جامعة الملك سعود
King Saud University

TIME: 3 Hours **DEPARTMENT OF MATHEMATICS** **FULL MARKS: 80**

M - 107 **Final Examination (Second Semester 1437-1438)**

Question: 1: (a) What conditions must a, b and c satisfy in order for the system

$$\begin{aligned} 3x - y + 3z &= a \\ x - y + 2z &= b \\ 2x - 6y + 10z &= c \end{aligned}$$

[6+6+6] to be consistent.

(b) Let A be a 3x3 matrix with $\det(A) = -4$. Use properties of determinant to evaluate $\det(2A) + \det(A^2) + \det\left(2A^{-1}\right)$.

(c) Find inverse of matrix A by method of cofactors

$$A = \begin{bmatrix} 7 & -3 & 1 \\ 1 & -2 & 4 \\ -3 & 1 & 0 \end{bmatrix}$$

Question:2 (a) Given three points $P(2,2,-1)$, $Q(3,-1,2)$ and $R(1,2,1)$.

[6+6+6]

i. Find the equation of the line passing through point P and normal to plane containing P, Q and R.

ii. Find the distance from R to this normal line through P.

(b) Determine whether the lines

$$l_1: \ x = 1 + 2t, \ y = 1 - 4t, \ \ z = 5 - t$$
$$l_2: \ x = 4 - v, \ y = -1 + 6v, \ z = 4 + v$$

are intersecting, if so, find the point of intersection, and also find the angle between the lines.

(c) Show that the points $A(0,-1,0)$, $B(2,1,-1)$ $C(1,1,1)$ and $D(3,3,0)$ are coplanar.

Question: 3. (a) Identify the surface $x^2 + 4y^2 = z$.

[6+6+8] Write the names and the equations of the traces of the surface on the co-ordinate planes, and sketch the surface.

(b) An object starts from rest at the point (2, 1, 3) and moves with acceleration $a(t) = i + 2tj + 3t^2k$. Find the location of the object after 2 seconds.

(c) The position vector of a moving particle at time t is given by $r(t) = ti + 2\cos tj + 3\sin tk$. Find the tangential and the normal components of acceleration at any time t. Also find curvature at any time t.

Question: 4. (a) Use implicit differentiations to find $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$, if $xy^2 + z^3 + \sin(xyz) = 0$.

[6+6] **(b)** The electric potential V is given by $V(x,y,z) = x^4yz - xy^3 + z$

i. Find the rate of change of V at P(1, 1, -3) in direction from P to origin.

ii. In what direction V increases most rapidly?

iii. What is the maximum value of the rate of change of V at P?

Question: 5. (a) Show that $f(x,y) = x^3 e^{-\frac{y}{x}}$ satisfies $y\dfrac{\partial^2 f}{\partial y^2} - 3\dfrac{\partial f}{\partial y} = \dfrac{\partial f}{\partial x}$

[6+6] **(b)** If $f(x,y) = 2x^4 - x^2 + y^2$, find local extrema and saddle points of $f(x,y)$.

Q1(a). 6

$$[A|B] \equiv \left[\begin{array}{ccc|c} 3 & -1 & 3 & a \\ 1 & -1 & 2 & b \\ 2 & -6 & 10 & c \end{array}\right] \equiv \left[\begin{array}{ccc|c} 1 & -1 & 2 & b \\ 3 & -1 & 3 & a \\ 2 & -6 & 10 & c \end{array}\right]$$

$$\equiv \left[\begin{array}{ccc|c} 1 & -1 & 2 & b \\ 0 & 2 & -3 & a-3b \\ 0 & -4 & 6 & c-2b \end{array}\right] \begin{array}{l} \\ -3R_1+R_2 \\ -2R_1+R_3 \end{array}$$

$$\equiv \left[\begin{array}{ccc|c} 1 & -1 & 2 & b \\ 0 & 2 & -3 & a-3b \\ 0 & 0 & 0 & c-8b+2a \end{array}\right] \begin{array}{l} \\ \\ 2R_2+R_3 \end{array}$$

system will be consistent if $c-8b+2a=0$

$\Rightarrow 8b = c+2a$

(b) 6

$\det A = -4$, A is 3×3

$\det 2A = 2^3 \det A = 8(-4) = -32$ (2)

$\det A^2 = \det \cdot \det A = (-4)(-4) = 16$ (1)

$\det(2A^{-1}) = \dfrac{2^3}{\det A} = \dfrac{8}{-4} = -2$ (2)

$\det 2A + \det A^2 + \det(2A^{-1}) = -32+16-2$ (1)

$= -18$.

(c) 6

$c_{11} = -4$, $c_{12} = -12$, $c_{13} = -5$ (1)

$c_{21} = 1$, $c_{22} = 3$, $c_{23} = 2$, $\det A = 3 \neq 0$

$c_{31} = -10$, $c_{32} = -27$, $c_{33} = -11$

Matrix of cofactors $C = \begin{bmatrix} -4 & -12 & -5 \\ 1 & 3 & 2 \\ -10 & -27 & -11 \end{bmatrix}$ (3)

$\operatorname{adj} A = C^T = \begin{bmatrix} -4 & 1 & -10 \\ -12 & 3 & -27 \\ -5 & 2 & -11 \end{bmatrix}$ (1)

$A^{-1} = \dfrac{1}{\det A} \operatorname{adj} A = \dfrac{1}{3} \begin{bmatrix} -4 & 1 & -10 \\ -12 & 3 & -27 \\ -5 & 2 & -11 \end{bmatrix}$. (1)

Q2(a) $\overrightarrow{PQ} = \langle 1, -3, 3\rangle$, $\overrightarrow{PR} = \langle -1, 0, 2\rangle$ (2)

$\underline{6}$

$$n = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 1 & -3 & 3 \\ -1 & 0 & 2 \end{vmatrix} = -6i - 5j - 3k$$ (3)

(i) Equation of line passing through $P^{(2,2,-1)}$ and normal to plane are (1)

$x = 2-6t$, $y = 2-5t$, $z = -1-3t$, $t \in R$

(ii) Distance of $R(1,2,1)$ for the line

$$D = \frac{\|\overrightarrow{PR} \times n\|}{\|n\|} \qquad \overrightarrow{PR} \times n = \begin{vmatrix} i & j & k \\ -1 & 0 & 2 \\ -6 & -5 & -3 \end{vmatrix}$$ (2)

$\|\overrightarrow{PR} \times n\| = \sqrt{350}$, $\|n\| = \sqrt{70}$ $= 10i - 15j + 5k.$

$$D = \sqrt{\frac{350}{70}} = \sqrt{5}$$

Q2 (b) $1+2t_0 = 4-v_0$ $\Rightarrow$ $2t_0 + v_0 = 3$ $\to E_1$

$\underline{6}$

$1-4t_0 = -1+6v_0$ $\Rightarrow$ $-4t_0 - 6v_0 = -2$ $\to E_2$

$5-t_0 = 4+v_0$ $\Rightarrow$ $-t_0 - v_0 = -1$ $\to E_3$

$E_1 + E_3 \Rightarrow t_0 = 2$, put in E_1, $4+v_0 = 3$, $v_0 = -1$

Substituting $t_0 = 2$, $v_0 = -1$ in $E_2 \Rightarrow -4(2) - 6(-1)$ (3)

$= -8+6$

$= -2$

$\Rightarrow$ Lines intersects

Point of intersection is let $t_0 = 2$

$x_0 = 1+4 = 5$

$y_0 = 1-8 = -7.$ (1)

$z_0 = 5-2 = 3$

$(5, -7, 3)$

$a = \langle 2, -4, -1\rangle$, $b = \langle -1, 6, 1\rangle$, $a \cdot b = -27$ (2)

$$\cos\theta = \frac{a \cdot b}{\|a\|\|b\|} = \frac{-27}{\sqrt{21}\sqrt{38}} \Rightarrow \theta = \cos^{-1}\left[\frac{-27}{\sqrt{21}\sqrt{38}}\right].$$

(c) Points are coplanar if Volume of parallelopiped made by AB, AC, AD is Zero that is

$\underline{6}$

$$|\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD})| = 0$$

$\overrightarrow{AB} = \langle 2, 2, -1\rangle$, $\overrightarrow{AC} = \langle 1, 2, 1\rangle$, $\overrightarrow{AD} = \langle 3, 4, 0\rangle$ (2)

$$\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = \begin{vmatrix} 2 & 2 & -1 \\ 1 & 2 & 1 \\ 3 & 4 & 0 \end{vmatrix} = 2(0-4) - 2(0-3) - 1(4-6)$$ (3)

$= -8+6+2$

$= 0$

$\Rightarrow$ A, B, C and D are coplanar.

(1)

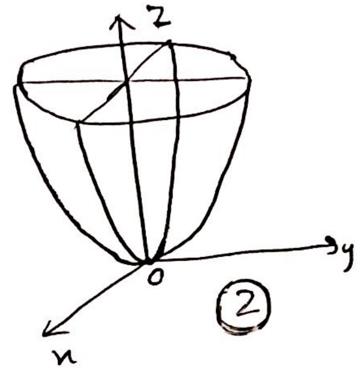
3 (a) The surface is paraboloid. 1, $x^2+4y^2=z$ 3

Trace	Equation	name
yz - plane $x=0$	$4y^2=z$	Parabola
xz - plane $y=0$	$x^2=z$	Parabola
xy - plane $z=0$	$x^2+4y^2=0$,	Point.
$z>0$	$x^2+4y^2=c$, $c>0$	Ellips.

6

4

3



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(b) $V(0)=\langle 0,0,0\rangle$, $r(0)=\langle 2,1,3\rangle$

6

$a(t)= i+2tj+3t^2k$

$V(t)=\int a(t)\,dt=\int (i+2tj+3t^2k)\,dt$

$=ti+t^2j+t^3k+c_1i+c_2j+c_3k$

$V(0)= c_1i+c_2j+c_3k=0 \Rightarrow c_1=0,\ c_2=0,\ c_3=0$

$V(t)= ti+t^2j+t^3k$ 2

$r(t)=\int (ti+t^2j+t^3k)\,dt$

$=\frac{t^2}{2}i+\frac{t^3}{3}j+\frac{t^4}{4}k+c_4i+c_5j+c_6k$

$r(0)= c_4i+c_5j+c_6k=2i+j+3k$

$c_4=2,\ c_5=1,\ c_6=3$

$r(t)=(\frac{t^2}{2}+2)i+(\frac{t^3}{3}+1)j+(\frac{t^4}{4}+3)k$ 2

Location at $t=2$

$r(2)=(\frac{4}{2}+2)i+(\frac{8}{3}+1)j+(\frac{16}{4}+3)k$

$=4i+\frac{11}{3}j+7k.$

object is at point $(4, 3.6, 7)$. 2

(c) $r(t)=ti+2\cos t\,j+3\sin t\,k$, $r'(t)=i-2\sin t\,j+3\cos t\,k$ 1

$r''(t)=-2\cos t\,j-3\sin t\,k.$ 1

8

1 $r'\cdot r''=4\sin t\cos t-9\sin t\cos t=-5\sin t\cos t$

2 $r'\times r''=6i+3\sin t\,j-2\cos t\,k$

$\|r'(t)\|=\sqrt{1+4\sin^2 t+9\cos^2 t}$, $\|r'\times r''\|=\sqrt{36+9\sin^2 t+4\cos^2 t}$

1 $a_T=\frac{|r'\cdot r''|}{\|r'(t)\|}=\frac{|-5\sin t\cos t|}{\sqrt{1+4\sin^2 t+9\cos^2 t}}=\frac{5\sin t\cos t}{\sqrt{1+4\sin^2 t+9\cos^2 t}}$

1 $a_N=\frac{\|r'\times r''\|}{\|r'(t)\|}=\sqrt{\frac{36+9\sin^2 t+4\cos^2 t}{1+4\sin^2 t+9\cos^2 t}}$, 1 $K=\sqrt{\frac{36+9\sin^2 t+4\cos^2 t}{(1+4\sin^2 t+9\cos^2 t)^3}}$

4 (a). $F(x,y,z) = xy^2 + z^3 + \sin(xyz) = 0$ 4

6 $$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{y^2 + yz\cos(xyz)}{3z^2 + xy\cos(xyz)}$$ 3

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{2xy + xz\cos(xyz)}{3z^2 + xy\cos(xyz)}$$ 3

4 (b)

6 (i) The rate of change of V at P in direction from P to origin is $D_u V(1,1,-3)$. , $\overrightarrow{PO} = \langle -1,-1,3\rangle$

$$u = \frac{\overrightarrow{PO}}{\|\overrightarrow{PO}\|} = \left\langle -\frac{1}{\sqrt{11}}, -\frac{1}{\sqrt{11}}, \frac{3}{\sqrt{11}} \right\rangle$$

$$\nabla V = \langle 4x^3yz - y^3,\ x^4z - 3xy^2,\ x^4y + 1\rangle$$

4 $$\nabla V(1,1,-3) = \langle 4(1)(1)(-3) - (1)^3,\ (1)(-3) - 3(1)(1),\ (1)(1)+1\rangle$$
$$= \langle -13, -6, 2\rangle$$

$$D_u V(1,1,-3) = \langle -13, -6, 2\rangle \cdot \left\langle -\frac{1}{\sqrt{11}}, -\frac{1}{\sqrt{11}}, \frac{3}{\sqrt{11}}\right\rangle = \frac{25}{\sqrt{11}}$$

1 (ii) V increases most rapidly in the direction of

$$\nabla V(1,1,-3) = -13i - 6j + 2k.$$

1 (iii) The maximum rate of change of V at P is

$$\|\nabla V(1,1,-3)\| = \sqrt{13^2 + 6^2 + 2^2} = \sqrt{209}$$

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Q.5 (a) $f(x,y) = x^3 e^{-\frac{y}{x}}$ (5)

6

(2) $\frac{\partial f}{\partial x} = 3x^2 e^{-\frac{y}{x}} - x^3 . \left(-\frac{y}{x^2}\right) e^{-\frac{y}{x}}$

$= 3x^2 e^{-\frac{y}{x}} + xy e^{-\frac{y}{x}} \longrightarrow 1$

(1) $\frac{\partial f}{\partial y} = x^3 e^{-\frac{y}{x}} . -\frac{1}{x} = - x^2 e^{-\frac{y}{x}}$

(1) $\frac{\partial^2 f}{\partial y^2} = x e^{-\frac{y}{x}}$

(2) $y \frac{\partial^2 f}{\partial y^2} - 3\frac{\partial f}{\partial y} = xy e^{-\frac{y}{x}} + 3x^2 e^{-\frac{y}{x}} = \frac{\partial f}{\partial x}$ //

5 (b). $f(x,y) = 2x^4 - x^2 + y^2$

6

$f_x = 8x^3 - 2x$, $f_y = 2y$

$f_{xx} = 24x^2 - 2$

(1) $f_{xy} = 0$

$f_{yy} = 2$

critical points: $f_x = 0$, $f_y = 0$

$8x^3 - 2x = 0$ $2y = 0 \Rightarrow y = 0$

$2x(4x^2 - 1) = 0$

(2) $x = 0$, $x = \frac{1}{2}$, $x = -\frac{1}{2}$

Points are $(0,0)$, $(-\frac{1}{2}, 0)$, $(\frac{1}{2}, 0)$

(1) Discriminant $D(x,y) = (24x^2 - 2)(2) = 48x^2 - 2$

(1) $D(0,0) = -2 < 0$ saddle point , $f(0,0) = 0$

(1) $D(-\frac{1}{2}, 0) = 8 > 0$ and $f_{yy} = 2 > 0 \Rightarrow$ Local min. at $(-\frac{1}{2}, 0)$

$D(\frac{1}{2}, 0) = 8 > 0$ and $f_{yy} = 2 > 0 \Rightarrow$ Local min. at $(\frac{1}{2}, 0)$.

$f(-\frac{1}{2}, 0) = 4$

$f(\frac{1}{2}, 0) = 4$.