King Saud University	College of Sciences	Department of Mathematics

Final Examination	Math 228	Semester 2 (1443)	Time: 3 Hours

The Examination contains 2 pages

Question 1: (3+3+3)

1. Find the area A of the region in the xy-plane bounded by the graphs $y = 4x - x^2$ and $y = -x$.
2. Evaluate the integral $\displaystyle\int_0^2 \int_0^y y^4 \cos(xy^2)\, dx\, dy$.
3. Evaluate the integral $\displaystyle\iint_D e^{x^2+y^2}\, dA$, where D is the disc of center $(0,0)$ and radius 2.

Question 2: (3+3+3+3) 12

1. Find the volume of the tetrahedron bounded by the coordinates $x=0$, $y=0$, $z=0$ and the plane $x+y+z=1$.
2. Evaluate the integral by changing to spherical coordinates
$$\int_0^2 \int_0^{\sqrt{4-y^2}} \int_0^{\sqrt{4-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\, dx\, dy$$
3. Use cylindrical coordinates to evaluate $\displaystyle\iiint_Q \sqrt{x^2+y^2}dV$, where Q is the solid bounded by the paraboloid $z = 1-(x^2+y^2)$ and the xy-plane.
4. Find the volume of the region that lies between two spheres $x^2+y^2+z^2 = 9$ and $x^2+y^2+z^2 = 4$.

Question 3: (3+2+2+3+3) = 13

1. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \left[\frac{1}{(n+3)(n+4)} + \frac{1}{2^n}\right]$.
2. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{100^n}{n!}$ converges or diverges. Justify you answer.
3. Use integral test to determine whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n)}{n}$ converges or diverges.
4. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+4}$ is absolutely convergent, conditionally convergent, or divergent.

5. Find the interval of convergence and radius of convergence of the power series $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{n4^n}$.

Question 4: (3+3) 6

1. Find the power series representation of $f(x) = \ln(1+x)$, $|x| < 1$, and use first three non zeros terms to find the value of $\ln(1.2)$.

2. Gives the Maclaurin's series for the function $f(x) = \cos x$ and prove that it represents $\cos x$ for all x. hence approximate the integral $\int_0^1 x\cos(x^3)\, dx$ up to four decimal places by using the first three non-zeros terms.

Answer final math 228

semester 2, 1443

Q 1:

1.)

$y = 4x - x^2 ; \ y = -x.$

$\Rightarrow 4x - x^2 = -x \Rightarrow x^2 - 5x = 0$

$\Rightarrow x(x-5) = 0 \Rightarrow x = 0$ or $x = 5.$

$\Rightarrow 0 \le x \le 5 ; \ -x \le y \le 4x - x^2$

$$\Rightarrow A = \int_0^5 \int_{-x}^{4x-x^2} dy\, dx$$

$$= \int_0^5 [y]_{-x}^{4x-x^2} dx = \int_0^5 (4x - x^2 + x)\, dx$$

$$= \int_0^5 (5x - x^2)\, dx$$

$$= \left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_0^5$$

$$= \frac{5^3}{2} - \frac{5^3}{3} = \frac{5^3}{6} = \frac{125}{6} \simeq 20{,}8.$$

2)

$$\int_0^2 \int_0^y y^4 \cdot \cos(xy^2)\, dx\, dy$$

$$= \int_0^2 y^4 \cdot \left[\frac{1}{y^2} \sin xy^2\right]_0^y dy.$$

$u = y^3 \Rightarrow du = 3y^2\, dy.$ $\quad y = 0 \Rightarrow u = 0$, $y = 2 \Rightarrow y = 8$

$$= \int_0^2 y^2 \sin y^3\, dy = \frac{1}{3}\int_0^8 \sin u\, du = -\frac{1}{3}[\cos u]_0^8 = \frac{1}{3}[1 - \cos 8].$$

3) $\iint_D e^{x^2+y^2}\, dA = \int_0^2 \int_0^{2\pi} e^{r^2} r\, d\theta\, dr.$

$= \int_0^2 r e^{r^2} dr \int_0^{2\pi} d\theta.$

$= \left[\frac{1}{2} e^{r^2}\right]_0^2 . 2\pi.$

$= \frac{1}{2}(e^4 - 1). 2\pi$

$= \pi(e^4 - 1).$

Question 2:

1) $x+y+z=1 \Rightarrow z = 1-x-y$; $z=0$

$\Rightarrow x+y=1 \Rightarrow y = 1-x.$

$\Rightarrow V = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx.$

$= \int_0^1 \int_0^{1-x} (1-x-y)\, dy\, dx.$

$= \int_0^1 \left[y - xy - \frac{y^2}{2}\right]_0^{1-x} dx$

$= \int_0^1 \left((1-x) - x(1-x) - \frac{(1-x)^2}{2}\right) dx.$

$= \int_0^1 (1-x)\left(1-x-\frac{1}{2}(1-x)\right) dx.$

$= \int_0^1 (1-x)\left(\frac{1}{2}(1-x)\right) dx$

$= \frac{1}{2}\int_0^1 (1-x)^2 dx = \frac{1}{2}\left[-\frac{1}{3}(1-x)^3\right]_0^1 = \frac{1}{6}.$

3

2)

$$\int_0^2 \int_0^{\sqrt{4-y^2}} \int_0^{\sqrt{4-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\, dx\, dy$$

$0 \leq \theta \leq \frac{\pi}{2}$.
$0 \leq \varphi \leq \frac{\pi}{2}$.
$0 \leq \rho \leq 2$.

$$= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 \rho\, \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\pi/2} \sin\varphi\, d\varphi \int_0^2 \rho^3\, d\rho.$$

3

$$= \frac{\pi}{2} \cdot \left[-\cos\varphi\right]_0^{\frac{\pi}{2}} \left[\frac{\rho^4}{4}\right]_0^2$$

$$= \frac{\pi}{2} \left[-0+1\right] \cdot \frac{16}{4} = 2\pi.$$

3)

$z=0$; $z = 1-(x^2+y^2) \Rightarrow x^2+y^2=1$
$\Rightarrow 0 \leq r \leq 1$
$0 \leq \theta \leq 2\pi$.

$0 \leq z \leq 1-r^2$.

$$\Rightarrow \iiint_Q \sqrt{x^2+y^2}\, dV = \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} \sqrt{r^2}\, r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 \int_0^{1-r^2} r^2\, dz\, dr$$

3

$$= 2\pi \int_0^1 r^2 (1-r^2)\, dr.$$

$$= 2\pi \int_0^1 (r^2 - r^4)\, dr$$

$$= 2\pi \left[\frac{r^3}{3} - \frac{r^5}{5}\right]_0^1 = 2\pi\left(\frac{1}{3}-\frac{1}{5}\right)$$

$$= 2\pi\, \frac{2}{15} = \frac{4\pi}{15}.$$

4) Using spherical coordinates,

$0 \leq \theta \leq 2\pi$; $0 \leq \varphi \leq \pi$; $2 \leq \rho \leq 3$.

$$V = \int_0^{2\pi}\int_0^{\pi}\int_2^3 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi \, d\varphi \int_2^3 \rho^2 \, d\rho .$$

$$= 2\pi \left[-\cos\varphi\right]_0^{\pi} \left[\frac{\rho^3}{3}\right]_2^3$$

3

$$= 2\pi\left(-\left[(-1)-1\right]\right)\left(\frac{27-8}{3}\right)$$

$$= \frac{19}{3} \times 4\pi = \frac{76\pi}{3} .$$

Question 3:

1) $\sum_{n=1}^{\infty}\left[\frac{1}{(n+3)(n+4)} + \frac{1}{2^n}\right] = \sum_{n=1}^{\infty} \frac{1}{(n+3)(n+4)} + \sum_{n=1}^{\infty} \frac{1}{2^n}$.

3

$$\sum_{n=1}^{\infty} \frac{1}{(n+3)(n+4)} = \sum_{n=1}^{\infty} \frac{1}{n+3} - \frac{1}{n+4} .$$

$$S_n = \frac{1}{4} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{5}} - \cancel{\frac{1}{6}} + \cancel{\cdots} + \cancel{\frac{1}{n+3}} - \frac{1}{n+4} .$$

$$S_n = \frac{1}{4} - \frac{1}{n+4} .$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{(n+3)(n+4)} = \lim_{n \to \infty} S_n = \frac{1}{4} .$$

$\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with $S = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$.

$$\Rightarrow \sum_{n=1}^{\infty}\left[\frac{1}{(n+3)(n+4)} + \frac{1}{2^n}\right] = \frac{5}{4} .$$

2) $\sum_{n=1}^{\infty} \frac{100^n}{n!}$

By ratio test, $a_n = \frac{100^n}{n!}$;

$\Rightarrow \frac{a_{n+1}}{a_n} = \frac{100^{n+1}}{(n+1)!} \cdot \frac{n!}{100^n} = \frac{100}{n+1}$.

$\Rightarrow \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{100}{n+1} = 0 < 1$.

2

$\Rightarrow \sum_{n=1}^{\infty} \frac{100^n}{n!}$ converges.

3) $\sum_{n=2}^{\infty} \frac{\ln(n)}{n}$,

$f(x) = \frac{\ln x}{x} > 0$ on $[2, \infty)$.

• $f(x)$ continue on $[2, \infty)$.

• $f'(x) = \frac{\frac{1}{x} \cdot x - 1 \cdot \ln x}{x^2} = \frac{1 - \ln x}{x^2} \leq 0$, $\forall x \in [2, \infty)$.

$\Rightarrow f$ decreasing.

$\int_2^{\infty} \frac{\ln x}{x} dx = \lim_{t \to \infty} \int_2^t \ln x \cdot \frac{1}{x} dx = \frac{1}{2} \cdot \lim_{t \to \infty} \left[(\ln x)^2 \right]_2^t$.

$= \frac{1}{2} \lim_{t \to \infty} (\ln t)^2 - (\ln 2)^2 = \infty$.

$\Rightarrow$ diverges.

2

4) $\cdot \sum_{n=1}^{\infty} |u_{n+1}| = \sum_{n=1}^{\infty} \frac{n}{n^2+4} \cdot a_n$.

comparaison test with $\frac{1}{n} = b_n$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n}{n^2+4}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^2+4} = 1 \neq 0.$$

$\Rightarrow$ is not absolutely convergent.

$\cdot$ A.S.T ; $a_n = \frac{n}{n^2+4}$.

3

$\cdot \lim_{n\to\infty} a_n = 0$.

$\cdot f(x) = \frac{x}{x^2+4}$; $f'(x) = \frac{-x^2+4}{(x^2+4)^2} < 0$

$\Rightarrow$ decreasing

$\Rightarrow a_n \geq a_{n+1}$.

$\Rightarrow$ converges.

5)

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{(n+1)\,4^{n+1}} \cdot \frac{n\,4^n}{x^n}\right|$$

$$= \frac{n}{4(n+1)} |x|$$

$\Rightarrow \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \frac{1}{4}|x| \Rightarrow \frac{1}{4}|x| < 1 \Rightarrow |x| < 4$

$\Rightarrow -4 \leq x < 4$.

3

$\underline{x = -4:}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n(-4^n)}{n\,4^n} = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverge.}$$

$\underline{x = 4:}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{n\,4^n} = \cancel{4} \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n} \text{ converges}$$

$\Rightarrow$ the interval is $(-4, 4]$.

Question 4 :

1) $\ln(1+x) = \int_0^x \left(\frac{1}{1+t}\right) dt = \int_0^x \left[1 - t + t^2 - t^3 + \dots\right] dt$

$$= \left[t - \frac{t^2}{2} + \frac{t^3}{3} + \dots\right]_0^x \simeq x - x^2 + \frac{x^3}{3}$$

$\ln(1.2) = \ln(1+0,2) = (0,2) - \frac{(0,2)^2}{2} + \frac{(0,2)^3}{3}$

$$= 0,2 - 0,02 + 0,00267 \simeq 0,183.$$

3

2).

$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \dots + (-1)^{n-1} \frac{x^{2n}}{(2n)!} + \dots$

$\cos(x^3) = 1 - \frac{x^6}{2} + \frac{x^{12}}{4!} + \dots$

$x\cos(x^3) = x - \frac{x^7}{2} + \frac{x^{13}}{4!} + \dots$

$\int_0^1 x\cos(x^3)\, dx = \int_0^1 \left[x - \frac{x^7}{2} + \frac{x^{13}}{4!}\right] dx$

$$= \left[\frac{x^2}{2} - \frac{x^8}{16} + \frac{x^{14}}{336}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{16} + \frac{1}{336}$$

3

$$= \frac{168 - 21 + 1}{336} = \frac{148}{336} \simeq 0,4405.$$