Exercise 1 :
Consider the differential equation

$$y^{''} - 2y' + y = e^x(x + \cos x)$$

The general solution of the homogenous equation is

$$y_c = (ax + b)e^x, \quad a, b \in \mathbb{R}$$

Let $z = e^{-x}y$, $y^{''} - 2y' + y = e^x z^{''} = e^x(x + \cos x) \iff z^{''} = x + \cos x$. Then $y_p = (\frac{x^3}{6} - \cos x)e^x$ and $y = e^x(ax + b + \frac{x^3}{6} - \cos x)$.

Exercise 2 :
Consider the differential equation

$$y^{''} + y' + y = \cos x$$

The general solution of the homogenous equation is

$$y_c = e^{-\frac{1}{2}x}(a\cos(\frac{\sqrt{3}}{2}x) + b\sin(\frac{\sqrt{3}}{2}x)), \quad a, b \in \mathbb{R}$$

$y_p = \sin x$ and

$$y = e^{-\frac{1}{2}x}(a\cos(\frac{\sqrt{3}}{2}x) + b\sin(\frac{\sqrt{3}}{2}x)) + \sin x, \quad a, b \in \mathbb{R}.$$

Exercise 3 :
Consider the differential equation

$$y^{''} - 3y' + 2y = e^x + xe^{2x}$$

The general solution of the homogenous equation is

$$y_c = ae^x + be^{2x}, \quad a, b \in \mathbb{R}$$

$y_p = -xe^x + (\frac{1}{2}x^2 - x)e^{2x}$ and

$$y = (a - x)e^x + (\frac{1}{2}x^2 - x + b)e^{2x}, \quad a, b \in \mathbb{R}.$$

Exercise 4 :
Consider the differential equation

$$x^2y^{''} + xy' + y = 0, \qquad x > 0.$$

If $x = e^t$ and $z(t) = y(e^t)$.
The differential equation is equivalent to $z^{''} + z = 0$. Then $z = a\cos(t) + b\sin(t)$ and

$$y = a\cos(\ln(x)) + b\sin(\ln(x)).$$

Exercise 5 :
Consider the differential equation

$$2x^2y^{''} + 5xy' + y = 0, \qquad x > 0.$$

If $x = e^t$ and $z(t) = y(e^t)$.
The differential equation is equivalent to $2z^{''} + 3z' + z = 0$. Then $z = ae^{-t} + be^{-\frac{1}{2}t}$ and

$$y = \frac{a}{x} + \frac{b}{\sqrt{x}}.$$

Exercise 6 :
Consider the differential equation

$$y^{''} + xy' + 3y = 0.$$

Let $y = \sum_{n=0}^{+\infty} a_n x^n$.
$xy' = \sum_{n=0}^{+\infty} n a_n x^n$, $y = \sum_{n=0}^{+\infty} (n+1)(n+2)a_{n+2}x^n$. If y is a solution, then

$$(n+1)(n+2)a_{n+2} + (n+3)a_n = 0, \qquad \forall n \geq 0.$$

Let $u_n = a_{2n}$ and $v_n = a_{2n+1}$.

$$u_n = -\frac{2n+1}{2n(2n-1)}u_{n-1} = (-1)^n \frac{2n+1}{2^n n!} a_0$$

and

$$v_n = -\frac{n+1}{n(2n+1)}v_{n-1} = (-1)^n \frac{2^n(n+1)!}{(2n+1)!} a_1$$

Then

$$y = a\sum_{n=0}^{+\infty} (-1)^n \frac{2n+1}{2^n n!} x^{2n} + b\sum_{n=0}^{+\infty} (-1)^n \frac{2^n(n+1)!}{(2n+1)!} x^{2n+1}.$$