2).$$

This may remind some students of the *Euler theorem for homogeneous functions* in multivariate calculus.

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5. Interest rates

In McDonald, interest rates are usually continuously compounded rates (forces of interest). Exceptions are Appendix 11.B, the nonannualized interest rate $R_t(T, T+s)$ in Section 25.1, and the "Black-Derman-Toy Model" in Section 25.4 and Appendix 25.A.

6. Market price of risk

The no-arbitrage argument in Section 20.5 is an important insight in finance. Let us rewrite (20.26), (20.27) and the equation in footnote 8 on page 618 as follows: for $j = 1, 2$,

$$\frac{dS_j}{S_j} = (\alpha_j - \delta_j)dt + \upsilon_j dZ,$$

where δ_j is the (not necessarily constant) dividend yield of asset j, and

$$|\upsilon_j| = \sigma_j.$$

Note that the price dynamics of both assets are driven by the same Brownian motion $\{Z(t)\}$, and that α_j, δ_j and υ_j can depend on t. In this more general setting, equation (20.28) takes the form

$$\frac{1}{\upsilon_1 S_1}(dS_1 + S_1\delta_1 dt) - \frac{1}{\upsilon_2 S_2}(dS_2 + S_2\delta_2 dt) + \left(\frac{1}{\upsilon_2} - \frac{1}{\upsilon_1}\right)rdt = \left(\frac{\alpha_1 - r}{\upsilon_1} - \frac{\alpha_2 - r}{\upsilon_2}\right)dt.$$

Therefore, to preclude arbitrage, we must have

$$\frac{\alpha_1 - r}{\upsilon_1} = \frac{\alpha_2 - r}{\upsilon_2}. \tag{13}$$

Many authors use the term "market price of risk" for the ratio $(\alpha_j - r)/\upsilon_j$. The Sharpe ratio is the same as the market price of risk if the denominator, υ_j, is positive.

Two important applications of the no-arbitrage condition (13) are

$$\frac{\alpha - r}{\sigma} = \frac{\alpha_{\text{option}} - r}{\text{sgn}(\Omega)\sigma_{\text{option}}}, \tag{14}$$

which is a corrected version of (21.21), and

$$\frac{\alpha(r(t),t,T_1)-r(t)}{-q(r(t),t,T_1)} = \frac{\alpha(r(t),t,T_2)-r(t)}{-q(r(t),t,T_2)},$$

which is equivalent to (25.24). Here, sgn(Ω) denotes the sign of Ω, the option's elasticity. In a Black-Scholes model, the market price of risk for every traded asset is the stock's Sharpe ratio, $(\alpha - r)/\sigma$, which does not depend on time or stock price. In the interest rate model in Section 25.2, the market price of risk at time t for every bond is $-\phi(r, t)$, where r is the value of the short-term interest rate at that time.

Besides (21.21), formulas (21.17), (21.18) and (21.19) also contain typographical errors. The expectation operator E in (21.17) and (21.18) should be written as E_t, because the expectation is conditional on information up to time t. The two terms on the left-hand side of (21.18) should be switched. In (21.19), sgn(Ω) is missing. The corrected (21.18) and (21.19) are:

$$\frac{dV}{V} - \frac{E_t(dV)}{V} = \frac{SV_S}{V}\sigma dZ \qquad (21.18)$$

$$= \text{sgn}(\Omega)\sigma_{\text{option}}dZ \qquad (21.19)$$

Also, the unnumbered equation between (21.29) and (21.30) on page 637 should be $E_t^*(dS) = S \times (r-\delta)dt.$

It follows from (14) or from (21.22) (or from (12.10)) that

$$\alpha_{\text{option}} - r = \Omega \times (\alpha - r).$$

For a put option, delta at time t is $-e^{-\delta(T-t)}N(-d_1)$, which is negative. Hence, sgn(Ω) is negative. If $\alpha > r$ (which seems to be a natural assumption), we have

$$\alpha_{\text{put option}} < r.$$

This means that if one wants to value a put option using the actual probability measure, then for discounting, one needs to use an interest rate that is lower than the risk-free interest rate or even a negative interest rate. This may seem strange to some actuaries.

7. Risk-neutral valuation

It is a rather amazing result that, under the assumption of no arbitrage (and the securities market being frictionless), the price of each derivative security is the expected present value of its payoffs. Of course, actuaries have been calculating expected present values for over two centuries. However, the expected present value in finance is different in two respects: (i) the discounting is calculated using the risk-free interest rate; (ii) the expectation is taken with respect to a so-called risk-neutral probability measure.

For an option or a claim that pays

$$V(S(T),T)$$

at time T, its price at time t, $t \leq T$, is

$$E_t^*[e^{-r(T-t)}V(S(T),T)]. \qquad (15)$$

With $t = 0$, (15) is the right-hand side of (19.1) and (20.31). The asterisk in (15) signifies that the expectation is taken with respect to the risk-neutral probability measure, and the subscript t means that the expectation is taken conditional on information up to time t, in particular, the stock price at time t, $S(t)$. Note that the right-hand side of the first displayed equation on page 639 is the integral form of (15) for the time-T payoff

$$V(S(T),T) = \max[0, S(T) - K].$$

If the risk-free interest rate is not constant, then the discount factor $e^{-r(T-t)}$ is replaced by $e^{-\int_t^T r(s)\,ds}$; see formula (25.9). For example, consider a zero-coupon bond maturing for 1 at time T. Then, its price at time t, $t \le T$, is

$$\mathrm{E}_t^*[e^{-\int_t^T r(s)\,ds} 1],$$

which is the right-hand side of (25.10). (Note that both E^* on page 759 should be changed to E_t^*.)

The notion of prepaid forward prices does not seem to be in other finance textbooks. In any case, (15) gives you a formula for determining the prepaid forward price. That is, the time-t *prepaid forward price* for delivery of $V(S(T),T)$ at time T is (15). As an application, let us derive the relationship of *put-call parity*. It follows from the identity

$$\max[x, 0] \; - \; \max[-x, 0] \; = \; x$$

that

$$\max[S(T) - K, 0] \; - \; \max[K - S(T), 0] \; = \; S(T) \; - \; K.$$

Discounting each term with the risk-free interest rate and taking expectation with respect to the risk-neutral probability measure, we obtain the identity

$$C(K, T) \; - \; P(K, T) \; = \; F_{0,T}^P(S) \; - \; \mathrm{PV}_{0,T}(K),$$

which is (3.1).

Section 20.6, entitled "Risk-neutral valuation," is an important section, but it has several typographical errors. In (20.29), $\tilde{Z}(t)$ should be $Z(t)$. In (20.30), $Z(t)$ should be $\tilde{Z}(t)$. Note that (20.29) and (20.30) are the same equation:

$$\frac{dS(t)}{S(t)} = (\alpha - \delta)dt + \sigma dZ(t)$$

$$= (r - \delta)dt + \sigma\left[\frac{\alpha - r}{\sigma}dt + dZ(t)\right]$$

$$= (r - \delta)dt + \sigma d\tilde{Z}(t),$$

because

$$\tilde{Z}(t) = \frac{\alpha - r}{\sigma}t + Z(t). \tag{16}$$

Under the risk-neutral probability measure, the stochastic process $\{\tilde{Z}(t)\}$ is a standard Brownian motion. Note that the fraction in (16) is the market price of risk. (As pointed out in the last section, the market price of risk in Section 25.2 is $-\phi(r, t)$. Analogous to (16), $\{\tilde{Z}(t)\}$ in Section 25.2 is given by

$$d\tilde{Z}(t) = -\phi(r(t),t)dt + dZ(t).$$

You can check this by comparing (25.26) with (25.13).)

Every Z on page 619 and 620 should be changed to $\tilde{Z}$. The expectation operator E in the middle of page 619 should be written as E^*, or more precisely, as E_0^*. The word "expected" in the third sentence of the second paragraph on page 619 should be deleted.

To obtain the first displayed, but un-numbered, formula on page 619 (with $Z(T)$ changed to $\tilde{Z}(T)$), we can start with (20.22). By applying (16) to (20.22), we have $S(t) = S(0)e^{(r-\delta-\frac{1}{2}\sigma^2)t+\sigma\tilde{Z}(t)}$. Then, replace t by T, and raise both sides to power a.

Here is an interesting result that follows from (20.32) in Proposition 20.3. If a is a number such that

$$-r + a(r - \delta) + \tfrac{1}{2}a(a - 1)\sigma^2 \;=\; 0, \tag{17}$$

then

$$F_{t,T}^P(S^a) = [S(t)]^a. \tag{18}$$

Equation (17) is a quadratic equation in a. Thus, there are two values of a for which (18) holds. They are given in Exercise 21.2 with $\gamma = 0$. (They are also given by h_1 and h_2 on page 373 and page 695. h_1 and h_2 have appeared in Question 16 of the May 2007 MFE examination, but Section 12.6 is not on the MFE syllabus anymore.)

8. Further references

This section is NOT part of the MFE syllabus.

A popular textbook that treats topics similar to those in McDonald is John Hull, *Options, Futures, and Other Derivatives*, Prentice-Hall. On the Internet, you can obtain an early edition of this book rather cheaply. For a deeper understanding of the material in the MFE syllabus, here are two of many fine books:

- Martin Baxter and Andrew Rennie (1996) *Financial Calculus: An Introduction to Derivative Pricing*, Cambridge University Press.
- Tomas Björk (2009) *Arbitrage Theory in Continuous Time*, 3rd ed., Oxford University Press.

The following survey paper can be downloaded via JSTOR

- G. Malliaris (1983) "Ito's calculus in financial decision making," *SIAM Review* 25 no. 4, 481-496.

There are now numerous lecture notes and videos available on the Internet. The University of Chicago has a Master of Science in Financial Mathematics program. Introductory lectures can be found in
http://www-finmath.uchicago.edu/students/sept_review.shtml
In particular, I recommend viewing Professor Robert Fefferman's lectures on probability. Also, you may want to read the following 2-page article about Professor Kyosi Itô, whose "lemma" is the theme of Section 20.4.
http://www.ams.org/notices/200706/tx070600744p.pdf
Note the sentence: "Indeed, one can argue that most of applied mathematics traditionally comes down to changes of variables and Taylor-type expansions."

Finally, there is a time-honored tool in actuarial science called the "Esscher transform." It turns out to be a very efficient method for pricing financial options, especially those involving multiple assets.
http://www.soa.org/library/research/transactions-of-society-of-actuaries/1990-95/1994/january/tsa94v467.aspx