Math 280
3^{rd} semester 1444

1. Question [2+2]

   (a) Let A be a non-empty, lower bounded subset of $\mathbb{R}$ and $\alpha \in \mathbb{R}$. Show that: $\alpha = \inf A$, if and only if, for every $\varepsilon > 0$ there exists $a \in A$ such that $\alpha \leq a < \alpha + \varepsilon$.

   (b) Prove that if $x_n \to x$, then there is a positive real number M such that

   $$|x_n| \leqslant M \qquad \text{for all} \qquad n \in \mathbb{N}$$

2. Question [3+3+3+3]

   (a) If $f : (-1, 1) \to \mathbb{R}$ satisfies $|f(x)| \leq 2|x|$, prove that f is continuous at $x = 0$.

   (b) If f, $g : [a, b] \to \mathbb{R}$ are two continuous functions such that $f(a) < a^2$ and $f(b) > b^2$, prove that there exists $c \in (a, b)$ such that $f(c) = c^2$.

   (c) Show that the function $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

   (d) Show that the function $g(x) = \cos x$ is uniformly continuous on $\mathbb{R}$.

3. Question [2+2+3+3+3].

   (a) Show that if a series $\sum a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

   (b) Give an example of a convergent series which is not absolutely convergent.

   (c) Test the following series for convergence:

      1. $\sum \frac{1}{(n+1)(n+2)}$.
      2. $\sum \frac{n}{2^n}$.
      3. $\sum n^n e^{-n}$.

4. Question, [3+3+3]

   (a) If the function f has an extremum on the open interval (a, b) at the point $c \in (a, b)$ and if f is differentiable at c, show that $f'(c) = 0$.

   (b) If the function f satisfies $|f(x)| \leq |x|^2$, for all $x \in [-1, 1]$, prove that f is differentiable at 0 and find $f'(0)$.

   (c) Consider the function defined by

   $$f(x) = \begin{cases} x^2 & \text{if} \quad x < 1, \\ 3x - 2 & \text{if} \quad x \geq 1. \end{cases}$$

   Show that f is continuous on $\mathbb{R}$, but not differentiable at $x = 1$.

5. Question[3+2]

   (a) If f is continuous on $[a, b]$, show that there exists a point c in (a, b) such that

   $$\int_a^b f(x)\, dx = f(c)(b - a).$$

   (b) Give an example of a function f such that $|f| \in \Re(a, b)$ and $f \notin \Re(a, b)$.