2.6 Exercises

1. Use bisection method to find solutions accurate to within 10^{-4} on the interval $[-5, 5]$ of the following functions:
   (a) $f(x) = x^5 - 10x^3 - 4$, *(b)* $f(x) = 2x^2 + \ln(x) - 3$, *(c)* $f(x) = \ln(x) + 30e^{-x} - 3$.

2. The following equations have a root in the interval $[0, 1.6]$. Determine these with an error less than 10^{-4} using bisection method. (a) $2x - e^{-x} = 0$; (b) $e^{-3x} + 2x - 2 = 0$.

3. Estimate the number of iterations needed to achieve an approximation with accuracy 10^{-4} to the solution of $f(x) = x^3 + 4x^2 + 4x - 4$ lying in the interval $[0, 1]$ using bisection method.

4. Use the bisection method for $f(x) = x^3 - 3x + 1$ in $[1, 3]$ to find the first eight approximation to the root of the given equation. Compute an error estimate $|\alpha - x_8|$.

5. The cubic equation $x^3 - 3x - 20 = 0$ can be written as

   $$(a)\quad x = \frac{(x^3 - 20)}{3}, \quad (b)\quad x = \frac{3}{(x^3 - 3)}, \quad (c)\quad x = (3x + 20)^{1/3}.$$

   Choose the form which satisfies the condition $|g'(x)| < 1$ on $[3, 4]$ and then find third approximation x_3 when $x_0 = 3.5$.

6. Find value of k such that the iterative scheme $x_{n+1} = \dfrac{x_n^2 - 4kx_n + 7}{4}$, $n \geq 0$ converges to 1. Also, find the rate of convergence of the iterative scheme.

7. Write the equation $x^2 - 6x + 5 = 0$ in the form $x = g(x)$, where $x \in [0, 2]$, so that the iteration $x_{n+1} = g(x_n)$ will converge to the root of the given equation for $x_0 \in [0, 2]$.

8. Which of the following iterations

   $$(a)\quad x_{n+1} = \frac{1}{4}\left(x_n^2 + \frac{6}{x_n}\right), \qquad (b)\quad x_{n+1} = \left(4 - \frac{6}{x_n^2}\right)$$

   is suitable to find a root of the equation $x^3 = 4x^2 - 6$ in the interval $[3, 4]$? Estimate the number of iterations required to achieve 10^{-3} accuracy, starting from $x_0 = 3$.

9. Let $f(x) = e^x + 3x^2$. Find Newton's formula $g(x_k)$. Start with $x_0 = 4$ and $x_0 = -0.5$, find x_4.

10. Use Newton's formula for the reciprocal of square root of a number 15 and then find the 3rd approximation of number, with $x_0 = 0.05$.

11. Find Newton's formula for $f(x) = x^3 - 3x + 1$ in $[1, 3]$ to calculate x_3, if $x_0 = 1.5$. Also, find the rate of convergence of the method.

12. Rewrite the nonlinear equation $g(x) = \dfrac{1}{2}e^{0.5x}$ which defined in the interval $[0, 1]$ in the equivalent form $f(x) = 0$ and then use the Newton's method with $x_0 = 0.5$ to find third approximation x_3.

13. Find x_4 for $x^3 - 2x - 5 = 0$ by secant method using $x_0 = 2$ and $x_1 = 3$.

14. Solve the equation $e^{-x} - x = 0$ by secant method,using $x_0 = 0$ and $x_1 = 1$, accurate to 10^{-4}.

15. Use secant method to find a solution accurate to within 10^{-4} for $\ln(x) + x - 5 = 0$ on $[3, 4]$.

16. Find the root of multiplicity of the function $f(x) = (x-1)^2 \ln(x)$ at $\alpha = 1$.

17. Show that the root of multiplicity of the function $f(x) = x^4 - x^3 - 3x^2 + 5x - 2$ is 3 at $\alpha = 1$. Estimate the number of iterations required to solve the problem with accuracy 10^{-4}, start with the starting value $x_0 = 0.5$ by using: *(a)* Newton's method; *(b)* First modified Newton's method; *(c)* Second modified Newton's method.

18. If $f(x), f'(x)$ and $f''(x)$ are continuous and bounded on a certain interval containing $x = \alpha$ and if both $f(\alpha) = 0$ and $f'(\alpha) = 0$ but $f''(\alpha) \neq 0$, show that $x_{n+1} = x_n - 2\dfrac{f(x_n)}{f'(x_n)}$ will converge quadratically if x_n is in the interval.

19. Show that iterative scheme $x_{n+1} = 1 + x_n - \dfrac{x_n^2}{2}$, $n \geq 0$ converges to $\sqrt{2}$. Find the rate of convergence of the sequence.

20. Solve the following system using the Newton's method:

$$4x^3 + y = 6 \quad \text{and} \quad x^2 y = 1.$$

Use $x_0 = y_0 = 1$ and stop when successive iterates differ by less than 10^{-7}.

21. Solve the following system using the Newton's method:

$$x + e^y = 68.1 \quad \text{and} \quad \sin x - y = -3.6.$$

Start with initial approximation $x_0 = 2.5,\ y_0 = 4$, compute the first three approximations.