King Saud University, Department of Mathematics
Math 280 (Real Analysis)
Final Exam 31/05/2022

Question 1.[2+2+2]

(a) Show that for every $x \in \mathbb{R},\ x > 0$ there is a natural number n such that $0 < \frac{1}{n} < x$.

(b) Find, if there exist, the supremum, the infimum, the maximum and the minimum of the following sets.
$A = \left\{ \frac{1}{2n-1} : n \in \mathbb{N}^{+} \right\}$

(c) Determine the interior, closer and the boundary of $(2, 3] \cup (4, 5)$.

Question 2.[3+3]

(a) Show that "If a sequence a_n is converges, then a_n is bounded".

(b) If a_n a sequence such that $a_n \geq 0$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} a_n = a$, **prove that** $\lim\limits_{n\to\infty} \sqrt{a_n} = \sqrt{a}$.

Question 3.[3+3]

(a) If the series $\sum\limits_{n=1}^{\infty} |a_n|$ converges, **prove that** the series $\sum\limits_{n=1}^{\infty} a_n$ is converges.

(b) If $\sum\limits_{n=1}^{\infty} a_n$ with $a_n > 0$ is convergent, and if $b_n = \frac{a_1+a_2+a_3+\cdots+a_n}{n}$ for $n \in \mathbb{N}$,
then **show that** $\sum\limits_{n=1}^{\infty} b_n$ is always divergent.

Question 4.[6+3]

(a) Show whether each of the following statements is true or false, and explain or give prove for the false one. 1- The function $f(x) = \frac{1}{x}$ is not uniformly continuous on $[\frac{1}{2}, \frac{3}{2}]$.
2- The function $f(x) = e^{-x}$ is not uniformly continuous on $[a, \infty)$.
3- The function $f(x) = \sqrt{x}$ is not uniformly continuous on $[0, \infty)$.

(b) Let f be defined in a neighborhood I of x_0. **Prove that** if f is differentiable at x_0, then f is continuous at x_0.

Question 5.[4+2] Let $f : [a, b] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [a, b] \\ 0, & x \notin \mathbb{Q} \cap [a, b] \end{cases}$

(i) Choose uniform partition P_n for the interval $[a, b]$ and calculate $U(f, P)$ and $L(f, P)$

(ii) Prove that $f \notin \mathcal{R}[a, b]$

Question 6.[3+4]

(a) Consider the sequence of functions $f_n(x) = x^n$. **Show that** $f_n(x) = x^n$ uniform convergence on any interval $[0, \alpha]$, for $0 < \alpha < 1$.

(b) Use M-Test to show the uniform convergence of the series $\sum\limits_{n=1}^{\infty} \frac{\sin nx}{n^p}$ for $p > 1$.