King Saud University **Math 280**
Mathematics Department Final Exam
first semester 1444 Time: 3 hours

1. The first question.(7 marks)
   (a) Let $A = \left\{1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}, 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}, ...\right\}$ Find $\sup A$ and $\inf A$ if they exist.
   (b) Let A be a non-empty subset of $\mathbb{R}$. For any b in $\mathbb{R}$, define $A+b = \{a+b : a \in A\}$. If A is bounded below, prove that $\inf(A+b) = \inf(A) + b$.
   (c) If $0 < b < 1$, show that $\lim_{n\to\infty} nb^n = 0$.

2. The second question.(9 marks)
   (a) Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be two series, and suppose that $a_k = b_k$ whenever $k > 42$. Show that $\sum_{k=1}^{\infty} b_k$ converges if $\sum_{k=1}^{\infty} b_k$ converges.
   (b) Decide whether the following series converge or diverge:
      1. $\sum_{k=1}^{\infty} \frac{1}{k^2+k}$
      2. $\sum_{k=1}^{\infty} \frac{3^k+4^k}{6^k}$
      3. $\sum_{k=1}^{\infty} \frac{k}{2k^2-1}$.

3. The third question. (12 marks)
   (a) Use definition to show that $f(x) = \frac{1}{x}$ is uniformly continuous on $[2, \infty)$.
   (b) Show that the function $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.
   (c) Use Mean Value Theorem to prove that
   $$|\cos x - \cos y| \leq |x - y| \text{ for all } x, y \in \mathbb{R}.$$
   (d) If the function f has an extremum on the open interval (a, b) at the point $c \in (a, b)$, and if f is differentiable at c, show that $f'(c) = 0$.

4. The fourth question. (12 marks)
   (a) Consider the function $f(x) = \begin{cases} x^2 & x < 1 \\ 3x - 2 & x \geq 1 \end{cases}$ show that f is continuous but not differentiable at $x = 1$.
   (b) Use Taylor's theorem with $n = 4$ to obtain a suitable approximation of the number e.
   (c) Give an example of a bounded function which is not integrable. Justify your answer.
   (d) Suppose that f is Riemann integrable on $[a, b]$, and let $F : [a, b] \to \mathbb{R}$ be defined by $F(x) = \int_a^x f(t)\, dt$. Prove that if f is continuous at $c \in [a, b]$, then F is differentiable at c and $F'(c) = f(c)$.

Good Luck