Final Exam, Semester II, 1445
Dept. of Mathematics, College of Science, KSU
Math: 280 — Full Mark: 40 — Time: 3H

immediate

Question 1[3+3]

1. Prove that for every real number, there exists an integer n such that $n-1 \leq x < n$. Find such n if $x = -\frac{17}{5}$.
2. Determine $\sup(A)$ and $\inf(A)$ where $A = \{x \in \mathbb{R} : x^2 - 9 < 0\}$, and justify your answer.

Question 2 [2+2+3]

Use the definition of the limit to find the following if they exists.
1. $\lim_{n\to\infty} \frac{n^3}{2n^4+1}$.
2. $\lim_{n\to\infty} c^{\frac{1}{n}}$, where $c > 1$.
3. $\lim_{n\to\infty} na^n = 0$, where $0 < a < 1$.

Question 3[3+3]

Discuss the convergence of the following series:
(i) $\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n}}{n^2+1}$
(ii) $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$

Question 4[3+3]

1. Find the following limits, if they exist, and prove using the definition of the limit or sequence characterization:
$a)$ $\lim_{x\to 0} \frac{x^2}{|x|}$ (b) $\lim_{x\to\infty} \frac{x^2}{e^x}$.
2. Let

$$f(x) = \begin{cases} x^2 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$$

Prove that f is differentiable at $x = 0$, and evaluate $f'(0)$.

Question 5[3+3+3]

1. Determine a real interval of length $\frac{1}{2}$ where the equation

$$x^3 - 6x^2 + \frac{5}{2} = 0,$$

has a solution. Justify your answer.

2. Prove that if f is continuous on $[a, b]$ and has zero derivative on (a, b), then f is constant.

3.Use Taylor's theorem with $n = 3$ and $x_0 = 0$ to obtain a suitable approximation of the function $f(x) = \sqrt{1 - x}$ by a polynomial of degree 3.

Question 6[4+1+1]

Let

$$f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \cap [-2, 2] \\ -1 \text{ if } x \in \mathbb{Q}^c \cap [-2, 2] \end{cases}$$

i) Find the upper and the lower integral of f over $[-2, 2]$.

ii) Is f integrable on $[-2, 2]$? justify your answer.

iii) Is $|f|$ integrable on $[-2, 2]$? justify your answer.