

8. Functions of Random Variables

Chapter 6

- Method of Distribution Functions
- One-to-One Transformations.
- Two-to Two Transformations. (Joint distribution of Functions of Random Variables)
- Method of Moment-Generating Functions.

Q1) If $X \sim \text{Uniform}(0,1)$, find the pdf of $Y = -2\ln X$. Name the distribution and its parameter values.

Q2) If $X \sim \text{Uniform}(a, b)$, find the constants c and d such that $Y = c + dX \sim \text{Uniform}(0,1)$.

Q3) If $X \sim \text{Normal}(\mu, \sigma^2)$, find the pdf of $Y = e^X$.

Q4) If $X \sim \text{Exponential}(1)$, find the pdf of $Y = -\ln X$.

Q5) If $X \sim \text{Uniform}(0,1)$, find the pdf of $Y = \sqrt{X}$.

Q6) The pdf of X is given by $f_X(x) = \frac{1}{2}x$; $0 < x < 2$.

a. Find the pdf of $Y = X^3$.

b. Find $P\left(\frac{1}{2} < X < 1\right)$ and $P\left(\frac{1}{8} < Y < 1\right)$. Are they the same or different? Why?

Q7) If $X \sim \chi_4^2$, find $P(X > 5)$.

Q8) If $X \sim \text{Uniform}(0,1)$ independent of $Y \sim \text{Exponential}(1)$, find the distribution of $Z = X + Y$:

a. Using the pdf formula derived in class.

b. By first finding the cdf and then differentiating.

Q9) If $X \sim \text{Gamma}(2,3)$ independent of $Y \sim \text{Uniform}(0,2)$, and $Z \sim \text{Gamma}(5,3)$, what is the distribution of $X+Y+Z$ if X , Y and Z are independent?

Q10) If $X \sim \text{Normal}(2,3)$ independent of $Y \sim \text{Normal}(5,1)$, and $Z \sim \text{Normal}(20,21)$, with X , Y and Z independent, find $P(X+Y+Z < 25)$.

Q11) Let X and Y have joint pdf $f(x, y) = 1; -y < x < y, 0 < y < 1$.

- Find the conditional pdf of $X|Y=y$.
- Find $P(X < 0|Y = y)$.
- Find $P(X > \frac{1}{4}|Y = \frac{1}{3})$.
- Find $P(0 < X < \frac{1}{4}|Y = \frac{1}{2})$.

Q12) Let X and Y have joint pdf $f(x, y) = \frac{2}{5}(x + 4y); 0 < x < 1, 0 < y < 1$.

- Find the conditional pdf of $Y|X=x$.
- Find $P(Y < \frac{1}{3}|X = \frac{1}{2})$.

Q13) If $X \sim \text{Uniform}(0,1)$ independent of $Y \sim \text{Exponential}(1)$, find

- The joint density function of $Z=X+Y$ and $U=X/Y$.
- The density function of Z .
- The density function of U .

Q14) Let (X,Y) have joint pdf $f(x, y) = \frac{1}{x^2y^2}; x \geq 1, y \geq 1$.

- Find the joint density of $U=XY$ and $V=X/Y$.
- What are the marginal density of U and V ?

Q15) Let X_1 and X_2 be independent $Exp(\lambda)$ r.v. Find the joint density of

$$Y_1 = X_1 + X_2 \text{ and } Y_2 = e^{X_1}.$$

Q16) Let $X_1 \sim Exp(\lambda_1)$ independent of $X_2 \sim Exp(\lambda_2)$ r.v.. Find:

a. The cumulative distribution function of $Z = \frac{X_1}{X_2}$.

b. $P(X_1 < X_2)$.

Q17) The joint pdf of (X, Y) is given by $f(x, y) = \frac{e^{-y}}{y}; 0 < x < y, 0 < y < \infty$. Find $E(X)$, $E(Y)$, $V(X)$, $V(Y)$ and $Cov(X, Y)$.

Q18) Let X and Y be distributed as independent $Uniform(0,1)$ r.v.

a. Find the joint density function of $Z_1 = X+Y$ and $Z_2 = Y$.

b. Find the marginal pdf of Z_1 from the joint density.

Q19) Let X and Y be distributed as independent $Exp(1)$ r.v., find:

a. The joint density function of $Z = X + Y$ and $U = \frac{X}{X+Y}$.

b. Find the marginal pdf of U .

Q20) Let (X, Y) have joint density given by $f(x, y) = 24xy; 0 < x < 1, 0 < y < 1, x + y < 1$, find the pdf of $Z = XY^2$.

Q21) Let X and Y have independent $Gamma(\alpha, \lambda)$ distributions.

a. Find the joint pdf of $U = \frac{X}{X+Y}$ and $V = X + Y$.

b. Show that the marginal density of U is a Beta distribution.

Q22) Let (X, Y) have joint density given by $f(x, y) = 24xy; 0 < x < 1, 0 < y < 1, x + y < 1$, find:

- a. The marginal pdf's.
- b. The following expectations:
- i. $E(X)$ and $E(X^2)$.
 - ii. $E(Y)$ and $E(Y^2)$.
 - iii. $E(XY)$ and $E(X^2 Y^3)$.
 - iv. $V(X)$, $V(Y)$, $\text{Cov}(X,Y)$. Do X and Y have a positive or negative relationship?

Q23) Let joint pdf of (X,Y) given by $f(x,y) = \frac{1}{y} e^{-y} e^{-x/y}$; $x > 0, y > 0$, find:

- a. $E(X)$ and $E(X^2)$.
- b. $E(Y)$ and $E(Y^2)$.
- c. Show that $\text{Cov}(X,Y)=1$.
- d. $\rho(X,Y)$.

Q24) If $X, Y, Z \sim$ independent $\text{Exp}(1)$, derive the joint distribution of $U=X+Y$, $V=X+Z$, and $Z=Y+Z$.

- If X_i indpt. $\text{Exp}(\lambda)$, then the sum $\sum_{i=1}^{i=n} X_i \sim \text{Gamma}(n, \lambda)$
- If X_i indpt. $\text{Gamma}(\alpha_i, \beta)$, then the sum $\sum_{i=1}^{i=n} X_i \sim \text{Gamma}(\sum_{i=1}^{i=n} \alpha_i, \beta)$
- If X_i indpt. $\text{Normal}(\mu_i, \sigma_i^2)$, then the sum $\sum_{i=1}^{i=n} X_i \sim \text{Normal}(\sum_{i=1}^{i=n} \mu_i, \sum_{i=1}^{i=n} \sigma_i^2)$
- If X_i indpt. $\text{Normal}(\mu_0, \sigma_0^2)$, then the sum $\sum_{i=1}^{i=n} X_i \sim \text{Normal}(n\mu_0, n\sigma_0^2)$
- If $Z \sim \text{Normal}(0,1)$, then the $Z^2 \sim \chi_1^2$
- If $X \sim \chi_n^2$, ind. of $Y \sim \chi_m^2$, then the $X + Y \sim \chi_{n+m}^2$
- If $Z_1 \sim \text{Normal}(0,1)$, ind. of $Z_2 \sim \text{Normal}(0,1)$ then the $Z_1 + Z_2 \sim \chi_2^2$