M - 107 *FINAL EXAMONATION* *(SEMESTER I, 1431-1432)*

Question: 1 . (a) For what values of λ does the system of equations have
[<u>8</u>+4+10] (i) unique solution, (ii) infinitly many solutions and (iii) no solution.

$$\begin{aligned} 3x + \lambda z \quad\quad &= 2 \\ 3x + 3y \quad + \; 4z \; &= 4 \\ y + 2z &= 3 \end{aligned}$$

Solution: (a) Row Echelon Form

$$\begin{bmatrix} 3 & 0 & \lambda & 2 \\ 3 & 3 & 4 & 4 \\ 0 & 1 & 2 & \lambda \end{bmatrix} \xrightarrow{-R_1+R_2} \begin{bmatrix} 3 & 0 & \lambda & 2 \\ 0 & 3 & 4-\lambda & 2 \\ 0 & 1 & 2 & \lambda \end{bmatrix} \xrightarrow{3R_3-R_2} \begin{bmatrix} 3 & 0 & \lambda & 2 \\ 0 & 3 & 4-\lambda & 2 \\ 0 & 0 & \lambda+2 & 3\lambda-2 \end{bmatrix}$$

considering last row the matrix

$$0x + 0y + (\lambda+2)z = 3\lambda - 2$$

(i) If $\lambda = -2,$ then $0 = -8,$ which is not possible, so there is no solution.

(ii) If $\lambda \neq -2,$ then , we have three equations and three unkowns, so we have unique solution.

(iii) As both side of last row of the matrix will not have all zero value for any value of $\lambda,$ so system will not have infinitly many solutions.

(b) By using properties of determinant show that

$$\begin{vmatrix} y+z & z+x & y+x \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Solution

$$\begin{vmatrix} y+z & z+x & y+x \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} \approx \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} \approx (x+y+z) \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$Step:1.$ $R_1 + R_2,$

$Step:2.$ Take common form Row 1

Step: 3. Detereminant is zero as Row 1 and Row 3 are proportional or
If we subtract R_3 from R_1, we will get row with all entries zero .

(c) For the given system of linear equations:

$$x + 2y + 3z = 1$$
$$3x + y + 3z = 3$$
$$x + 2y + 4z = 1$$

i. **Write the system of equation in the form AX=B,**
ii. **Find adj (A),**
iii. **Use adj (A) to find A^{-1} , if exists, and**
iv. **Use A^{-1} to solve the given system.**

Solution: (i)

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \Rightarrow AX = B$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

(ii) Matrix of cofactors

$$c_{11} = -2, \quad c_{12} = -9, \quad c_{13} = 5$$
$$c_{21} = -2, \quad c_{22} = 1, \quad c_{23} = 0$$
$$c_{31} = 3, \quad c_{32} = 6, \quad c_{33} = -5$$

$$C = \begin{bmatrix} -2 & -9 & 5 \\ -2 & 1 & 0 \\ 3 & 6 & -5 \end{bmatrix}$$

$$(iii) AdjA = C^T = \begin{bmatrix} -2 & -2 & 3 \\ -9 & 1 & 6 \\ 5 & 0 & -5 \end{bmatrix}$$

(iv) Determinant of matriz A, $\det A = -5$

(v)Inverse of matrix A $A^{-1} = \dfrac{-1}{5}\begin{bmatrix} -2 & -2 & 3 \\ -9 & 1 & 6 \\ 5 & 0 & -5 \end{bmatrix}$

(vi) Solution of the system, $X = A^{-1}B = \dfrac{-1}{5}\begin{bmatrix} -2 & -2 & 3 \\ -9 & 1 & 6 \\ 5 & 0 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \dfrac{-1}{5}\begin{bmatrix} -2 \\ 6 \\ -5 \end{bmatrix}$

Question:2.(a) Find the comp_b^a and Proj_b^a

[6+6+6] if $a = -2i + j + k$ and $b = 4i - 3j + k$ **.**

Solution:

$$Comp_b^a = \frac{a.b}{\|b\|}$$

$$a.b = -8 - 3 + 1 = -10$$

$$\|b\| = \sqrt{16 + 9 + 1} = \sqrt{26}$$

$$Comp_b^a = \frac{-10}{\sqrt{26}}$$

$$\text{Proj}_b^a = \left(\frac{a.b}{\|b\|}\right).\frac{b}{\|b\|} = \frac{-10}{\sqrt{26}}\frac{< 4, -3, 1 >}{\sqrt{26}} = \frac{-5}{13} < 4, -3, 1 >$$

(b) Find the point at which the line

$x = 2 + 3t, \quad y = -4t, \quad z = 5 + t, \quad t \in R$

intersects the plane $4x + 5y - 2z = 18$**.**

Solution: As line intersects the plane so it must satisfy the equation of the plane.

$$4(2 + 3t) + 5(-4t) - 2(5 + t) = 18$$

$$8 + 12t - 20t - 10 - 2t = 18$$

$$-10t = 20$$

$$t = -2,$$

required point is

$x = 2 - 6 = -4, \quad y = -4(-2) = 8, \quad z = 5 - 2 = 3$

$(-4, 8, 3)$

(c) Find the direction cosines and direction angles of the vector

a = 2i + 3j - 6k

Solution:

1. Direction Cosine

$a = 2i + 3j - 6k$

$\|a\| = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$

$\cos\alpha = \frac{2}{7}, \cos\beta = \frac{3}{7}, \text{and } \cos\gamma = -\frac{6}{7}$

2. Direction Angles $\alpha = \cos^{-1}\left(\frac{2}{7}\right), \beta = \cos^{-1}\left(\frac{3}{7}\right), \text{and } \gamma = \cos^{-1}\left(\frac{-6}{7}\right)$

Question 3(a) Show that $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist**.** **[6+8+6]**

Solution:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

Path 1 $\quad y = x, \quad \lim_{(x,x)\to(0,0)} \frac{x^2}{x^2+x^2} = \lim_{(x,x)\to(0,0)} \frac{1}{2} = \frac{1}{2}$

Path 2 $\quad y = 2x, \quad \lim_{(x,2x)\to(0,0)} \frac{2x^2}{x^2+4x^2} = \lim_{(x,2x)\to(0,0)} \frac{2}{5} = \frac{2}{5}$

Limits on two paths is different, hence limit does not exist.

(b) A particle starts from position $r(1) = i + j$. Its velocity is $v(t) = 2ti + 3t^2 j + \sqrt{t}k$. Find its position at time t.

Solution:

$$r(t) = \int (2ti + 3t^2 j + \sqrt{t})dt$$

$$= t^2 i + t^3 j + \frac{2}{3} t^{\frac{3}{2}} k + c$$

$$At \quad t = 1, \quad i + j + \frac{2}{3}k + c = i + j, \quad c = -\frac{2}{3}k,$$

hence, $\quad r(t) = t^2 i + t^3 j + \frac{2}{3} t^{\frac{3}{2}} k - \frac{2}{3}k = t^2 i + t^3 j + \frac{2}{3}(t^{\frac{3}{2}} - 1)k$

(c) Find the parametric equations of the tangent line to the curve with parametric equations $x = t^5, y = t^4, z = t^3,$ **at the point (1, 1, 1).**

Solution:

$$r(t) = < t^5, t^4, t^3 >$$

$$r'(t) = < 5t^4, 4t^3, 3t^2 >$$

At point $(1,1,1), \quad t = 1,$

$$r'(1) = < 5, 4, 3 >$$

Equation of the tangent line is

$$x = 1 + 5t, y = 1 + 4t, z = 1 + 3t, \quad t \in R$$

Question: 4 .(a) Show that the function $f(x,y)=\ln(x^2+y^2) \quad x\neq 0, y\neq 0$

[6+6+10] **satisfies the Laplace equation** $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$

Solution:

$$f(x,y)=ln(x^2+y^2)$$

$$f_x=\frac{2x}{x^2+y^2}, \qquad f_{xx}=\frac{2(x^2+y^2)-4x^2}{\left(x^2+y^2\right)^2}$$

$$f_y=\frac{2y}{x^2+y^2}, \qquad f_{yy}=\frac{2(x^2+y^2)-4y^2}{\left(x^2+y^2\right)^2}$$

$$f_{xx}+f_{yy}=\frac{2(x^2+y^2)-4x^2+2(x^2+y^2)-4y^2}{\left(x^2+y^2\right)^2}=0.$$

(b) If $3xy^3+3x^2y-6y+9xy=8$ **find** $\frac{dy}{dx}$

Solution:

$$f(x,y)=3xy^3+3x^2y-6y+9xy-8=0$$

$$f_x=3y^3+6xy+9y$$

$$f_y=9xy^2+3x^2-6+9x$$

$$\frac{dy}{dx}=-\frac{f_x}{f_y}=-\frac{\left(3y^3+6xy+9y\right)}{\left(9xy^2+3x^2-6+9x\right)}$$

(c) Find the tangential and normal components of the acceleration of a particle moving along th curve ***x = t*** ^{3} ***, y = t+2, z = 4 , t*** $\in$ ***R*** **at** ***t =1*****. Also find the curvature at** ***t= 1.***

Solution:

$$r(t)=<t^3,t+2,4>$$

$$r'(t)=<3t^2,1,0>$$

$$r''(t)=<6t,0,0>$$

$$r'(t).r''(t)=18t^3$$

$$r'(t) \text{ x } r''(t)=\begin{vmatrix} i & j & k \\ 3t^2 & 1 & 0 \\ 6t & 0 & 0 \end{vmatrix}=<0,0,-6t>$$

$$\|r'(t)\|=\sqrt{9t^4+1}$$

$$\|r'(t) \text{ x } r''(t)\|=\sqrt{36t^2}=6t$$

Tangential component of acceleration, $a_T=\frac{r'(t).r''(t)}{\|r'(t)\|}=\frac{18t^2}{\sqrt{9t^4+1}}$

Normal component of acceleration, $a_N=\frac{\|r'(t) \text{ x } r''(t)\|}{\|r'(t)\|}=\frac{6t}{\sqrt{9t^4+1}}$

Curvature $\kappa=\frac{\|r'(t) \text{ x } r''(t)\|}{\left(\|r'(t)\|\right)^{\frac{3}{2}}}=\frac{6t}{\sqrt{\left(9t^4+1\right)^3}}$

at t = 1, $a_T=\frac{18}{\sqrt{10}}$, $a_N=\frac{6}{\sqrt{10}}$ and $\kappa=\frac{6}{10\sqrt{10}}$

Question 5. (a) The base radius and the height of the right circular cone are measured to be 10cm and 25cm respectively, with a possible error in measurement of 0.1cm in each. Use differentials to estimate the maximum error in the calculated volume of the cone $\left[V = \frac{1}{3}\pi r^2 h\right]$. [6]

Solution::

$Data \quad r = 10\ cm, \quad h = 25cm, \quad dr = \Delta r = .1cm, \quad dh = \Delta h = .1cm$

$Volume, \quad \text{V}=\frac{1}{3}\pi r^2 h$

$$\text{dV}= \frac{2}{3}\pi rhdr + \frac{1}{3}\pi r^2 dh$$

$$\text{dV}= \frac{2}{3}\pi(10)(25)(.1) + \frac{1}{3}\pi(10)^2(.1)$$

$$\text{dV}= 166.6\pi + 33.3\pi = 200\pi\ cm^3$$

(b) Find the directional derivative of $f(x,y) = xy$ at the point P(3, 2) in the direction of < 1, 2>. In which direction is the direction of the derivative maximum? What is the maximum value of the derivative? [8]

Solution::

$$\nabla f(x,y) = yi + xj$$

$$\nabla f(3,2) = 2i + 3j$$

$$a = i + 2j, \quad \|a\| = \sqrt{5}$$

$$u = \frac{a}{\|a\|} = \frac{1}{\sqrt{5}}i + \frac{2}{\sqrt{5}}j$$

$$(i)\ \ Df_a(3,2) = \nabla f(3,2).\ u = (2i+3j).\left(\frac{1}{\sqrt{5}}i + \frac{2}{\sqrt{5}}j\right) = \frac{2}{\sqrt{5}} + \frac{6}{\sqrt{5}} = \frac{8}{\sqrt{5}}$$

(ii) *Direction of Maximum rate of change occurs in*

direction of gradient is $\dfrac{2i+3j}{\sqrt{13}}$

(iii) *Maximum rate of change is magnitude of the gradient,* $\|\nabla f(3,2)\| = \sqrt{13}$

(c) Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = x + 2y$ **subject to constraint** $x^2 + y^2 = 5$**.** [6]

Solution:

$$f(x, y) = x + 2y, \qquad g(x, y) = x^2 + y^2 - 5 = 0$$

$$\nabla f = < 1, 2 > \qquad \nabla g = < 2x, 2y >$$

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

$$< 1, 2 > = \lambda < 2x, 2y >$$

$$1 = 2\lambda x, \qquad x = \frac{1}{2\lambda}$$

$$2 = 2\lambda y, \qquad y = \frac{1}{\lambda}$$

$$x^2 + y^2 - 5 = 0$$

substituting value of x and y in $g(x, y) = 0$

$$\frac{1}{4\lambda^2} + \frac{1}{\lambda^2} = 5 \quad \Rightarrow \frac{5}{4\lambda^2} = 5 \quad \Rightarrow \quad 4\lambda^2 = 1 \quad \Rightarrow \quad \lambda^2 = \frac{1}{4} \Rightarrow \quad \lambda = \pm\frac{1}{2}$$

$$x = \pm\frac{1}{2} x \frac{2}{1} = \pm 1$$

$$y = \pm 2$$

Values of the function at these points *are*

$$f(1, 2) = 5$$

$$f(-1, 2) = 3$$

$$f(1, -2) = -3$$

$$f(-1, -2) = -5$$

Largest *value is* maximum, *and smallest value is* minimum

Maximum value of the function, $f(1, 2) = 5$

Minmum value of the function, $f(-1-, 2) = -5$