SOCIETY OF ACTUARIES

EXAM FM FINANCIAL MATHEMATICS

EXAM FM SAMPLE SOLUTIONS

Interest Theory

This page indicates changes made to Study Note FM-09-05.

January 14, 2014:

Questions and solutions 58–60 were added.

June, 2014

Question 58 was moved to the Derivatives Markets set of sample questions.

Questions 61-73 were added.

Many of the questions were re-worded to conform to the current style of question writing. The substance was not changed.

December, 2014: Questions 74-76 were added.

January, 2015: Questions 77-93 were added.

May, 2015: Questions 94-133 were added.

Some of the questions in this study note are taken from past SOA examinations.

> These questions are representative of the types of questions that might be asked of candidates sitting for the Financial Mathematics (FM) Exam. These questions are intended to represent the depth of understanding required of candidates. The distribution of questions by topic is not intended to represent the distribution of questions on future exams.

The following model solutions are presented for educational purposes. Alternative methods of solution are, of course, acceptable.

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1. Solution: C

Given the same principal invested for the same period of time yields the same accumulated value, the two measures of interest $i^{(2)} = 0.04$ and δ must be equivalent, which means: $\left(1+\frac{i^{(2)}}{2}\right)^2 = e^{\delta}$ over a one-year period. Thus,

$$e^{\delta} = \left(1+\frac{i^{(2)}}{2}\right)^2 = 1.02^2 = 1.0404$$

$$\delta = \ln(1.0404) = 0.0396.$$

2. Solution: E

From basic principles, the accumulated values after 20 and 40 years are

$$100[(1+i)^{20} + (1+i)^{16} + \cdots + (1+i)^4] = 100\frac{(1+i)^4 - (1+i)^{24}}{1-(1+i)^4}$$

$$100[(1+i)^{40} + (1+i)^{36} + \cdots + (1+i)^4] = 100\frac{(1+i)^4 - (1+i)^{44}}{1-(1+i)^4}.$$

The ratio is 5, and thus (setting $x = (1+i)^4$)

$$5 = \frac{(1+i)^4 - (1+i)^{44}}{(1+i)^4 - (1+i)^{24}} = \frac{x - x^{11}}{x - x^6}$$

$$5x - 5x^6 = x - x^{11}$$

$$5 - 5x^5 = 1 - x^{10}$$

$$x^{10} - 5x^5 + 4 = 0$$

$$(x^5 - 1)(x^5 - 4) = 0.$$

Only the second root gives a positive solution. Thus

$$x^5 = 4$$

$$x = 1.31951$$

$$X = 100\frac{1.31951 - 1.31951^{11}}{1 - 1.31951} = 6195.$$

Annuity symbols can also be used. Using the annual interest rate, the equation is

$$100\frac{s_{\overline{40}|}}{a_{\overline{4}|}} = 5(100)\frac{s_{\overline{20}|}}{a_{\overline{4}|}}$$

$$\frac{(1+i)^{40}-1}{i} = 5\frac{(1+i)^{20}-1}{i}$$

$$(1+i)^{40} - 5(1+i)^{20} + 4 = 0$$

$$(1+i)^{20} = 4$$

and the solution proceeds as above.

3. Solution: C

Eric's (compound) interest in the last 6 months of the 8th year is $100\left(1+\frac{i}{2}\right)^{15}\frac{i}{2}$.

Mike's (simple) interest for the same period is $200\frac{i}{2}$.

Thus,

$$100\left(1+\frac{i}{2}\right)^{15}\frac{i}{2} = 200\frac{i}{2}$$

$$\left(1+\frac{i}{2}\right)^{15} = 2$$

$$1+\frac{i}{2} = 1.047294$$

$$i = 0.09459 = 9.46\%.$$

4. Solution: A

The periodic interest is 0.10(10,000) = 1000. Thus, deposits into the sinking fund are 1627.45-1000 = 627.45.

Then, the amount in sinking fund at end of 10 years is $627.45s_{\overline{10}|0.14} = 12{,}133$. After repaying the loan, the fund has 2,133, which rounds to 2,130.

5. Solution: E

The beginning balance combined with deposits and withdrawals is 75 + 12(10) – 5 – 25 – 80 – 35 = 50. The ending balance of 60 implies 10 in interest was earned.

The denominator is the average fund exposed to earning interest. One way to calculate it is to weight each deposit or withdrawal by the remaining time:

$$75(1)+10\left(\frac{11}{12}+\frac{10}{12}+\cdots+\frac{0}{12}\right)-5\frac{10}{12}-25\frac{6}{12}-80\frac{5}{24}-35\frac{2}{12}=90.833.$$

The rate of return is 10/90.833 = 0.11009 = 11.0%.

6. Solution: C

$$\begin{aligned}77.1&=v\left(Ia\right)_{\overline{n}|}+\frac{nv^{n+1}}{i}\\&=v\left[\frac{\ddot{a}_{\overline{n}|}-nv^{n}}{i}\right]+\frac{nv^{n+1}}{i}\\&=\frac{a_{\overline{n}|}}{i}-\frac{nv^{n+1}}{i}+\frac{nv^{n+1}}{i}\\&=\frac{a_{\overline{n}|}}{i}=\frac{1-v^{n}}{i^{2}}=\frac{1-v^{n}}{0.011025}\end{aligned}$$

$$0.85003=1-v^{n}$$

$$1.105^{-n}=0.14997$$

$$n=-\frac{\ln(0.14997)}{\ln(1.105)}=19.$$

To obtain the present value without remembering the formula for an increasing annuity, consider the payments as a perpetuity of 1 starting at time 2, a perpetuity of 1 starting at time 3, up to a perpetuity of 1 starting at time n + 1. The present value one period before the start of each perpetuity is 1/i. The total present value is $(1/i)(v+v^{2}+\cdots+v^{n})=(1/i)a_{\overline{n}|}$.

7. Solution: C

The interest earned is a decreasing annuity of 6, 5.4, etc. Combined with the annual deposits of 100, the accumulated value in fund Y is

$$6(Ds)_{\overline{10}|0.09} + 100s_{\overline{10}|0.09}$$
$$= 6\left(\frac{10(1.09)^{10} - s_{\overline{10}|0.09}}{0.09}\right) + 100(15.19293)$$
$$= 565.38 + 1519.29$$
$$= 2084.67.$$

8. Deleted

9. Solution: D

For the first 10 years, each payment equals 150% of interest due. The lender charges 10%, therefore 5% of the principal outstanding will be used to reduce the principal.

At the end of 10 years, the amount outstanding is $1000(1-0.05)^{10} = 598.74$.

Thus, the equation of value for the last 10 years using a comparison date of the end of year 10 is

$$598.74 = Xa_{\overline{10}|10\%} = 6.1446X$$
$$X = 97.44.$$

10. Solution: B

The book value at time 6 is the present value of future payments:

$$BV_6 = 10,000v^4 + 800a_{\overline{4}|\,0.06} = 7920.94 + 2772.08 = 10,693.$$

The interest portion is 10,693(0.06) = 641.58.

11. Solution: A

The value of the perpetuity after the fifth payment is 100/0.08 = 1250. The equation to solve is:

$$1250 = X(v + 1.08v^2 + \cdots + 1.08^{24}v^{25})$$
$$= X(v + v + \cdots + v) = X(25)/1.08$$
$$X = 50(1.08) = 54.$$

12. Solution: C

Equation of value at end of 30 years:

$$10(1-d/4)^{-40}(1.03)^{40}+20(1.03)^{30}=100$$

$$10(1-d/4)^{-40}=[100-20(1.03)^{30}]/1.03^{40}=15.7738$$

$$1-d/4=1.57738^{-1/40}=0.98867$$

$$d=4(1-0.98867)=0.0453=4.53\%.$$

13. Solution: E

The accumulation function is $a(t)=\exp\left[\int_0^t (s^2/100)ds\right]=\exp(t^3/300)$.

The accumulated value of 100 at time 3 is $100\exp(3^3/300)=109.41743$.

The amount of interest earned from time 3 to time 6 equals the accumulated value at time 6 minus the accumulated value at time 3. Thus

$$\left(109.41743+X\right)[a(6)/a(3)-1]=X$$

$$(109.41743+X)(2.0544332/1.0941743-1)=X$$

$$(109.41743+X)0.877613=X$$

$$96.026159=0.122387X$$

$$X=784.61.$$

14. Solution: A

$$167.50=10a_{\overline{5}|9.2\%}+10(1.092)^{-5}\sum_{t=1}^{\infty}\left[\frac{(1+k)}{1.092}\right]^t$$

$$167.50=38.86955+6.44001\frac{(1+k)/1.092}{1-(1+k)/1.092}$$

$$(167.50-38.86955)[1-(1+k)/1.092]=6.44001(1+k)/1.092$$

$$128.63045=135.07046(1+k)/1.092$$

$$1+k=1.0399$$

$$k=0.0399\Rightarrow K=3.99\%.$$

15. Solution: B

$\text{Option 1: } 2000 = Pa_{\overline{10}|0.0807}$

$P = 299 \Rightarrow \text{Total payments} = 2990$

$\text{Option 2: Interest needs to be } 2990 - 2000 = 990$

$990 = i[2000 + 1800 + 1600 + \cdots + 200]$

$= 11,000i$

$i = 0.09 = 9.00\%$

16. Solution: B

Monthly payment at time t is $1000(0.98)^{t-1}$.

Because the loan amount is unknown, the outstanding balance must be calculated prospectively. The value at time 40 months is the present value of payments from time 41 to time 60:

$$
\begin{aligned}
OB_{40} &= 1000[0.98^{40}v^1 + \cdots + 0.98^{59}v^{20}] \\
&= 1000\frac{0.98^{40}v^1 - 0.98^{60}v^{21}}{1 - 0.98v}, \; v = 1/(1.0075) \\
&= 1000\frac{0.44238 - 0.25434}{1 - 0.97270} = 6888.
\end{aligned}
$$

17. Solution: C

The equation of value is

$98S_{\overline{3n}|} + 98S_{\overline{2n}|} = 8000$

$\frac{(1+i)^{3n} - 1}{i} + \frac{(1+i)^{2n} - 1}{i} = 81.63$

$(1+i)^n = 2$

$\frac{8-1}{i} + \frac{4-1}{i} = 81.63$

$\frac{10}{i} = 81.63$

$i = 12.25\%$

18. Solution: B

Convert 9% convertible quarterly to an effective rate of *j* per month:

$$(1+j)^3 = \left(1+\frac{0.09}{4}\right) \text{ or } j = 0.00744.$$

Then

$$2(Ia)_{\overline{60}|0.00744} = 2\frac{\ddot{a}_{\overline{60}|0.00744} - 60v^{60}}{0.00744} = 2\frac{48.6136-38.4592}{0.00744} = 2729.7.$$

19. Solution: C

For Account K, the amount of interest earned is 125 – 100 – 2X + X = 25 – X.

The average amount exposed to earning interest is 100 – (1/2)X + (1/4)2X = 100. Then

$$i = \frac{25-X}{100}.$$

For Account L, examine only intervals separated by deposits or withdrawals. Determine the interest for the year by multiplying the ratios of ending balance to beginning balance. Then

$$i = \frac{125}{100}\frac{105.8}{125-X} - 1.$$

Setting the two equations equal to each other and solving for X,

$$\frac{25-X}{100} = \frac{13,225}{100(125-X)} - 1$$

$$(25-X)(125-X) = 13,225 - 100(125-X)$$

$$3,125 - 150X + X^2 = 13,225 - 12,500 + 100X$$

$$X^2 - 250X + 2,400 = 0$$

$$X = 10.$$

Then i = (25 – 10)/100 = 0.15 = 15%.

20. Solution: A

Equating present values:

$$100 + 200v^n + 300v^{3n} = 600v^{10}$$
$$100 + 200(0.76) + 300(0.76)^2 = 600v^{10}$$
$$425.28 = 600v^{10}$$
$$0.7088 = v^{10}$$
$$0.96617 = v$$
$$1.03501 = 1 + i$$
$$i = 0.035 = 3.5\%.$$

21. Solution: A

The accumulation function is:

$$a(t) = e^{\int_0^t \frac{1}{8+r}dr} = e^{\ln(8+r)|_0^t} = \frac{8+t}{8}.$$

Using the equation of value at end of 10 years:

$$20,000 = \int_0^{10} (8k + tk) \frac{a(10)}{a(t)} dt = k\int_0^{10} (8+t) \frac{18/8}{(8+t)/8} dt = k\int_0^{10} 18dt$$
$$= 180k \Rightarrow k = \frac{20,000}{180} = 111.$$

22. Solution: D

Let C be the redemption value and $v = 1/(1+i)$. Then

$$X = 1000ra_{\overline{2n}|i} + Cv^{2n}$$
$$= 1000r\frac{1-v^{2n}}{i} + 381.50$$
$$= 1000(1.03125)(1-0.5889^2) + 381.50$$
$$= 1055.11.$$

23. Solution: D

Equate net present values:

$$-4000+2000v+4000v^2=2000+4000v-Xv^2$$
$$\frac{4000+X}{1.21}=6000+\frac{2000}{1.1}$$
$$X=5460.$$

24. Solution: E

For the amortization method, the payment is determined by

$$20,000=Xa_{\overline{20|}0.065}=11.0185,\quad X=1815.13.$$

For the sinking fund method, interest is 0.08(2000) = 1600 and total payment is given as *X*, the same as for the amortization method. Thus the sinking fund deposit = *X* – 1600 = 1815.13 – 1600 = 215.13.

The sinking fund, at rate *j*, must accumulate to 20000 in 20 years. Thus, $215.13s_{\overline{20|}j}=20,000$, which yields (using calculator) *j* = 14.18%.

25. Solution: D

The present value of the perpetuity = *X*/*i*. Let *B* be the present value of Brian's payments.

$$B=Xa_{\overline{n|}}=0.4\frac{X}{i}$$
$$a_{\overline{n|}}=\frac{0.4}{i}\Rightarrow 0.4=1-v^n\Rightarrow v^n=0.6$$
$$K=v^{2n}\frac{X}{i}$$
$$K=0.36\frac{X}{i},$$

Thus the charity's share is 36% of the perpetuity's present value.

26. Solution: D

The given information yields the following amounts of interest paid:

$$\text{Seth} = 5000\left(\left(1+\frac{0.12}{2}\right)^{10}-1\right) = 8954.24-5000 = 3954.24$$

$$\text{Janice} = 5000(0.06)(10) = 3000.00$$

$$\text{Lori} = P(10)-5000 = 1793.40 \text{ where } P = \frac{5000}{a_{\overline{10}|6\%}} = 679.35$$

The sum is 8747.64.

27. Solution: E

For Bruce, $X = 100[(1+i)^{11}-(1+i)^{10}] = 100(1+\text{i})^{10}i$. Similarly, for Robbie, $X = 50(1+i)^{16}i$ .Dividing the second equation by the first gives $1 = 0.5(1+i)^6$ which implies $i = 2^{1/6}-1 = 0.122462$. Thus $X = 100(1.122462)^{10}(0.122462) = 38.879$.

28. Solution: D

Year t interest is $ia_{\overline{n-t+1}|i} = 1-v^{n-t+1}$.

Year t+1 principal repaid is $1-(1-v^{n-t}) = v^{n-t}$.

$$X = 1-v^{n-t+1}+v^{n-t} = 1+v^{n-t}(1-v) = 1+v^{n-t}d.$$

29. Solution: B

For the first perpetuity,

$$32 = 10(v^3+v^6+\cdots) = 10v^3/(1-v^3)$$

$$32-32v^3 = 10v^3$$

$$v^3 = 32/42.$$

For the second perpetuity,

$$X = v^{1/3}+v^{2/3}+\cdots = v^{1/3}/(1-v^{1/3}) = (32/42)^{1/9}/[1-(32/42)^{1/9}] = 32.599.$$

30. Solution: D

Under either scenario, the company will have 822,703(0.05) = 41,135 to invest at the end of each of the four years. Under Scenario A these payments will be invested at 4.5% and accumulate to $41{,}135 s_{\overline{4}|0.045} = 41{,}135(4.2782) = 175{,}984.$ Adding the maturity value produces 998,687 for a loss of 1,313. Note that only answer D has this value.

The Scenario B calculation is
$41{,}135 s_{\overline{4}|0.055} = 41{,}135(4.3423) = 178{,}621 + 822{,}703 - 1{,}000{,}000 = 1{,}324.$

31. Solution: D.

The present value is

$$5000[1.07v + 1.07^2 v^2 + \cdots + 1.07^{20} v^{20}]$$

$$= 5000 \frac{1.07v - 1.07^{21} v^{21}}{1 - 1.07v} = 5000 \frac{1.01905 - 1.48622}{1 - 1.01905} = 122{,}617.$$

32. Solution: C.

The first cash flow of 60,000 at time 3 earns 2400 in interest for a time 4 receipt of 62,400. Combined with the final payment, the investment returns 122,400 at time 4. The present value is $122{,}400(1.05)^{-4} = 100{,}699.$ The net present value is 699.

33. Solution: B.

Using spot rates, the value of the bond is:

$$60/1.07 + 60/1.08^2 + 1060/1.09^3 = 926.03.$$

34. Solution: E.

Using spot rates, the value of the bond is:

$60/1.07 + 60/1.08^2 + 1060/1.09^3 = 926.03.$ The annual effective rate is the solution to $926.03 = 60 a_{\overline{3}|i} + 1000(1+i)^{-3}$. Using a calculator, the solution is 8.9%.

35. Solution: C.

Duration is the negative derivative of the price multiplied by one plus the interest rate and divided by the price. Hence, the duration is –(–700)(1.08)/100 = 7.56.

36. Solution: C

The size of the dividend does not matter, so assume it is 1. Then the duration is

$$\frac{\sum_{t=1}^{\infty} tv^t}{\sum_{t=1}^{\infty} v^t} = \frac{(Ia)_{\overline{\infty}|}}{a_{\overline{\infty}|}} = \frac{\ddot{a}_{\overline{\infty}|}/i}{1/i} = \frac{1/(di)}{1/i} = \frac{1}{d} = \frac{1.1}{0.1} = 11.$$

37. Solution: B

Duration = $\dfrac{\sum_{t=1}^{\infty} tv^t R_t}{\sum_{t=1}^{\infty} v^t R_t} = \dfrac{\sum_{t=1}^{\infty} tv^t 1.02^t}{\sum_{t=1}^{\infty} v^t 1.02^t} = \dfrac{(Ia)_{\overline{\infty}|j}}{a_{\overline{\infty}|j}} = \dfrac{\ddot{a}_{\overline{\infty}|j}/j}{1/j} = \dfrac{1}{d}.$

The interest rate j is such that $(1+j)^{-1} = 1.02v = 1.02/1.05 \Rightarrow j = 0.03/1.02.$ Then the duration is $1/d = (1+j)/j = (1.05/1.02)/(0.03/1.02) = 1.05/0.03 = 35.$

45. Solution: A

For the time weighted return the equation is:

$$1+0 = \frac{12}{10}\frac{X}{12+X} \Rightarrow 120+10X = 12X \Rightarrow 120 = 2X \Rightarrow X = 60.$$

Then the amount of interest earned in the year is 60 – 60 – 10 = –10 and the weighted amount exposed to earning interest is 10(1) + 60(0.5) = 40. Then Y = –10/40 = –25%.

46. Solution: A

The outstanding balance is the present value of future payments. With only one future payment, that payment must be 559.12(1.08) = 603.85. The amount borrowed is $603.85a_{\overline{4}|0.08} = 2000.$ The first payment has 2000(0.08) = 160 in interest, thus the principal repaid is 603.85 – 160 = 443.85.

Alternatively, observe that the principal repaid in the final payment is the outstanding loan balance at the previous payment, or 559.12. Principal repayments form a geometrically decreasing sequence, so the principal repaid in the first payment is $559.12/1.08^3 = 443.85.$

47. Solution: B

Because the yield rate equals the coupon rate, Bill paid 1000 for the bond. In return he receives 30 every six months, which accumulates to $30s_{\overline{20}|j}$ where *j* is the semi-annual interest rate. The equation of value is $1000(1.07)^{10} = 30s_{\overline{20}|j} + 1000 \Rightarrow s_{\overline{20}|j} = 32.238.$ Using a calculator to solve for the interest rate produces *j* = 0.0476 and so $i = 1.0476^2 - 1 = 0.0975 = 9.75\%.$

48. Solution: A

To receive 3000 per month at age 65 the fund must accumulate to 3,000(1,000/9.65) = 310,880.83. The equation of value is $310,880.83 = X\ddot{s}_{\overline{300}|0.08/12} = 957.36657X \Rightarrow 324.72.$

49. Solution: D

(A) The left-hand side evaluates the deposits at age 0, while the right-hand side evaluates the withdrawals at age 17.

(B) The left-hand side has 16 deposits, not 17.

(C) The left-hand side has 18 deposits, not 17.

(D) The left-hand side evaluates the deposits at age 18 and the right-hand side evaluates the withdrawals at age 18.

(E) The left-hand side has 18 deposits, not 17 and 5 withdrawals, not 4.

50. Deleted

51. Solution: D

Because only Bond II provides a cash flow at time 1, it must be considered first. The bond provides 1025 at time 1 and thus 1000/1025 = 0.97561 units of this bond provides the required cash. This bond then also provides 0.97561(25) = 24.39025 at time 0.5. Thus Bond I must provide 1000 – 24.39025 = 975.60975 at time 0.5. The bond provides 1040 and thus 975.60975/1040 = 0.93809 units must be purchased.

52. Solution: C

Because only Mortgage II provides a cash flow at time two, it must be considered first. The mortgage provides $Y / a_{\overline{2}|0.07} = 0.553092Y$ at times one and two. Therefore, 0.553092*Y* = 1000 for *Y* = 1808.02. Mortgage I must provide 2000 – 1000 = 1000 at time one and thus *X* = 1000/1.06 = 943.40. The sum is 2751.42.

53. Solution: A

Bond I provides the cash flow at time one. Because 1000 is needed, one unit of the bond should be purchased, at a cost of 1000/1.06 = 943.40.

Bond II must provide 2000 at time three. Therefore, the amount to be reinvested at time two is 2000/1.065 = 1877.93. The purchase price of the two-year bond is $1877.93/1.07^2 = 1640.26$.

The total price is 2583.66.

54. Solution: C

Given the coupon rate is greater than the yield rate, the bond sells at a premium. Thus, the minimum yield rate for this callable bond is calculated based on a call at the earliest possible date because that is most disadvantageous to the bond holder (earliest time at which a loss occurs). Thus, X, the par value, which equals the redemption value because the bond is a par value bond, must satisfy

Price = $1722.25 = 0.04Xa_{\overline{30}|0.03} + Xv_{0.03}^{30} = 1.196X \Rightarrow X = 1440.$

55. Solution: B

Because 40/1200 is greater than 0.03, for early redemption the earliest redemption should be evaluated. If redeemed after 15 years, the price is $40a_{\overline{30}|0.03} + 1200/1.03^{30} = 1278.40$. If the bond is redeemed at maturity, the price is $40a_{\overline{40}|0.03} + 1100/1.03^{40} = 1261.80$. The smallest value should be selected, which is 1261.80.

56. Solution: E

Given the coupon rate is less than the yield rate, the bond sells at a discount. Thus, the minimum yield rate for this callable bond is calculated based on a call at the latest possible date because that is most disadvantageous to the bond holder (latest time at which a gain occurs). Thus, X, the par value, which equals the redemption value because the bond is a par value bond, must satisfy

Price = $1021.50 = 0.02Xa_{\overline{20}|0.03} + Xv_{0.03}^{20} = 0.851225X \Rightarrow X = 1200.$

57. Solution: B

Given the price is less than the amount paid for an early call, the minimum yield rate for this callable bond is calculated based on a call at the latest possible date. Thus, for an early call, the effective yield rate per coupon period, *j*, must satisfy Price = $1021.50 = 22a_{\overline{19}|j} + 1200v_j^{19}$. Using the calculator, *j* = 2.86%. We also must check the yield if the bond is redeemed at maturity. The equation is $1021.50 = 22a_{\overline{20}|j} + 1100v_j^{20}$. The solution is *j* = 2.46% Thus, the yield, expressed as a nominal annual rate of interest convertible semiannually, is twice the smaller of the two values, or 4.92%.

58. Moved to Derivatives section

59. Solution: C

First, the present value of the liability is $PV = 35,000a_{\overline{15}|6.2\%} = 335,530.30.$

The duration of the liability is:

$$\bar{d} = \frac{\sum tv^t R_t}{\sum v^t R_t} = \frac{35,000v + 2(35,000)v^2 + \cdots + 15(35,000)v^{15}}{335,530.30} = \frac{2,312,521.95}{335,530.30} = 6.89214.$$

Let X denote the amount invested in the 5 year bond.

Then, $\frac{X}{335,530.30}(5) + \left(1 - \frac{X}{335,530.30}\right)(10) = 6.89214 \Rightarrow X = 208,556.$

60. Solution: A

The present value of the first eight payments is:

$$PV = 2000v + 2000(1.03)v^2 + ... + 2000(1.03)^7 v^8 = \frac{2000v - 2000(1.03)^8 v^9}{1 - 1.03v} = 13,136.41.$$

The present value of the last eight payments is:

$$PV = 2000(1.03)^7 0.97v^9 + 2000(1.03)^7 (0.97)^2 v^{10} + \cdots + 2000(1.03)^7 (0.97^8)v^{16}$$

$$= \frac{2000(1.03)^7 0.97v^9 - 2000(1.03)^7 (0.97)^9 v^{17}}{1 - 0.97v} = 7,552.22.$$

Therefore, the total loan amount is *L* = 20,688.63.

61. Solution: E

$$2000 = 500\exp\left(\int_0^t \frac{\frac{r^2}{100}}{3+\frac{r^3}{150}}\,dr\right)$$

$$4 = \exp\left(0.5\int_0^t \frac{\frac{r^2}{50}}{3+\frac{r^3}{150}}\,dr\right) = \exp\left[0.5\ln\left(3+\frac{r^3}{150}\right)\Big|_0^t\right]$$

$$4 = \exp\left[0.5\ln\left(1+\frac{t^3}{450}\right)\right] = \left(1+\frac{t^3}{450}\right)^{\frac{1}{2}}$$

$$16 = \left(1+\frac{t^3}{450}\right)$$

$$t = 18.8988$$

62. Solution: E

Let F, C, r, and i have their usual interpretations. The discount is $(Ci - Fr)a_{\overline{n}|}$ and the discount in the coupon at time t is $(Ci - Fr)v^{n-t+1}$. Then,

$$194.82 = (Ci - Fr)v^{26}$$

$$306.69 = (Ci - Fr)v^{21}$$

$$0.63523 = v^5 \Rightarrow v = 0.91324 \Rightarrow i = 0.095$$

$$(Ci - Fr) = 194.82(1.095)^{26} = 2062.53$$

$$\text{Discount} = 2062.53a_{\overline{40}|0.095} = 21{,}135$$

63. Solution: A

$$699.68 = Pv^{8-5+1}$$

$$P = 842.39 \text{ (annual payment)}$$

$$P_1 = \frac{699.68}{1.0475^4} = 581.14$$

$$I_1 = 842.39 - 581.14 = 261.25$$

$$L = \frac{261.25}{0.0475} = 5500 \text{ (loan amount)}$$

$$\text{Total interest} = 842.39(8) - 5500 = 1239.12$$

64. Solution: D

$$OB_{18} = 22,000(1.007)^{18} - 450.30 s_{\overline{18}|0.007} = 16,337.10$$

$$16,337.10 = P a_{\overline{24}|0.004}$$

$$P = 715.27$$

65. Solution: C

If the bond has no premium or discount, it was bought at par so the yield rate equals the coupon rate, 0.038.

$$d = \frac{\frac{1}{2}\left(1(190)v + 2(190)v^2 + \cdots + 14(190)v^{14} + 14(5000)v^{14}\right)}{190v + 190v^2 + \cdots + 190v^{14} + 5000v^{14}}$$

$$d = \frac{95\left(Ia\right)_{\overline{14}|} + 7(5000)v^{14}}{190 a_{\overline{14}|} + 5000v^{14}}$$

$$d = 5.5554$$

Or, taking advantage of a shortcut:

$d = \ddot{a}_{\overline{14}|0.038} = 11.1107$. This is in half years, so dividing by two, $d = \frac{11.1107}{2} = 5.5554$.

66. Solution: A

$$\bar{v} = \frac{7.959}{1.072} = 7.425$$

$$P(0.08) = P(0.072)\left[1 - (\Delta i)\bar{v}\right]$$

$$P(0.08) = 1000\left[1 - (0.008)(7.425)\right] = 940.60$$

67. Solution: E

$$(1+s_3)^3 = (1+s_2)^2(1+ {}_2f_1)$$

$$0.85892 = \frac{1}{(1+\mathrm{s}_3)^3}, s_3 = 0.052$$

$$0.90703 = \frac{1}{(1+s_2)^2}, s_2 = 0.050$$

$$1.052^3 = 1.050^2(1+ {}_2f_1)$$

$${}_2f_1 = 0.056$$

68. Solution: C

Let d_0 be the Macaulay duration at time 0.

$$d_0 = \ddot{a}_{\overline{8}|0.05} = 6.7864$$
$$d_1 = d_0 - 1 = 5.7864$$
$$d_2 = \ddot{a}_{\overline{7}|0.05} = 6.0757$$
$$\frac{d_1}{d_2} = \frac{5.7864}{6.0757} = 0.9524$$

This solution employs the fact that when a coupon bond sells at par the duration equals the present value of an annuity-due. For the duration just before the first coupon the cash flows are the same as for the original bond, but all occur one year sooner. Hence the duration is one year less.

Alternatively, note that the numerators for d_1 and d_2 are identical. That is because they differ only with respect to the coupon at time 1 (which is time 0 for this calculation) and so the payment does not add anything. The denominator for d_2 is the present value of the same bond, but with 7 years, which is 5000. The denominator for d_1 has the extra coupon of 250 and so is 5250. The desired ratio is then 5000/5250 = 0.9524.

69. Solution: A

Let N be the number of shares bought of the bond as indicated by the subscript.

$$N_C(105) = 100, N_C = 0.9524$$
$$N_B(100) = 102 - 0.9524(5), N_B = 0.9724$$
$$N_A(107) = 99 - 0.9524(5), N_A = 0.8807$$

70. Solution: B

All are true except B. Immunization requires frequent rebalancing.

71. Solution: D

Set up the following two equations in the two unknowns:

$$A(1.05)^2 + B(1.05)^{-2} = 6000$$

$$2A(1.05)^2 - 2B(1.05)^{-2} = 0.$$

Solving simultaneously gives:

$$A = 2721.09$$

$$B = 3307.50$$

$$|A - B| = 586.41.$$

72. Solution: A

Set up the following two equations in the two unknowns.

$$(1) \quad 5000(1.03)^3 + B(1.03)^{-b} = 12,000 \Rightarrow$$

$$5463.635 + B(1.03)^{-b} = 12,000 \Rightarrow B(1.03)^{-b} = 6536.365$$

$$(2) \quad 3(5000)(1.03)^3 - bB(1.03)^{-b} = 0 \Rightarrow 16,390.905 - b6536.365 = 0$$

$$b = 2.5076$$

$$B = 7039.27$$

$$\frac{B}{b} = 2807.12$$

73. Solution: D

$$P_A = A(1+i)^{-2} + B(1+i)^{-9}$$

$$P_L = 95,000(1+i)^{-5}$$

$$P_A' = -2A(1+i)^{-3} - 9B(1+i)^{-10}$$

$$P_L' = -5(95,000)(1+i)^{-6}$$

Set the present values and derivatives equal and solve simultaneously.

$$0.92456A + 0.70259B = 78,083$$

$$-1.7780A - 6.0801B = -375,400$$

$$B = \frac{78,083(1.7780/0.92456) - 375,400}{0.70259(1.7780/0.92456) - 6.0801} = 47,630$$

$$A = [78,083 - 0.70259(47,630)]/0.92456 = 48,259$$

$$\frac{A}{B} = 1.0132$$

74. Solution: D

Throughout the solution, let $j = i/2$.

For bond A, the coupon rate is $(i + 0.04)/2 = j + 0.02$.

For bond B, the coupon rate is $(i - 0.04)/2 = j - 0.02$.

The price of bond A is $P_A = 10,000(j + 0.02)a_{\overline{20}|j} + 10,000(1 + j)^{-20}$.

The price of bond B is $P_B = 10,000(j - 0.02)a_{\overline{20}|j} + 10,000(1 + j)^{-20}$.

Thus,

$$P_A - P_B = 5,341.12 = [200 - (-200)]a_{\overline{20}|j} = 400a_{\overline{20}|j}$$
$$a_{\overline{20}|j} = 5,341.12 / 400 = 13.3528.$$

Using the financial calculator, $j = 0.042$ and $i = 2(0.042) = 0.084$.

75. Solution: D

The initial level monthly payment is

$$R = \frac{400,000}{a_{\overline{15\times12}|0.09/12}} = \frac{400,000}{a_{\overline{180}|0.0075}} = 4,057.07.$$

The outstanding loan balance after the 36th payment is

$$B_{36} = Ra_{\overline{180-36}|0.0075} = 4,057.07a_{\overline{144}|0.0075} = 4,057.07(87.8711) = 356,499.17.$$

The revised payment is 4,057.07 – 409.88 = 3,647.19.

Thus,

$$356,499.17 = 3,647.19a_{\overline{144}|j/12}$$
$$a_{\overline{144}|j/12} = 356,499.17 / 3,647.19 = 97.7463.$$

Using the financial calculator, $j/12 = 0.575\%$, for $j = 6.9\%$.

76. Solution: D

The price of the first bond is

$$1000(0.05/2)a_{\overline{30\times2}|0.05/2} + 1200(1 + 0.05/2)^{-30\times2} = 25a_{\overline{60}|0.025} + 1200(1.025)^{-60}$$
$$= 772.72 + 272.74 = 1,045.46.$$

The price of the second bond is also 1,045.46. The equation to solve is

$$1,045.46 = 25a_{\overline{60}|j/2} + 800(1 + j/2)^{-60}.$$

The financial calculator can be used to solve for $j/2 = 2.2\%$ for $j = 4.4\%$.

77. Solution: E

Let n = years. The equation to solve is

$$1000(1.03)^{2n} = 2(1000)(1.0025)^{12n}$$

$$2n\ln 1.03 + \ln 1000 = 12n\ln 1.0025 + \ln 2000$$

$$0.029155n = 0.69315$$

$$n = 23.775.$$

This is 285.3 months. The next interest payment to Lucas is at a multiple of 6, which is 288 months.

78. Solution: B

The ending balance is 5000(1.09) + 2600sqrt(1.09) = 8164.48.

The time-weighted rate of return is (5200/5000) x [8164.08/(5200 + 2600)] – 1 = 0.0886.

79. Solution: A

Equating the accumulated values after 4 years provides an equation in K.

$$10\left(1+\frac{K}{25}\right)^4 = 10\exp\left(\int_0^4 \frac{1}{K+0.25t}\,dt\right)$$

$$4\ln(1+0.04K) = \int_0^4 \frac{1}{K+0.25t}\,dt = 4\ln(\mathrm{K}+0.25\,\mathrm{t})\Big|_0^4 = 4\ln(K+1) - 4\ln(K) = 4\ln\frac{K+1}{K}$$

$$1+0.04K = \frac{K+1}{K}$$

$$0.04K^2 = 1$$

$$K = 5.$$

Therefore, $X = 10(1+5/25)^4 = 20.74.$

80. Solution: C

To repay the loan, the sinking fund must accumulate to 1000. The deposit is 2(1000i). Therefore,

$$1000 = 2000is_{\overline{5}|0.8i}$$

$$0.5 = i\frac{(1+0.8i)^5 - 1}{0.8i}$$

$$(1+0.8i)^5 = 1.4$$

$$1+0.8i = 1.0696$$

$$i = 0.0696/0.8 = 0.087.$$

81. Solution: D

The outstanding balance at time 25 is $100(Da)_{\overline{25|}} = 100\dfrac{25-a_{\overline{25|}}}{i}$. The principle repaid in the 26th payment is $X = 2500 - i(100)\dfrac{25-a_{\overline{25|}}}{i} = 2500 - 2500 + 100a_{\overline{25|}} = 100a_{\overline{25|}}$. The amount borrowed is the present value of all 50 payments, $2500a_{\overline{25|}} + v^{25}100(Da)_{\overline{25|}}$. Interest paid in the first payment is then

$$\begin{aligned}
&i\left[2500a_{\overline{25|}} + v^{25}100(Da)_{\overline{25|}}\right] \\
&= 2500(1-v^{25}) + 100v^{25}(25-a_{\overline{25|}}) \\
&= 2500 - 2500v^{25} + 2500v^{25} - v^{25}100a_{\overline{25|}} \\
&= 2500 - Xv^{25}.
\end{aligned}$$

82. Solution: A

The exposure associated with *i* produces results quite close to a true effective rate of interest as long as the net amount of principal contributed at time *t* is small relative to the amount in the fund at the beginning of the period.

83. Solution: E

The time-weighted weight of return is

j = (120,000 / 100,000) x (130,000 / 150,000) x (100,000 / 80,000) – 1 = 30.00%.

Note that 150,000 = 120,000 + 30,000 and 80,000 = 130,000 – 50,000.

84. Solution: C

The accumulated value is $1000\ddot{s}_{\overline{20|}0.816} = 50,382.16$. This must provide a semi-annual annuity-due of 3000. Let *n* be the number of payments. Then solve $3000\ddot{a}_{\overline{n|}0.04} = 50,382.16$ for *n* = 26.47. Therefore, there will be 26 full payments plus one final, smaller, payment. The equation is $50,382.16 = 3000\ddot{a}_{\overline{26|}0.04} + X(1.04)^{-26}$ with solution *X* = 1430. Note that the while the final payment is the 27th payment, because this is an annuity-due, it takes place 26 periods after the annuity begins.

85. Solution: D

For the first perpetuity,

$$\frac{1}{(1+i)^2-1}+1=7.21$$

$$\frac{1}{6.21}=(1+i)^2-1$$

$i=0.0775.$

For the second perpetuity,

$$R\left[\frac{1}{(1.0775+0.01)^3-1}+1\right](1.0875)^{-1}=7.21$$

$$1.286139R=7.21(1.0875)(0.286139)$$

$R=1.74.$

86. Solution: E

$$10,000=100(Ia)_{\overline{5|}}+Xv^5a_{\overline{15|}}=100\left(\frac{\ddot{a}_{\overline{5|}}-5v^5}{0.05}\right)+Xv^5a_{\overline{15|}}$$

$$10,000=1256.64+8.13273X$$

$$1075=X$$

87. Solution: C

$$5000=Xs_{\overline{10|}0.06}(1.05)^5$$

$$X=\frac{5000}{13.1808(1.2763)}=297.22$$

88. Solution: E

The monthly payment on the original loan is $\frac{65,000}{a_{\overline{180|}8/12\%}}=621.17$. After 12 payments the outstanding balance is $621.17a_{\overline{168|}8/12\%}=62,661.40$. The revised payment is $\frac{62,661.40}{a_{\overline{168|}6/12\%}}=552.19$.

89. Solution: E

At the time of the final deposit the fund has $750s_{\overline{18}|0.07} = 25,499.27.$ This is an immediate annuity because the evaluation is done at the time the last payments is made (which is the end of the final year). A tuition payment of $6000(1.05)^{17} = 13,752.11$ is made, leaving 11,747.16. It earns 7%, so a year later the fund has 11,747.16(1.07) = 12,569.46. Tuition has grown to 13,752.11(1.05) = 14,439.72. The amount needed is 14,439.72 – 12,569.46 = 1,870.26

90. Solution: B

The coupons are 1000(0.09)/2 = 45. The present value of the coupons and redemption value at 5% per semiannual period is $P = 45a_{\overline{40}|0.05} + 1200(1.05)^{-40} = 942.61.$

91. Solution: A

For a bond bought at discount, the minimum price will occur at the latest possible redemption date. $P = 50a_{\overline{20}|0.06} + 1000(1.06)^{-20} = 885.30.$

92. Solution: C

$$\frac{1.095^5}{1.090^4} - 1 = 11.5\%$$

93. Solution: D

The accumulated value of the first year of payments is $2000s_{\overline{12}|0.005} = 24,671.12.$ This amount increases at 2% per year. The effective annual interest rate is $1.005^{12} - 1 = 0.061678.$ The present value is then

$$P = 24,671.12\sum_{k=1}^{25} 1.02^{k-1}(1.061678)^{-k} = 24,671.12\frac{1}{1.02}\sum_{k=1}^{25}\left(\frac{1.02}{1.061678}\right)^k$$

$$= 24,187.37\frac{0.960743 - 0.960743^{26}}{1 - 0.960743} = 374,444.$$

This is 56 less than the lump sum amount.

94. Solution: A

The monthly interest rate is 0.072/12 = 0.006. 6500 five years from today has value $6500(1.006)^{-60} = 4539.77$. The equation of value is

$$4539.77 = 1700(1.006)^{-n} + 3400(1.006)^{-2n}.$$

Let $x = 1.006^{-n}$. Then, solve the quadratic equation

$$3400x^2 + 1700x - 4539.77 = 0$$

$$x = \frac{-1700 + \sqrt{1700^2 - 4(3400)(-4539.77)}}{2(3400)} = 0.93225.$$

Then,

$$1.006^{-n} = 0.9325 \Rightarrow -n\ln(1.006) = \ln(0.93225) \Rightarrow n = 11.73.$$

To ensure there is 6500 in five years, the deposits must be made earlier and thus the maximum integral value is 11.

95. Solution: C

$$\frac{(1-d/2)^{-4}}{(1-d/4)^{-4}} = \left(\frac{39}{38}\right)^4 \Rightarrow \frac{1-d/2}{1-d/4} = \frac{38}{39} \Rightarrow 39 - 39(d/2) = 38 - 38(d/4)$$

$$d\left(39/2 - 38/4\right) = 39 - 38$$

$$d = 1/(19.5 - 9.5) = 0.1$$

$$1 + i = (1 - d/2)^{-2} = .95^{-2} = 1.108 \Rightarrow i = 10.8\%.$$

96. Solution: C

Thee monthly interest rate is 0.042/12 = 0.0035. The quarterly interest rate is $1.0035^3 - 1 = 0.0105$. The investor makes 41 quarterly deposits and the ending date is 124 months from the start. Using January 1 of year *y* as the comparison date produces the following equation:

$$X + \sum_{k=1}^{41} \frac{100}{1.0105^k} = \frac{1.9X}{1.0035^{124}}$$

Substituting $1.0105 = 1.0035^3$ gives answer (C).

97. Solution: D

Convert the two annual rates, 4% and 5%, to two-year rates as $1.04^2 - 1 = 0.0816$ and $1,05^2 - 1 = 0.1025$.

The accumulated value is

$$100\ddot{s}_{\overline{3}|0.0816}(1.05)^4 + 100\ddot{s}_{\overline{2}|0.1025} = 100(3.51678)(1.21551) + 100(2.31801) = 659.269.$$

With only five payments, an alternative approach is to accumulate each one to time ten and add them up.

The two-year yield rate is the solution to $100\ddot{s}_{\overline{5}|i} = 659.269$. Using the calculator, the two-year rate is 0.093637. The annual rate is $1.093637^{0.5} - 1 = 0.04577$ which is 4.58%.

Solve for the 2-year yield and then convert to annual yield:

98. Solution: C

$$(1.08)^{1/12} - 1 = 0.006434$$

$$\frac{1}{1.08^{15}} 25,000\ddot{a}_{\overline{4}|8\%} = X\ddot{a}_{\overline{216}|0.6434\%}$$

$$X = \frac{25,000(3.57710)}{3.17217(117.2790)} = 240.38$$

99. Solution: B

$$\text{PV}_{\text{perp.}} = \left[\frac{1}{0.1} + \frac{\frac{1}{0.08} - \frac{1}{0.1}}{1.1^{10}} \right](15,000) + 15,000$$

$$= 164,457.87 + 15,000 = 179,457.87$$

$$X\left(\ddot{a}_{\overline{10}|0.10} + \frac{\ddot{a}_{\overline{15}|0.08}}{1.10^{10}} \right) = 179,458$$

$$X\left(6.759 + \frac{9.244}{1.10^{10}} \right) = 179,458$$

$$X = 17,384$$

100. Solution: A

$$1050.50 = (22.50 + X)a_{\overline{14}|0.03} + X\left(\frac{a_{\overline{14}|0.03} - 14(1.03)^{-14}}{0.03}\right) + 300(1.03)^{-14}$$

$$1050.50 = (22.50 + X)11.2961 + X\left(\frac{11.2961 - 9.25565}{0.03}\right) + 198.335 => 79.3111X = 598 => X = 7.54$$

101. Solution: D

The amount of the loan is the present value of the deferred increasing annuity:

$$(1.03)^{-10}\left[500\ddot{a}_{\overline{30}|0.03} + 500(I\ddot{a})_{\overline{30}|0.03}\right] = (1.03^{-10})(500)\left[\ddot{a}_{\overline{30}|0.03} + \frac{\ddot{a}_{\overline{30}|0.03} - 30(1.03)^{-30}}{0.03/1.03}\right] = 64,257.$$

102. Solution: C

$$50,000\left[\frac{(1+i)^{30} - (1.03)^{30}}{(1+i)^{30}(i-0.03)}\right](1+i) = 5,000\left[\frac{(1+i)^{30} - (1.03)^{30}}{i-0.03}\right]$$

$$50,000/(1+i)^{29} = 5,000$$

$$(1+i)^{29} = 10$$

$$i = 10^{1/29} - 1 = 0.082637$$

The accumulated amount is

$$50,000\left[\frac{(1.082637)^{30} - (1.03)^{30}}{(1.082637)^{30}(i-0.03)}\right](1.082637) = 797,836.82$$

103. Solution: D

The first payment is 2,000, and the second payment of 2,010 is 1.005 times the first payment. Since we are given that the series of quarterly payments is geometric, the payments multiply by 1.005 every quarter.

Based on the quarterly interest rate, the equation of value is

$$100,000 = 2,000 + 2,000(1.005)v + 2,000(1.005)^2v^2 + 2,000(1.005)^3v^3 + \cdots = \frac{2,000}{1-1.005v}.$$

$$1 - 1.005v = 2,000/100,000 \Rightarrow v = 0.98/1.005.$$

The annual effective rate is $v^{-4} - 1 = (0.98/1.005)^{-4} - 1 = 0.10601 = 10.6\%$.

104. Solution: A

Present value for the first 10 years is $\dfrac{1-(1.06)^{-10}}{\ln(1.06)} = 7.58$

Present value of the payments after 10 years is

$$(1.06)^{-10}\int_0^\infty (1.03)^s (1.06)^{-s}\,ds = \frac{0.5584}{\ln(1.06)-\ln(1.03)} = 19.45$$

Total present value = 27.03

105. Solution: C

$$\left[1,000(1.06)^5 + X(1.06)^2\right]e^{\int_5^{10}\frac{1}{t+1}dt} = 75,000$$

$$(13,382.26 + 1.1236X)\frac{11}{6} = 75,000$$

$$1.1236X = 27,526.83$$

$$X = 24,498.78$$

106. Solution: D

The effective annual interest rate is $i = (1-d)^{-1} - 1 = (1-0.055)^{-1} - 1 = 5.82\%$

The balance on the loan at time 2 is $15,000,000(1.0582)^2 = 16,796,809.$

The number of payments is given by $1,200,000a_{\overline{n}|} = 16,796,809$ which gives n = 29.795 => 29 payments of 1,200,000. The final equation of value is

$$1,200,000a_{\overline{29}|} + X(1.0582)^{-30} = 16,796,809$$

$$X = (16,796,809 - 16,621,012)(5.45799) = 959,490.$$

107. Solution: C

$$1-v^2 = 0.525(1-v^4) \Rightarrow 1 = 0.525(1+v^2) \Rightarrow v^2 = 0.90476 \Rightarrow v = 0.95119$$

$$1-v^2 = 0.1427(1-v^n) \Rightarrow 1-\mathrm{v}^n = (1-0.90476)/0.1427 = 0.667414 \Rightarrow v^n = 0.332596$$

$$n = \ln(0.332596)/\ln(0.95119) = 22$$

108. Solution: B

Let X be the annual deposit on the sinking fund. Because the sinking fund deposits must accumulate to the loan amount, $L = Xs_{\overline{11}|0.047} = 13.9861X$. At time 7 the fund has $Xs_{\overline{7}|0.047} = 8.0681X$. This is 6241 short of the loan amount, so a second equation is $L = 8.0681X + 6241$. Combining the two equations gives $13.9861X - 8.0681X = 6241$ with implies X=1055.

109. Solution: C

The monthly payment is $200,000 / a_{\overline{360}|0.005} = 1199.10$. Using the equivalent annual effective rate of 6.17%, the present value (at time 0) of the five extra payments is 41,929.54 which reduces the original loan amount to 200,000 – 41,929.54 = 158,070.46. The number of months required is the solution to $158,070.46 = 1199.10a_{\overline{n}|0.005}$ which is 215.78. Using calculator, n = 215.78 months are needed to pay off this amount. So there are 215 full payments plus one fractional payment at the end of the 216th month, which is December 31, 2020.

110. Solution: D

The annual effective interest rate is 0.08/(1 – 0.08) = 0.08696. The level payments are $500,000 / a_{\overline{5}|0.08696} = 500,000 / 3.9205 = 127,535$. This rounds up to 128,000. The equation of value for X is

$$128,000a_{\overline{4}|0.08696} + X(1.08696)^{-5} = 500,000$$

$$X = (500,000 - 417,466.36)(1.51729) = 125,227.$$

111. Solution: B

The accumulated value is the reciprocal of the price. The equation is X[(1/0.94)+(1/0.95)+(1/0.96)+(1/0.97)+(1/0.98)+(1/0.99)] = 100,000.

X= 16,078

112. Solution: D

Let P be the annual payment. The fifth line is obtained by solving a quadratic equation.

$$P(1-v^{10}) = 3600$$

$$Pv^{10-6+1} = 4871$$

$$\frac{1-v^{10}}{v^5} = \frac{3600}{4871}$$

$$1-v^{10} = 0.739068v^5$$

$$v^5 = 0.69656$$

$$v^{10} = 0.485195$$

$$i = 0.485195^{-10} - 1 = 0.075$$

$$X = P\frac{1-v^{10}}{i} = \frac{3600}{0.075} = 48,000$$

113. Solution: A

Let j = periodic yield rate, r = periodic coupon rate, F = redemption (face) value, P = price, n = number of time periods, and $v_j = \frac{1}{1+j}$. In this problem, $j = (1.0705)^{\frac{1}{2}} - 1 = 0.03465$, r = 0.035, P = 10,000, and n = 50.

The present value equation for a bond is $P = Fv_j^{\ n} + Fra_{\overline{n|}j}$; solving for the redemption value F yields

$$F = \frac{P}{v_j^{\ n} + ra_{\overline{n|}j}} = \frac{10,000}{(1.03465)^{-50} + 0.035a_{\overline{50|}0.03465}} = \frac{10,000}{0.18211 + 0.035(23.6044)} = 9,918. \ .$$

114. Solution: B

Jeff's monthly cash flows are coupons of 10,000(0.09)/12 = 75 less loan payments of 2000(0.08)/12 = 13.33 for a net income of 61.67. At the end of the ten years (in addition to the 61.67) he receives 10,000 for the bond less a 2,000 loan repayment. The equation is

$$8000 = 61.67a_{\overline{120|}i^{(12)}/12} + 8000(1+i^{(12)}/12)^{-120}$$

$$i^{(12)}/12 = 0.00770875$$

$$i = 1.00770875^{12} - 1 = 0.0965 = 9.65\%.$$

115. Solution: B

The present value equation for a par-valued annual coupon bond is $P = Fv_i^n + Fra_{\overline{n}|i}$; solving for the coupon rate r yields $r = \frac{P - Fv_i^n}{Fa_{\overline{n}|i}} = \frac{P}{a_{\overline{n}|i}}\left(\frac{1}{F}\right) - \frac{v_i^n}{a_{\overline{n}|i}}$.

All three bonds have the same values except for F. We can write $r = x(1/F) + y$. From the first two bonds:

$0.0528 = x/1000 + y$ and $0.0440 = x/1100 + y$. Then,

$0.0528 – 0.044 = x(1/1000 – 1/1100)$ for $x = 96.8$ and $y = 0.0528 – 96.8/1000 = –0.044$. For the third bond, $r = 96.8/1320 – 0.044 = 0.2933 = 2.93\%$.

116. Solution: A

The effective semi-annual yield rate is $1.04 = \left(1 + \frac{i^{(2)}}{2}\right)^2 \Rightarrow \frac{i^{(2)}}{2} = 1.9804\%$. Then,

$$582.53 = c(1.02)v + c(1.02v)^2 + \cdots + c(1.02v)^{12} + 250v^{12}$$

$$= c\frac{1.02v - (1.02v)^{13}}{1 - 1.02v} + 250v^{12} = 12.015c + 197.579 \Rightarrow c = 32.04.$$

$$582.53 = c\frac{1.02v - (1.02v)^{13}}{1 - 1.02v} + 250v^{12} = 12.015c + 197.579 \Rightarrow c = 32.04$$

117. Solution: E

Book values are linked by BV3(1 + i) – Fr = BV4. Thus 1254.87(1.06) – Fr = 1277.38. Therefore, the coupon is Fr = 52.7822. The prospective formula for the book value at time 3 is

$$1254.87 = 52.7822\frac{1 - 1.06^{-(n-3)}}{0.06} + 1890(1.06)^{-(n-3)}$$

$$375.1667 = 1010.297(1.06)^{-(n-3)}$$

$$n - 3 = \frac{\ln(375.1667/1010.297)}{-\ln(1.06)} = 17.$$

Thus, $n = 20$. Note that the financial calculator can be used to solve for $n – 3$.

118. Solution: A

Book values are linked by BV3(1 + *i*) – *Fr* = BV4. Thus BV3(1.04) – 2500(0.035) = BV3 + 8.44. Therefore, BV3 = [2500(0.035) + 8.44]/0.04 = 2398.5. The prospective formula for the book value at time 3 is, where *m* is the number of six-month periods.

$$2398.5 = 2500(0.035)\frac{1-1.04^{-(m-3)}}{0.04} + 2500(1.04)^{-(m-3)}$$
$$211 = 312.5(1.04)^{-(m-3)}$$
$$m-3 = \frac{\ln(211/312.5)}{-\ln(1.04)} = 10.$$

Thus, *m* = 13 and *n* = *m*/2 = 6.5. Note that the financial calculator can be used to solve for *m* – 3.

119. Solution: C

$$S_1 = i_{0,1} = 0.04$$
$$i_{1,2} = 0.06 = \frac{(1+S_2)^2}{(1+S_1)} - 1 \text{ so } S_2 = \sqrt{(1.06)(1.04)} - 1 = 0.04995$$
$$i_{2,3} = 0.08 = \frac{(1+S_3)^3}{(1+S_2)^2} - 1 \text{ so } S_3 = [(1.08)(1.04995)^2]^{1/3} - 1 = 0.05987 = 6\%.$$

120. Solution: D

Interest earned is 55,000 – 50,000 – 8,000 + 10,000 = 7,000.

Equating the two interest measures gives the equation

$$\frac{7,000}{50,000 + (16,000/3) - 10,000(1-t)} = \frac{52}{50}\frac{62}{60}\frac{55}{52} - 1 = 0.13667$$
$$7,000 = 0.13667(55,333.33 - 10,000 + 10,000t)$$
$$t = [7,000 - 0.13667(45,333.33)]/1,366.7 = 0.5885.$$

121. Solution: B

The Macaulay duration of Annuity A is $0.93 = \frac{0(1)+1(v)+2(v^2)}{1+v+v^2} = \frac{v+2v^2}{1+v+v^2}$, which leads to the quadratic equation $1.07v^2 + 0.07v - 0.93 = 0$. The unique positive solution is v = 0.9.

The Macaulay duration of Annuity B is $\frac{0(1)+1(v)+2(v^2)+3(v^3)}{1+v+v^2+v^3} = 1.369.$

122. Solution: D

With v =1/1.07,

$$D = \frac{2(40,000)v^2 + 3(25,000)v^3 + 4(100,000)v^4}{40,000v^2 + 25,000v^3 + 100,000v^4} = 3.314.$$

123. Solution: C

The Macaulay duration of Bond A is $MacD^A = \ddot{a}_{\overline{3}|10\%} = 2.7355$

The modified duration of Bond A is $ModD^A = \frac{MacD^A}{1+0.1} = 2.4869$

The modified duration of Bond B is also 2.4869. The Macaulay duration of Bond B is $MacD^B = ModD^B(1+0.1/2)^2 = 2.74$.

124. Solution: C

$$30 = MacD = \frac{\sum_{n=0}^{\infty} nv^n}{\sum_{n=0}^{\infty} v^n} = \frac{Ia_{\overline{\infty}|}}{\ddot{a}_{\overline{\infty}|}} = \frac{1/(di)}{1/d} = \frac{(1+i)/i^2}{(1+i)/i} = \frac{1}{i}$$ and so i = 1/30.

Then, $ModD = \frac{MacD}{1+i} = \frac{30}{1+\frac{1}{30}} = 29.032.$

125. Solution: D

Let D be the next dividend for Stock J. The value of Stock F is $0.5D/(0.088 - g)$. The value of Stock J is $D/(0.088 + g)$. The relationship is

$$\frac{0.5D}{0.088-g} = 2\frac{D}{0.088+g}$$

$$0.5D(0.088+g) = 2D(0.088-g)$$

$$2.5g = 0.132$$

$$g = 0.0528 = 5.3\%.$$

126. Solution: B

I) False. The yield curve structure is not relevant.

II) True.

III) False. Matching the present values is not sufficient when interest rates change.

127. Solution: A

The present value function and its derivatives are

$$P(i) = X + Y(1+i)^{-3} - 500(1+i)^{-1} - 1000(1+i)^{-4}$$

$$P'(i) = -3Y(1+i)^{-4} + 500(1+i)^{-2} + 4000(1+i)^{-5}$$

$$P''(i) = 12Y(1+i)^{-5} - 1000(1+i)^{-3} - 20,000(1+i)^{-6}.$$

The equations to solve for matching present values and duration (at $i = 0.10$) and their solution are

$$P(0.1) = X + 0.7513Y - 1137.56 = 0$$

$$P'(0.1) = -2.0490Y + 2896.91 = 0$$

$$Y = 2896.91 / 2.0490 = 1413.82$$

$$X = 1137.56 - 0.7513(1413.82) = 75.36.$$

The second derivative is

$$P''(0.1) = 12(1413.82)(1.1)^{-5} - 1000(1.1)^{-3} - 20,000(1.1)^{-6} = -1506.34.$$

Redington immunization requires a positive value for the second derivative, so the condition is not satisfied.

128. Solution: D

This solution uses time 8 as the valuation time. The two equations to solve are

$$P(i) = 300,000(1+i)^2 + X(1+i)^{8-y} - 1,000,000 = 0$$

$$P'(i) = 600,000(1+i) + (8-y)X(1+i)^{7-y} = 0.$$

Inserting the interest rate of 4% and solving:

$$300,000(1.04)^2 + X(1.04)^{8-y} - 1,000,000 = 0$$

$$600,000(1.04) + (8-y)X(1.04)^{7-y} = 0$$

$$X(1.04)^{-y} = [1,000,000 - 300,000(1.04)^2]/1.04^8 = 493,595.85$$

$$624,000 + (8-y)(1.04)^7(493,595.85) = 0$$

$$y = 8 + 624,000/[493,595.85(1.04)^7] = 8.9607$$

$$\mathrm{X} = 493,595.85(1.04)^{8.9607} = 701,459.$$

129. Solution: A

This solution uses Macaulay duration and convexity. The same conclusion would result had modified duration and convexity been used.

The liabilities have present value $573/1.07^2 + 701/1.07^5 = 1000.$ Only portfolios A, B, and E have a present value of 1000.

The duration of the liabilities is $[2(573)/1.07^2 + 5(701)/1.07^5]/1000 = 3.5.$ The duration of a zero coupon bond is its term. The portfolio duration is the weighted average of the terms. For portfolio A the duration is [500(1) + 500(6)]/1000 = 3.5. For portfolio B it is [572(1) + 428(6)]/1000 = 3.14. For portfolio E it is 3.5. This eliminates portfolio B.

The convexity of the liabilities is $[4(573)/1.07^2 + 25(701)/1.07^5]/1000 = 14.5.$ The convexity of a zero-coupon bond is the square of its term. For portfolio A the convexity is [500(1) + 500(36)]/1000 = 18.5 which is greater than the convexity of the liabilities. Hence portfolio A provides Redington immunization. As a check, the convexity of portfolio E is 12.25, which is less than the liability convexity.

130. Solution: D

The present value of the liabilities is 1000, so that requirement is met. The duration of the liabilities is $402.11[1.1^{-1} + 2(1.1)^{-2} + 3(1.1)^{-3}]/1000 = 1.9365.$ Let X be the investment in the one-year bond. The duration of a zero-coupon is its term. The duration of the two bonds is then [X + (1000 – X)(3)]/1000 = 3 – 0.002X. Setting this equal to 1.9365 and solving yields X = 531.75.

131. Solution: A

Let *x*, *y*, and *z* represent the amounts invested in the 5-year, 15-year, and 20-year zero-coupon bonds, respectively. Note that in this problem, one of these three variables is 0.

The present value, Macaulay duration, and Macaulay convexity of the assets are, respectively,

$$x+y+z, \frac{5x+15y+20z}{x+y+z}, \frac{5^2x+15^2y+20^2z}{x+y+z}.$$

We are given that the present value, Macaulay duration, and Macaulay convexity of the liabilities are, respectively, 9697, 15.24, and 242.47.

Since present values and Macaulay durations need to match for the assets and liabilities, we have the two equations

$$x+y+z=9697, \quad \frac{5x+15y+20z}{x+y+z}=15.24.$$

Note that 5 and 15 are both less than the desired Macaulay duration 15.24, so z cannot be zero. So try either the 5-year and 20-year bonds (i.e. $y = 0$), or the 15-year and 20-year bonds (i.e. $x = 0$).

In the former case, substituting $y = 0$ and solving for *x* and *z* yields

$$x=\frac{(20-15.24)9697}{20-5}=3077.18 \text{ and } z=\frac{(15.24-5)9697}{20-5}=6619.82.$$

We need to check if the Macaulay convexity of the assets exceeds that of the liabilities.

The Macaulay convexity of the assets is $\frac{5^2(3077.18)+20^2(6619.82)}{9697}=281.00$, which exceeds the Macaulay convexity of the liabilities, 242.47. The company should invest 3077 for the 5-year bond and 6620 for the 20-year bond.

Note that setting $x = 0$ produces $y = 9231.54$ and $z = 465.46$ and the convexity is 233.40, which is less than that of the liabilities.

132. Solution: E

The correct answer is the lowest cost portfolio that provides for \$11,000 at the end of year one and provides for \$12,100 at the end of year two. Let *H*, *I*, and *J* represent the face amount of each purchased bond. The time one payment can be exactly matched with $H + 0.12J = 11,000$. The time two payment can be matched with $I + 1.12J = 12,100$. The cost of the three bonds is $H/1.1 + I/1.2321 + J$. This function is to be minimized under the two constraints. Substituting for *H* and *I* gives $(11,000 – 0.12J)/1.1 + (12,100 – 1.12J)/1.2321 + J = 19,820 – 0.0181J$. This is minimized by purchasing the largest possible amount of *J*. This is 12,100/1.12 = 10,803.57. Then, $H = 11,000 – 0.12(10,803.57) = 9703.57$. The cost of Bond H is 9703.57/1.1 = 8,821.43.

133. Solution: C

The strategy is to use the two highest yielding assets: the one year zero coupon bond and the two year zero coupon bond. The cost of these bonds is $25,000/1.0675+20,000/1.05^2=41,560$.