1st Midterm Exam Solution

استعن بالله وكن على يقين بأن كل ما ورد في هذه الورقة تعرفه جيدا وقد تدربت عليه بما فيه الكفاية

Student's Name		**Student's Number**	

Question #1:

Answer the following with ***True*** or ***False***:

FALSE 1. Simulation is always best used for problem that can be solved by common mathematical or analytically solution.

TRUE 2. Simulation modeling is not good if there is less data or no estimates available.

TRUE 3. It is always best to use simulation models to evaluate alternatives for complex or highly expensive systems.

FALSE 4. A system is a set of separate components that work independently towards a common goal.

TRUE 5. The sequence of random numbers generated from a given seed is called a random number a *Stream*.

FALSE 6. Inverse transform is used in simulation to generate random numbers between 0 and 1 only.

FALSE 7. In using *Acceptance/Rejection* method to generate from a function $f(x)$, each U(0,1) random number gives one random follows the function $f(x)$.

TRUE 8. It is possible to generate a random from Binomial distribution using *Convolution Method*.

TRUE 9. The function VLOOKUP in Excel always used to generate discrete numbers.

FALSE 10. The LCG ($X_n = (aX_{n-1} + c) \bmod(m)$) it is possible to find values for: X_0 , a , c and m to generate more than m pseudo-random numbers.

TRUE 11. The LCG is always used to generate pseudo-random numbers between 0 and 1.

Question #2:

Conceder the following LCG generator: $X_n = (13\,X_{n-1} + 13)\ mod\ (16)$, $X_0 = 14$
Answer the following:

a) Generate all possible uniform pseudo-random numbers from the above LCG.

b) Does this generator achieve the ***Full Cycle***? Justify your answer.

(a) all possible uniform pseudo-random is listed below:

n	X_{n-1}	$X_n = (13\,X_{n-1} + 13)\ mod\ (16)$	$U_n = X_n/16$
1	14	X_1 = (13(14)+13) mod 16 = 3	0.1875
2	3	X_2 = (13(3)+13) mod 16 =4	0.2500
3	4	X_3 = (13(4)+13) mod 16 = 1	0.0625
4	1	X_4 = (13(1)+13) mod 16 =10	0.6250
5	10	X_5 = (13(10)+13) mod 16 = 15	0.9375
6	15	X_6 = (13(15)+13) mod 16 = 0	0.0000
7	0	X_7 = (13(0)+13) mod 16 = 13	0.8125
8	13	X_8 = (13(13)+13) mod 16 = 6	0.3750
9	6	X_9 = (13(16)+13) mod 16 = 11	0.6875
10	11	X_{10} = (13(11)+13) mod 16 = 12	0.7500
11	12	X_{11} = (13(12)+13) mod 16 = 9	0.5625
12	9	X_{12} = (13(9)+13) mod 16 = 2	0.1250
13	2	X_{13} = (13(2)+13) mod 16 = 7	0.4375
14	7	X_{14} = (13(7)+13) mod 16 = 8	0.5000
15	8	X_{15} = (13(8)+13) mod 16 = 5	0.3125
16	5	X_{16} = (13(5)+13) mod 16 = 14	0.8750

The starting value X_n = 14 is repeated. Then STOP calculations

(b) The given LCG has a FULL CYCLE because number of generated pseudo-numbers is equal to m = 16 number.

Question #3:

Consider the following discrete distribution of the random variable X whose probability mass function is p(x).

X	-2	-1	0	2	8
P(X=x)	0.1	0.25	0.3	0.2	0.15
CDF(X)	**0.1**	**0.35**	**0.65**	**0.85**	**1.0**

(a)

a) Determine the CDF F(x) for the random variable, X.
b) Determine the average and variance of the random variable, X.
c) Find the inverse transform function to generate random numbers from P(X=x).
d) Generate the values of X using the following sequence of (0,1) random numbers.

n	1	2	3	4	5	6
$U_n(0,1)$	0.3237	0.6723	0.5649	0.9804	0.0356	0.3807
$\boldsymbol{F^{-1}(U)}$	**-1**	**2**	**0**	**8**	**-2**	**0**

(d)

(b) average X = E[x] = $\sum_1^8 xP(x)$

= (0.1)(-2) +(0.25)(-1) + (0.3)(0) + (0.2)(2) + (0.15)(8) = 1.15

variance X = Var[x] = $\sum_1^8 x(x - E[x])^2 P(x)$

= $(0.1)(-2-1.15)^2 + (0.25)(-1-1.15)^2 + (0.3)(0-1.15)^2$

$+ (0.2)(2-1.15)^2 + (0.15)(8-1.15)^2$ = 9.7275

(c) inverse transform function of P(X)

$$X = F^{-1}(u) = \begin{cases} -2 & ; \quad 0 \le u \le 0.1 \\ -1 & ; \quad 0.1 < u \le 0.35 \\ 0 & ; \quad 0.35 < u \le 0.65 \\ 2 & ; \quad 0.65 < u \le 0.85 \\ 8 & ; \quad 0.85 < u \le 0.1 \end{cases}$$

Question #4:

Consider the following set of pseudo-random numbers.

1	0.7551
2	0.8469
3	0.2268
4	0.6964
5	0.842
6	0.8075
7	0.5480
8	0.4047
9	0.6545
10	0.3427

α	0.10	0.05	0.025	0.01	0.005	0.001
D_α	1.22	1.36	1.48	1.63	1.73	1.95
$\chi^2_{(\alpha,9)}$	4.865	3.940	3.247	2.558	2.156	1.479

a) Test the hypothesis that these numbers are drawn from a U (0, 1) at a 95% confidence level using the Chi-squared goodness of fit test using **4 intervals**.
b) Test the hypothesis that these numbers are drawn from a U (0, 1) at a 95% confidence level using K-S Test.

(a) Chi-squared goodness

n=10 and k=4 then $p_j = 1/4 = 0.25$

j	b_{j-1}	b_j	C_j	np_j	$(c_j-np_j)^2/ np_j$
1	0.00	0.25	1	2.5	0.9
2	0.25	0.50	3	2.5	0.1
3	0.50	0.75	3	2.5	0.1
4	0.75	1.00	3	2.5	0.1

$\chi_0^2 = \frac{\sum_0^4 (c_j - n\,p_j)}{n\,p_j}$ =1.2 We have $\chi_\alpha^2 = 3.94$.

Since , $\chi_0^2 < \chi_\alpha^2$ Then *ACCEPT H_0*

which means that the data are taken from U[0,1]

(b) K-S Test

j	$X_{(j)}$	j/n	(j-1)/n	$F(X_{(j)})$	(j/n)- $F(X_{(j)})$	$F(X_{(j)})$- (j-1/n)
1	0.2268	0.1	0.0	0.2268	-0.1268	0.2268
2	0.3427	0.2	0.1	0.3427	-0.1427	0.2427
3	0.4047	0.3	0.2	0.4047	-0.1047	0.2047
4	0.5480	0.4	0.3	0.5480	-0.148	0.248
5	0.6545	0.5	0.4	0.6545	-0.1545	0.2545
6	0.6964	0.6	0.5	0.6964	-0.0964	0.1964
7	0.7551	0.7	0.6	0.7551	-0.0551	0.1551
8	0.8075	0.8	0.7	0.8075	-0.0075	0.1075
9	0.842	0.9	0.8	0.842	0.058	0.042
10	0.8469	1.0	0.9	0.8469	0.1531	-0.0531

We have $D^+ = 0.1531$ and $D^- = 0.2545$.

Then $D_n = \max\{ D^+ = 0.1531 ,\ D^- = 0.2545\} = 0.2545$

Also we have $D_\alpha = 1.36$

Since , $D_n < D_\alpha$ Then ***ACCEPT*** H_0

which means that the data are taken from U[0,1]

Question #5:

Consider the following probability function:

$$f(x) = \begin{cases} \frac{4}{7}x^3 ; & -1 \leq x \leq 2 \\ 0 ; & otherwise \end{cases}$$

n	1	2	3	4	5	6	7	8
U_n(0,1)	0.3045	0.6964	0.1709	0.3387	0.9804	0.1246	0.842	0.6557
X	**1.3302**	**1.5568**	**1.2173**	**1.3549**	**1.6745**			

a) Find the inverse transform of the probability and **generate 5 random** numbers from *f*(*x*) using the table of U(0,1) random numbers above.
b) Use acceptance/rejection method to **generate 2 random numbers** from from *f*(*x*) using the table of U(0,1) random numbers above.

(a) Inverse transform

1. Get the CDF of *f(x)*

$$f(x) = \int_{-1}^{x} \frac{4}{7}y^3 \, dy = \frac{4}{7}\left[\frac{y^4}{4}\right]_{-1}^{x} = \frac{1}{7}(x^4 - 1)$$

2. Let u = F(x) and solve for x

$$u = \frac{1}{7}(x^4 - 1) \iff 7u = (x^4 - 1) \iff 7u + 1 = x^4$$

$$Then \ \ x = \sqrt[4]{7u + 1}$$

(b) acceptance/rejection method

1. Get the function G(x) = max *f(x)*

$$\frac{d}{dx}f(x) = \frac{12}{7}x^2 = 0$$

Then , G(x) = max{ f(0) = 0 , f(-1) = 0.57 , f(2) = 4.58} = 4.58

2. Get the value of constant *c*

$$c = \int_{-1}^{2} 4.58 \; dx = [4.58\,x]_{-1}^{2} = 4.58(2+1) = 13.74$$

3. Get the pdf h(x) :

$$h(x) = \frac{G(x)}{c} = \frac{4.58}{13.74} = \frac{1}{3} \quad for \quad -1 \le x \le 2$$

4. Get the inverse function of h(x)

Let w~h(x) then: W = -1 + 3u where $u \sim U[0,1]$

5. Start generating random numbers from f(x):

First number from f(x):

- Generate u_1=0.3045
- Evaluate W = -1 + 3u_1 = -0.0865
- Evaluate f(W) = $\frac{4}{7}(-0.0865)^3 = 0.00004$
- Generate new u_2 = 0.6964
- If f(W)/G(W) < u_2 accept W , then we have -0.00004/4.58 < u_2 Then the 1st value from f(x) is W= -0.0865

Second number from f(x):

- Generate u_1=0.1709
- Evaluate W = -1 + 3u_1 = -0.4873
- Evaluate f(W) = $\frac{4}{7}(-0.4873)^3 = -0.066$
- Generate new u_2 = 0.3387
- If f(W)/G(W) < u_2 accept W , then we have -0.066/4.58 < u_2 Then the 1st value from f(x) is W= -0.4873

Question #6:

Customers arrive to a minimarket according to a Poisson process with arrival rate λ = 15 customers per hour. The arriving customers come to a single server checkout counter after they finish shopping. It is estimated that the checkout sever takes a random amount of time to finish the checkout for a customer. The service time follows an exponential distribution with mean 5 minutes. The server calculated that customers purchase from the market with an amount between 10 SR and 50 SR following the probability function

X	10	20	30	40	50
$P(X=x)$	0.2	0.25	0.3	0.15	0.1
$P(X<=x)$	**0.2**	**0.45**	**0.75**	**0.9**	**1.0**

1. Complete the following Simulation table in the next page?
   See next Page
   - **Col. # 3 [Time between arrivals (min)]=**
     Generate from Exponential (λ=15 cust./hr) :
     ***t* (min) = -(60/15)ln(1-u) where u~U[0,1]**
   - **Col. #4 [Arrival time (min)] = [Arrival time (min) of last customer] + [Time between arrivals (min)]**
   - **Col. # 6[Service time (min)] =**
     Generate from Exponential (λ=1/5 cust./min) :
     ***t* (min) = -(5)ln(1-u) where u~U[0,1]**
   - **Col. #7 [Service start (min)]**
     - **If waiting time = 0 then service start = arrival time of the same customer**
     - **if waiting time = 0 then service start = departure time of the last customer**
   - **Col. # 8[Cust. WITE?]**
     - **If service start = arrival time then Cust. WITE? = 0**
     - **If service start > arrival time then Cust. WITE? = 1**
   - **Col. #10 [Departure time (min)] = [Arrival time (min) of the customer] + [service Time (min)] + [Wait Time (min)]**
   - **Col. #13 [Money Spent (SR)] =**
     Generate from discrete function P(X=x)

2. What is the average waiting time (W)?
   waiting time (W) is a simple average = average of the [Waiting Time] column = 19.01/10 = 1.901 min.

3. What is the total money collected during the simulation (TM)?
   total money collected during the simulation (TM) = sum of last column [Money Spent (SR)] = 230 SR

4. From the simulation run, what is the average money spent by any customer?
   the average money spent per customer is a simple average = average of the last column [Money Spent (SR)] = 230/10 = 23 SR.

5. What is the percentage of customers spending 30 SR or less in the simulation run?
   percentage of customers spending 30 SR or less = count (# of 30SR) and (# of 20 SR) and (# of 10 SR) and divide by 10 = 10/10 = 1.00

6. What is the probability that the cashier is IDLE during the ***simulation Time***?
   probability that the cashier is IDLE is a time average
   = (sum of intervals that server is idle)/ (Total Simulation Time)
   = (0.24 + 0.36 + 3.48 + 7.77)/(39.56) = 11.85/39.56 = 0.2995

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Question #6:

Col#1	Col#2	Col#3	Col#4	Col#5	Col#6	Col#7	Col#8	Col#9	Col#10	Col#11	Col#12	Col#13
Cust. #	U[0,1]	Time between arrivals (min)	Arrival time (min)	U[0,1]	Service time (min)	Service start (min)	Cust. WITE?	Wait Time (min)	Departure time (min)	Cashire Idle Time (min)	U[0,1]	Money Spent (SR)
1	0.059	0.24	0.24	0.105	0.33	0.24	0	0.00	0.58	0.24	0.736	30
2	0.159	0.69	0.93	0.503	2.10	0.93	0	0.00	3.03	0.36	0.376	20
3	0.186	0.82	1.76	0.958	9.51	3.03	1	1.27	12.54	0.00	0.734	30
4	0.852	7.63	9.39	0.759	4.27	12.54	1	3.15	16.81	0.00	0.386	20
5	0.550	3.19	12.58	0.755	4.22	16.81	1	4.23	21.03	0.00	0.372	20
6	0.342	1.67	14.26	0.377	1.42	21.03	1	6.77	22.45	0.00	0.349	20
7	0.716	5.03	19.29	0.152	0.49	22.45	1	3.16	22.94	0.00	0.527	30
8	0.554	3.23	22.52	0.399	1.53	22.94	1	0.42	24.47	0.00	0.185	10
9	0.742	5.42	27.94	0.527	2.25	27.94	0	0.00	30.19	3.48	0.350	20
10	0.918	10.01	37.96	0.415	1.61	37.96	0	0.00	39.56	7.77	0.613	30