Laplace Transformation and Differential equations

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March 17, 2020

Exercise

Use the Laplace transform to solve the initial-value problem

$$y' + y = e^{-x} + e^{x} + \cos x + \sin x, \quad y(0) = 1.$$

Solution 1 :

We begin by taking the Laplace transform of both sides to achieve

$$\mathcal{L}\left[y'\right] + \mathcal{L}[y] = \mathcal{L}(e^{-x} + e^{x} + \cos x + \sin x).$$

We know that

$\mathcal{L}(e^{-x} + e^{x} + \cos x + \sin x) = \dfrac{1}{s+1} + \dfrac{1}{s-1} + \dfrac{s}{s^2+1} + \dfrac{1}{s^2+1}.$

Denote $Y(s) = \mathcal{L}[y]$, then

$$\begin{aligned} Y(s) &= \frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right) + \frac{s}{(s+1)(s^2+1)} \\ &\quad + \frac{1}{(s+1)(s^2+1)} \\ &= \frac{1}{2(s+1)} + \frac{1}{(s+1)^2} + \frac{1}{2(s-1)} + \frac{1}{s^2+1}. \end{aligned}$$

The solution of the differential equation:

$$y = \frac{1}{2}e^{-x} + xe^{-x} + \frac{1}{2}e^{x} + \sin x.$$

Exercise

Use Laplace transforms to solve the initial-value problem

$$y' - 2y = 5 + \cos x + e^{2x} + e^{-x}, \quad y(0) = 4.$$

Solution 2 :

By taking the Laplace transform of both sides of the differential equation, we get: $sY - 4 - 2Y = \dfrac{5}{s} + \dfrac{s}{s^2+1} + \dfrac{1}{s-2} + \dfrac{1}{s+1}$.

Then

$$\begin{aligned}
Y(s) &= \frac{4}{s-2} + \frac{5}{s(s-2)} + \frac{s}{(s-2)(s^2+1)} + \frac{1}{(s-2)^2} \\
&\quad + \frac{1}{(s+1)(s-2)} \\
&= \frac{4}{s-2} + \frac{5}{2}\left(\frac{1}{s-2} - \frac{1}{s}\right) + \frac{2}{5}\frac{1}{s-2} + \frac{1}{5}\frac{-2s+1}{s^2+1} \\
&\quad + \frac{1}{(s-2)^2} + \frac{1}{3}\frac{1}{s-2} - \frac{1}{3}\frac{1}{s+1} \\
&= \frac{217}{30(s-2)} - \frac{5}{2s} - \frac{2}{5}\frac{s}{s^2+1} + \frac{1}{5}\frac{1}{s^2+1} \\
&\quad + \frac{1}{(s-2)^2} - \frac{1}{3}\frac{1}{s+1}
\end{aligned}$$

and

$$y(x) = \frac{217}{30}e^{2x} - \frac{5}{2} - \frac{2}{5}\cos x + \frac{1}{5}\sin x + xe^{2x} - \frac{1}{3}e^{-x}.$$

Exercise

Use the Laplace transform to solve the initial-value problem

$$y'' - 2y' + 2y = \cos x, \quad y(0) = 1, y'(0) = -1$$

Solution 3 :

Using the Laplace transform of both sides of the differential equation, we get:

$$s^2 Y(s) - s + 1 - 2(sY(s) - 1) + 2Y(s) = \frac{s}{s^2+1}.$$

Then $Y(s) = \dfrac{s-3}{(s-1)^2+1} + \dfrac{s}{(s^2+1)((s-1)^2+1)}$. Solving for $Y(s)$, we find:

$$\begin{aligned} Y(s) &= \frac{s-1}{(s-1)^2+1} - \frac{2}{(s-1)^2+1} + \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1} \\ &\quad -\frac{1}{5}\frac{s-1}{(s-1)^2+1} + \frac{3}{5}\frac{1}{(s-1)^2+1}. \end{aligned}$$

Then

$$\begin{aligned} y &= e^x\cos x - 2e^x \sin x + \frac{1}{5}\cos x - \frac{2}{5}\sin x - \frac{1}{5}e^x\cos x + \frac{3}{5}e^x \sin x \\ &= \frac{4}{5}e^x\cos x - \frac{7}{5}e^x\sin x + \frac{1}{5}\cos x - \frac{2}{5}\sin x \end{aligned}$$

Exercise

Use Laplace transforms to solve the initial-value problem

$$y' + y = 5H(x-1) + e^x H(x-1) + H(x-1)\cos x, \quad y(0) = 2.$$

Solution 4 :

Using the Laplace transform of both sides of the differential equation, we get:

$(s+1)Y - 2 = \mathcal{L}\left[5H(x-1) + e^x H(x-1) + H(x-1)\cos x\right].$

Since $\mathcal{L}[H(x-1)] = \dfrac{1}{s}e^{-s}$, $\mathcal{L}[e^x H(x-1)] = \dfrac{e^{-(s-1)}}{s-1}$ and

$\mathcal{L}[H(x-1)\cos x] = e^{-s}\dfrac{\sin 1 + \cos 1}{s^2+1}$. We have:

$$
\begin{aligned}
Y(s) &= \frac{2}{s+1} + \frac{5e^{-s}}{s(s+1)} + \frac{e^{-(s-1)}}{(s-1)(s+1)} + \frac{(\sin 1 + \cos 1)e^{-s}}{(s+1)(s^2+1)} \\
&= \frac{2}{s+1} + \frac{5e^{-s}}{s} - \frac{e^{-s}}{s+1} + \frac{1}{2}\frac{e^{-(s-1)}}{s-1} \\
&\quad - \frac{1}{2}\frac{e^{-(s-1)}}{s+1} + \frac{(\sin 1 + \cos 1)}{2}\frac{e^{-s}}{s+1} \\
&\quad - \frac{(\sin 1 + \cos 1)}{2}\frac{se^{-s}}{s^2+1} + \frac{(\sin 1 + \cos 1)}{2}\frac{e^{-s}}{s^2+1}
\end{aligned}
$$

We have $\mathcal{L}^{-1}\left[\frac{4}{s+1}\right] = 4e^{-x}$,
$\mathcal{L}^{-1}\left[\dfrac{5}{s}e^{-s}\right] = 5H(x-1)$ and
$\mathcal{L}^{-1}\left[\dfrac{5}{s+1}e^{-s}\right] = 5H(x-1)e^{-(x-1)}$. Then

$$y(x) = 4e^{-x} + 5H(x-1) - 5H(x-1)e^{-(x-1)}.$$

Exercise

Use the Laplace transform to solve the initial-value problem

$$y'' + 2y' + 5y = H(x-2), \quad y(0) = 1, y'(0) = 0$$

Solution 5 :
Using the Laplace transform of both sides of the differential equation, we get:

$$s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + 5Y(s) = \frac{e^{-2s}}{s}.$$

Then $Y(s)\left(s^2+2s+5\right) = s+2+\dfrac{e^{-2s}}{s}$. Solving for $Y(s)$, we find:

$$Y(s) = \frac{s+2}{s^2+2s+5} + e^{-2s}\frac{1}{s\left(s^2+2s+5\right)}.$$

We have $\dfrac{s+2}{s^2+2s+5} = \dfrac{s+1}{(s+1)^2+4} + \dfrac{1}{(s+1)^2+4}$ and
$\mathcal{L}^{-1}\left[\dfrac{s+1}{(s+1)^2+4}\right] = e^{-x}\cos(2x)$ and
$\mathcal{L}^{-1}\left[\dfrac{2}{(s+1)^2+4}\right] = e^{-x}\sin(2x)$.

$$\frac{e^{-2s}}{s\left(s^2+2s+5\right)} = \frac{1}{5}e^{-2s}\left(\frac{1}{s} - \frac{(s+1)+1}{(s+1)^2+4}\right) = \frac{1}{5}e^{-2s}\left(\frac{1}{s} - \frac{s+1}{(s+1)^2+4} + \frac{1}{(s+1)^2+4}\right).$$

$$\begin{aligned}\mathcal{L}^{-1}\left[e^{-2s}\left(\frac{1}{s} - \frac{s+2}{s^2+2s+5}\right)\right] &= H(x-2)[1 \\ &\quad -e^{-(x-2)}\left(\cos 2(x-2) + \frac{1}{2}\sin 2(x-2)\right)].\end{aligned}$$

The solution $y(x)$ to the initial-value problem is

$$\begin{aligned}y(x) &= e^{-x}\cos(2x) + \frac{e^{-x}}{2}\sin(2x) + \frac{1}{5}H(x-2)[1 \\ &\quad -e^{-(x-2)}\cos 2(x-2) + \frac{e^{-(x-2)}}{2}\sin 2(x-2)]\end{aligned}$$

Exercise

Solve the differential equation

$$y' + 3y = 13\sin(2t), \quad y(0) = 6.$$

Solution 6 :

We take the Laplace transform of each member of the differential equation:

$\mathcal{L}(y') + 3\mathcal{L}(y) = 13\mathcal{L}(\sin(2t))$. Then $(s+3)Y(s) - 6 = 6 + \dfrac{26}{s^2+4}$.

$Y(s) = \dfrac{6}{s+3} + \dfrac{26}{(s+3)(s^2+4)} = \dfrac{6}{s+3} + \dfrac{-2s+6}{s^2+4}$ and

$y = 6\mathcal{L}^{-1}(\dfrac{1}{s+3}) - 2\mathcal{L}^{-1}(\dfrac{s}{s^2+4}) + 6\mathcal{L}^{-1}(\dfrac{1}{s^2+4}) =$

$6e^{-3t} - 2\cos(2t) + 3\sin(2t)$.

Exercise

Solve the differential equation

$$y'' - 3y' + 2y = e^{-4t}, \quad y(0) = 1, y'(0) = 5.$$

Solution 7 :

We take the Laplace transform of each member of the differential equation:

$\mathcal{L}(y^{''}) - 3\mathcal{L}(y') + 2\mathcal{L}(y) = \mathcal{L}(e^{-4t})$. Then

$Y(s) = \dfrac{s^2 + 6s + 9}{(s-1)(s-2)(s+4}$ and $y = -\dfrac{16}{5}e^{t} + \dfrac{25}{5}e^{2t} + \dfrac{1}{30}e^{-4t}$.

Exercise

Solve the differential equation

$$y'' - 6y' + 9y = t^2e^{3t}, \quad y(0) = 2,\ y'(0) = 17.$$

Solution 8 :

We take the Laplace transform of each member of the differential equation:

$Y(s) = \dfrac{2s+5}{(s-3)^2} + \dfrac{2}{(s-3)^5}.$

$y = 2e^{3t} + 11te^{3t}\frac{1}{12}t^4e^{3t}.$

Exercise

Solve the following differential equation: $y' - 2y = f(x)$, *with*$y(0) = 3$, $f(x) = 3\cos x$ *for* $x \geq 1$ *and* $f(x) = 0$, *for* $0 \leq x \leq 1$.

Solution 9 :

$\mathcal{L}(f(x)) = -\frac{3s}{s^2+1}e^{-s}$. Then $sF(s) - 3 - 2F(s) = -\frac{3s}{s^2+1}e^{-s}$ and

$$F(s) = \frac{1}{s-2}\left(3 - \frac{3s}{s^2+1}e^{-s}\right) =$$

$$\frac{3}{s-2} + \frac{6}{5}\frac{s}{s^2+1}e^{-s} - \frac{3}{5}\frac{1}{s^2+1}e^{-s}.$$

$\mathcal{L}^{-1}\left(\frac{3}{s-2}\right) = 3e^{2t}$,

$\mathcal{L}^{-1}\left(\frac{6}{5}\frac{s}{s^2+1}e^{-s}\right) = \frac{6}{5}\cos(t-1)H(t-1)$,

$\mathcal{L}^{-1}\left(\frac{3}{5}\frac{1}{s^2+1}e^{-s}\right) = \frac{3}{5}\sin(t-1)H(t-1)$.

$$y(t) = 3e^{2t} + \frac{6}{5}\cos(t-1)H(t-1) - \frac{3}{5}\sin(t-1)H(t-1).$$

Solve the initial value problem

$$y'' + y' + y = \sin(x), \quad y(0) = 1,\ y'(0) = -1.$$

$$\mathcal{L}\{y'(x)\} = sY(s) - y(0) = sY(s) - 1, \quad \mathcal{L}\{y''(x)\} = s^2Y(s) - sy(0) - y'(0$$

Taking Laplace transforms of the differential equation, we get $(s^2+s+1)Y(s) - s = \dfrac{1}{s^2+1}$. Then
$Y(s) = \dfrac{s}{s^2+s+1} + \dfrac{1}{(s^2+s+1)(s^2+1)}$.

$$\frac{s}{s^2+s+1} = \frac{s}{(s+\frac{1}{2})^2+\frac{3}{4}} = \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+(\frac{1}{2}\sqrt{3})^2} - \frac{1}{\sqrt{3}}\frac{\frac{1}{2}\sqrt{3}}{(s+\frac{1}{2})^2+(\frac{1}{2}$$

Finding the inverse Laplace transform. Since

$$\frac{s}{s^2+s+1} = \frac{s}{(s+\frac{1}{2})^2+\frac{3}{4}} = \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+(\frac{1}{2}\sqrt{3})^2} - \frac{1}{\sqrt{3}}\frac{\frac{1}{2}\sqrt{3}}{(s+\frac{1}{2})^2+(\frac{1}{2}\sqrt{3})}$$

we have

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+s+1}\right\} = e^{-\frac{1}{2}x}\cos(\frac{1}{2}\sqrt{3}x) - \frac{1}{\sqrt{3}}e^{-\frac{1}{2}x}\sin(\frac{1}{2}\sqrt{3}x).$$

Also, we have

$$\frac{1}{(s^2+s+1)(s^2+1)} = \frac{s+1}{s^2+s+1} - \frac{s}{s^2+1},$$

$$\frac{1}{s^2+s+1} = \frac{1}{(s+\frac{1}{2})^2+\frac{3}{4}}, \text{ and}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+s+1}\right\} = \frac{2}{\sqrt{3}}e^{-\frac{1}{2}x}\sin(\frac{1}{2}\sqrt{3}x).$$

Then

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+s+1)(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+s+1}\right\} -$$

We obtain

$$y(x) = 2e^{-\frac{1}{2}x}\cos(\frac{1}{2}\sqrt{3}x) - \cos(x).$$

Solve the system of linear differential equation:

$$\begin{cases} \frac{dx}{dt} &= -2x + y, \\ \frac{dy}{dt} &= x - 2y \end{cases}$$ with the initial conditions $x(0) = 1$, $y(0) = 2$.

Taking the Laplace transform of the equations, we get

$$\begin{cases} sX(s) - 1 &= -2X(s) + Y(s), \\ sY(s) - 2 &= X(s) - 2Y(s), \end{cases}$$

where $X(s) = \mathcal{L}\{x(x)\}$, $Y(s) = \mathcal{L}\{y(x)\}$. then

$$\begin{cases} (s+2)X(s) - Y(s) &= 1, \\ -X(s) + (s+2)Y(s) &= 2 \end{cases}$$

The solutions of the linear system of equations on X and Y are $X(s) = \frac{s+4}{s^2+4s+3}, \quad Y(s) = \frac{2s+5}{s^2+4s+3}$.

Using the inverse Laplace transform, we have

$$\frac{s+4}{(s+1)(s+3)} = \frac{\frac{3}{2}}{s+1} - \frac{\frac{1}{2}}{s+3}, \quad \frac{2s+5}{(s+1)(s+3)} = \frac{\frac{3}{2}}{s+1} + \frac{\frac{1}{2}}{s+3},$$

we obtain

$$x(x) = \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}, \quad y(x) = \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t}.$$