Eigen values and Eigen vectors :

Def : Let $T: V \to V$ be a Linear transformation.
$0 \neq v \in V$ is called an eigen vector iff
$T(v) = \lambda v$ for some $\lambda \in \mathbb{R}$. and,

the scalar $\lambda \in \mathbb{R}$ is called an eigen value iff
there exists $0 \neq v \in V$ such that $T(v) = \lambda v$.

Remarks

① The eigen vector always doesn't equal to zero vector. But, the eigen value might equal to zero.

② If $T: V \to V$ is Linear transformation, then it is not necessary that there are eigen vectors and eigen values.

Example

① Let $I: V \to V$ be the identity Linear transformation, i.e., $I(v) = v$ for every $v \in V$. Then Every $v \in V$ is an eigen vector, and only 1 is the eigen value.

② Let $O: V \to V$ be the zero-mp, i.e., $O(v) = 0$ for all $v \in V$. The only eiger vector is $0 \in V$, ~~and the~~ ~~only eigen value is $1 \in \mathbb{R}$ because~~ and the eigen values are all $\lambda \in \mathbb{R}$ for $O(0) = 0 \iff O(0) = \underbrace{\lambda}_{\in \mathbb{R}} \cdot 0$

How to find Eigen values and Eigen vectors:

first of all, we need to the following definition.

Def: Let $T: V \longrightarrow V$ be a Linear transform- where A is the standered matrix of T.

Δ = The characterstic Polynomial

$= |A - \lambda I|$ where $\lambda \in \mathbb{R}$ and I is the unite matrix.

for example

Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ where $T(x,y) = (2x-y, 4x)$.

The standered basis of $\mathbb{R}^2 = \{(1,0), (0,1)\}$

$$\left.\begin{array}{l} T(1,0) = (2,4) \\ T(0,1) = (-1,0) \end{array}\right\} \Rightarrow A = \begin{bmatrix} 2 & -1 \\ 4 & 0 \end{bmatrix}$$

$$\text{So, } \Delta = \left| \begin{bmatrix} 2 & -1 \\ 4 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} 2-\lambda & -1 \\ 4 & -\lambda \end{vmatrix}$$

$$\cancel{= (2-\lambda) + (4-\lambda) = 6-2}$$

$$= (2-\lambda)(-\lambda) + 4 = \lambda^2 - 2\lambda + 4$$

Notice that; it is a polynomial

To find eigen values

Put $\Delta = 0$

To find eigen ~~values~~ vectors

STEP1 find the eigen value λ.

STEP2 the eigen vector respects to λ is $(A - \lambda I)X = 0$ where A is the standered matrix.

Example : If $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$ then find the eigen values and the eigen vectors of A ?

Solution : STEP 1 : we will find the eigen values of A

Put $\Delta = 0 \iff |A - \lambda I| = 0$

$$\iff \left| \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\iff \lambda^2 + 3\lambda + 2 = 0$$

$$\iff \boxed{\lambda = -1} \text{ or } \boxed{\lambda = -2}$$

STEP 2 : we will find the eigen vector respects to $\lambda = -1$:-

Put $(A - \lambda I) X = 0$

$$\iff \left(\begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\iff \begin{bmatrix} 3 & -12 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By solving the system

$x_1 = t \quad x_2 = \frac{1}{4} t$

so, $X = \begin{bmatrix} 1 \\ 1/4 \end{bmatrix} t$

where $\boxed{t \neq 0}$

Because the eigen vector $\neq 0$

STEP 3 : we will find the eigen vector respects to $\lambda = -2$:

The same calculations of step 2

(complete !!)

(Ex) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a Linear transfor where $T(x,y) = (2x-y, 4x)$. Find the eigen values and the eigen vectors of T ?

Solution

STEP 1 : we will find the standered matrix A :-

$$B_{\mathbb{R}^2} = \{(1,0), (0,1)\}$$

$T(1,0) = (2,4)$ $\quad$ $T(0,1) = (-1,0)$

So, $A = \begin{bmatrix} 2 & -1 \\ 4 & 0 \end{bmatrix}$

STEP 2 : To find the eigen values :

Put $\Delta = 0 \iff |A - \lambda I| = 0$

$$\iff \begin{vmatrix} 2-\lambda & -1 \\ 4 & -\lambda \end{vmatrix} = 0$$

$$\iff \lambda^2 - 2\lambda + 4 = 0$$

$$\iff \lambda = \frac{2 \pm \sqrt{-12}}{2} \notin \mathbb{R}$$

So, there are no eigen values, and there there are no eigen vectors !!

(Ex) Let $T : P_1(x) \to P_1(x)$ where $T(a+bx) = -b + ax$. find the eigen values of T ?

Solution step 1 we will find the standered matrix A :

$$B_{P_1(x)} = \{1 + x\}$$

$T(1) = T(1 + 0x) = x = 0(1) + 1x$

$T(x) = T(0 + x) = -1 = -1(1) + 0x$

Therefore

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad (\text{matrix of coordinates})$$

STEP 2 To find the eigen values, Put $\Delta = 0$

So, $|A - \lambda I| = 0 \iff \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$

$\iff \lambda^2 = -1 \Rightarrow \lambda = \pm\sqrt{-1}$

$\Rightarrow \lambda \notin \mathbb{R}$

$\Rightarrow$ there are no eigen values.

(Ex) Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Find the eigen values and the eigen vectors of A ?

Solution (STEP 1) we will find the eigen values

Put $\Delta = 0 \iff |A - \lambda I| = 0$

$\iff \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$

$\iff (2-\lambda)^2 = 0 \Rightarrow \boxed{\lambda = 2}$

(STEP 2) To find the eigen vector respects to $\lambda = 2$, Put

$(A - 2I)X = 0$

$\iff \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\iff x_2 = 0$

So, the solution $X = \begin{bmatrix} r \\ 0 \\ s \end{bmatrix}$

$= \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} r + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} s$

$\boxed{\text{where } X \neq 0}$

Eigen space: It is the space of all eigen vectors respects to λ.

for example: Let $T: \mathbb{R}^3 \to \mathbb{R}^3$; $T(x,y,z) = (x,y,-2z)$

STEP1 $B_{\mathbb{R}^3} = \{(1,0,0), (0,1,0), (0,0,1)\}$

$$\left.\begin{array}{l} T(1,0,0) = (1,0,0) \\ T(0,1,0) = (0,1,0) \\ T(0,0,1) = (0,0,-2) \end{array}\right\} \Rightarrow A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

STEP2 To find the eigen values:

Put $\Delta = 0 \Leftrightarrow |A - \lambda I| = 0 \Leftrightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$

Called multiplicity of λ

$\Leftrightarrow (1-\lambda)^2 (-2-\lambda)^1 = 0$

$\Leftrightarrow \boxed{\lambda = 1}$ or $\boxed{\lambda = -2} \in \mathbb{R}$

STEP3 To find the eigen vectors respects to $\lambda = 1$:

Put $(A - \lambda I) X = 0$

$$\Leftrightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Leftrightarrow -3x_3 = 0 \Rightarrow x_3 = 0$

So, $S = \left\{ \begin{bmatrix} r \\ s \\ 0 \end{bmatrix} ; r, s \in \mathbb{R} \right\}$

$= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} r + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} s ; r, s \in \mathbb{R} \right\}$

S is called the eigen space of $\lambda = 1$

Rule $\leftarrow$ $\boxed{\text{Dim}(S) = 2 \xleftrightarrow{\leq} \text{the multicipility of } \lambda}$

STEP4: Do the same for $\lambda = -2$.