King Saud University | **Math 280**
Mathematics Department | **Final Exam**
First semester 1444 | **Time: 3 hours**

1. The first question.(7 marks)
   (a) Let $A=\left\{1,1+\frac{1}{2},1+\frac{1}{2}+\frac{1}{3},1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4},...\right\}$ Find $\sup A$ and $\inf A$ if they exist.
   (b) Let A be a non-empty subset of $\mathbb{R}$. For any b in $\mathbb{R}$, define $A+b=\{a+b: a\in A\}$. If A is bounded below, prove that $\inf(A+b)=\inf(A)+b$.
   (c) If $0<b<1$, show that $\lim_{n\to\infty} nb^n=0$.

2. The second question.(9 marks)
   (a) Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be two series, and suppose that $a_k=b_k$ whenever $k>42$. Show that $\sum_{k=1}^{\infty} b_k$ converges if $\sum_{k=1}^{\infty} b_k$ converges.
   (b) Decide whether the following series converge or diverge:
      1. $\sum_{k=1}^{\infty}\frac{1}{k^2+k}$
      2. $\sum_{k=1}^{\infty}\frac{3^k+4^k}{6^k}$
      3. $\sum_{k=1}^{\infty}\frac{k}{2k^2-1}$.

3. The third question. (12 marks)
   (a) Use definition to show that $f(x)=\frac{1}{x}$ is uniformly continuous on $[2,\infty)$.
   (b) Show that the function $f(x)=x^2$ is not uniformly continuous on $\mathbb{R}$.
   (c) Use Mean Value Theorem to prove that
   $$|\cos x-\cos y|\leq |x-y| \text{ for all } x,y\in\mathbb{R}.$$
   (d) If the function f has an extremum on the open interval (a,b) at the point $c\in(a,b)$, and if f is differentiable at c, show that $f'(c)=0$.

4. The fourth question. (12 marks)
   (a) Consider the function $f(x)=\begin{cases} x^2 & x<1 \\ 3x-2 & x\geq 1\end{cases}$ show that f is continuous but not differentiable at $x=1$.
   (b) Use Taylor's theorem with $n=4$ to obtain a suitable approximation of the number e.
   (c) Give an example of a bounded function which is not integrable. Justify your answer.
   (d) Suppose that f is Riemann integrable on $[a,b]$, and let $F:[a,b]\to\mathbb{R}$ be defined by $F(x)=\int_a^x f(t)\,dt$. Prove that if f is continuous at $c\in[a,b]$, then F is differentiable at c and $F'(c)=f(c)$.

Good Luck