Financial Mathematics for Actuaries

Chapter 5

Loans and Costs of Borrowing

Learning Objectives

1. Loan balance: prospective method and retrospective method

2. Amortization schedule

3. Sinking fund

4. Varying installments and varying interest rates

5. Quoted rate of interest and equivalent nominal rate of interest in monthly rest

6. Flat rate loan and flat rate discount loan

7. Annual percentage rate, annual percentage yield, effective rate of interest, and comparison rate of interest

5.1 Loan Balance: Prospective and Retrospective Methods

- Consider a loan with a fixed term of maturity, to be redeemed by a series of repayments.

- If the repayments prior to maturity are only to offset the interests, the loan is called an **interest-only loan**.

- If the repayments include both payment of interest and partial redemption of the principal, the loan is called a **repayment loan**.

- We consider two approaches to compute the balance of the loan: the **prospective method** and the **retrospective method**.

- The prospective method is forward looking. It calculates the loan balance as the present value of all future payments to be made.

- The retrospective method is backward looking. It calculates the loan balance as the accumulated value of the loan at the time of evaluation minus the accumulated value of all installments paid up to the time of evaluation.

- Let the loan amount be L, and the rate of interest per payment period be i. If the loan is to be paid back in n installments of an annuity-immediate, the installment amount A is given by

$$A = \frac{L}{a_{\overline{n}|i}}. \qquad (5.1)$$

- We also denote $L = B_0$, which is the loan balance at time 0.

- Immediately after the mth payment has been made the loan is redeemable by a $n - m$ annuity-immediate. We denote the loan balance

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after the mth installment by B_m, which is

$$
\begin{aligned}
B_m &= A\, a_{\overline{n-m}|i} \\
&= \frac{L\, a_{\overline{n-m}|i}}{a_{\overline{n}|i}}.
\end{aligned}
\tag{5.2}
$$

- To use the retrospective method, we first calculate the accumulated loan amount at time m, which is $L(1+i)^m$.

- The accumulated value of the installments is $A s_{\overline{m}|i}$.

- Thus, the loan balance can also be written as

$$
L(1+i)^m - A s_{\overline{m}|i}.
\tag{5.3}
$$

- To show that the two methods are equivalent, we simplify (5.3).

Thus, we have

$$Aa_{\overline{n}|i}(1+i)^m - As_{\overline{m}|i} = A\left[\frac{1-v^n}{i}(1+i)^m - \frac{(1+i)^m-1}{i}\right]$$

$$= A\left[\frac{1-v^{n-m}}{i}\right]$$

$$= A\,a_{\overline{n-m}|i}.$$

Example 5.1: A housing loan of \$400,000 was to be repaid over 20 years by monthly installments of an annuity-immediate at the nominal rate of 5% per year. After the 24th payment was made, the bank increased the interest rate to 5.5%. If the lender was required to repay the loan within the same period, how much would be the increase in the monthly installment. If the installment remained unchanged, how much longer would it take to pay back the loan?

Solution: We first demonstrate the use of the prospective and retrospective methods for the calculation of the loan balance after the 24th payment. From (5.1), the amount of the monthly installment is

$$
\begin{aligned}
A &= \frac{400{,}000}{a_{\overline{240}|0.05/12}} \\
&= \frac{400{,}000}{151.525} \\
&= \$2{,}639.82.
\end{aligned}
$$

By the prospective method, after the 24th payment the loan would be redeemed with a 216-payment annuity-immediate so that the balance is

$$
\begin{aligned}
A\, a_{\overline{216}|0.05/12} &= 2{,}639.82 \times 142.241 \\
&= \$375{,}490.
\end{aligned}
$$

By the retrospective method, the balance is

$$400{,}000 \left(1 + \frac{0.05}{12}\right)^{24} - 2{,}639 \, s_{\overline{24}|0.05/12} \;=\; 441{,}976.53 - 2{,}639.82 \times 25.186$$

$$=\; \$375{,}490.$$

Hence, the two methods give the same answer. After the increase in the rate of interest, if the loan is to be repaid within the same period, the revised monthly installment is

$$\frac{375{,}490}{a_{\overline{216}|0.055/12}} \;=\; \frac{375{,}490}{136.927}$$

$$=\; \$2{,}742.26,$$

so that the increase in installment is \$102.44. Let m be the remaining number of installments if the amount of installment remained unchanged. Thus,

$$a_{\overline{m}|0.055/12} = a_{\overline{216}|0.05/12} = 142.241,$$

from which we solve for $m = 230.88 \approx 231$. Thus, it takes 15 months more to pay back the loan. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Example 5.2: A housing loan is to be repaid with a 15-year monthly annuity-immediate of \$2,000 at a nominal rate of 6% per year. After 20 payments, the borrower requests for the installments to be stopped for 12 months. Calculate the revised installment when the borrower starts to pay back again, so that the loan period remains unchanged. What is the difference in the interest paid due to the temporary stoppage of installments?

Solution: The loan is to be repaid over $15 \times 12 = 180$ payments. After 20 payments, the loan still has 160 installments to be paid. Using the prospective method, the balance of the loan after 20 payments is

$$2,000 \times a_{\overline{160}|0.005} = \$219,910.$$

Note that if we calculate the loan balance using the retrospective method, we need to compute the original loan amount. The full calculation using the retrospective method is

$$2{,}000\, a_{\overline{180}|0.005}\, (1.005)^{20} - 2{,}000\, s_{\overline{20}|0.005} \;=\; 2{,}000\,(130.934 - 20.979)$$
$$=\; \$219{,}910.$$

Due to the delay in payments, the loan balance 12 months after the 20th payment is

$$219{,}910\,(1.005)^{12} = \$233{,}473,$$

which has to be repaid with a 148-payment annuity-immediate. Hence, the revised installment is

$$\frac{233{,}473}{a_{\overline{148}|0.005}} \;=\; \frac{233{,}473}{104.401}$$
$$=\; \$2{,}236.31.$$

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The difference in the interest paid is

$$2{,}236.31 \times 148 - 2{,}000 \times 160 = \$10{,}973.$$

□

Example 5.3: A man borrows a housing loan of $500,000 from Bank A to be repaid by monthly installments over 20 years at nominal rate of interest of 4% per year. After 24 installments Bank B offers the man a loan at rate of interest of 3.5% to be repaid over the same period. However, if the man wants to re-finance the loan he has to pay Bank A a penalty equal to 1.5% of the outstanding balance. If there are no other re-financing costs, should the man re-finance the loan?

Solution: The monthly installment paid to Bank A is

$$\frac{500{,}000}{a_{\overline{240}|0.04/12}} = \frac{500{,}000}{165.02}$$

$$= \quad \$3{,}029.94.$$

The outstanding balance after paying the 24th installment is

$$3{,}029.94\, a_{\overline{216}|0.04/12} \quad = \quad 3{,}029.94 \times 153.80$$
$$= \quad \$466{,}004.$$

If the man re-finances with Bank B, he needs to borrow

$$466{,}004 \times 1.015 = \$472{,}994,$$

so that the monthly installment is

$$\frac{472{,}994}{a_{\overline{216}|0.035/12}} \quad = \quad \frac{472{,}994}{160.09}$$
$$= \quad \$2{,}954.56.$$

As this is less than the installments of \$3,029.94 he pays to Bank A, he should re-finance. $\square$

5.2 Amortization

- If a loan is repaid by the amortization method, each installment is first used to offset the interest incurred since the last payment.

- The remaining part of the installment is then used to reduce the principal.

- Consider a loan to be repaid by a n-payment unit annuity-immediate. The principal is $a_{\overline{n}|i}$ and the interest incurred in the first payment period is $i\,a_{\overline{n}|i} = 1 - v^n$.

- Thus, out of the unit payment, $1 - v^n$ goes to paying the interest and the principal is reduced by the amount v^n.

- The principal is then reduced to

$$
\begin{aligned}
a_{\overline{n}|i} - v^n &= \frac{1 - v^n}{i} - v^n \\
&= \frac{1 - v^n(1 + i)}{i} \\
&= \frac{1 - v^{n-1}}{i} \\
&= a_{\overline{n-1}|i}.
\end{aligned}
$$

- Making use of this result we can construct an **amortization schedule** that separates each installment into the interest and principal components.

Table 5.1: Amortization of a loan of $a_{\overline{n}|i}$ by a n-payment unit annuity-immediate at effective interest rate of i per period

Time	Installment	Interest payment	Principal payment	Outstanding balance
0				$a_{\overline{n}|}$
1	1	$i\,a_{\overline{n}|} = 1 - v^n$	v^n	$a_{\overline{n}|} - v^n = a_{\overline{n-1}|}$
.	.	.	.	.
.	.	.	.	.
t	1	$i\,a_{\overline{n-t+1}|} = 1 - v^{n-t+1}$	v^{n-t+1}	$a_{\overline{n-t+1}|} - v^{n-t+1} = a_{\overline{n-t}|}$
.	.	.	.	.
.	.	.	.	.
n	1	$i\,a_{\overline{1}|} = 1 - v$	v	$a_{\overline{1}|} - v = 0$
Total	n	$n - a_{\overline{n}|}$	$a_{\overline{n}|}$	

Example 5.4:   Construct an amortization schedule of a loan of $5,000 to be repaid over 6 years with a 6-payment annuity-immediate at effective rate of interest of 6% per year.

Solution: The annual payment is

$$\frac{5,000}{a_{\overline{6}|0.06}} = \frac{5,000}{4.9173}$$
$$= \$1,016.81.$$

Using Table 5.1 with installments of 1,016.81 in place of 1, we obtain the amortization schedule in Table 5.2.

Table 5.2: Amortization of a loan of $5,000 by a 6-payment annuity-immediate at effective interest rate of 6% per year

Year	Installment	Interest payment	Principal payment	Outstanding balance
0				5,000.00
1	1,016.81	300.00^a	716.81^b	4,283.19^c
2	1,016.81	256.99^d	759.82	3,523.37
3	1,016.81	211.40	805.41	2,717.96
4	1,016.81	163.08	853.73	1,864.23
5	1,016.81	111.85	904.96	959.26
6	1,016.81	57.55	959.26	0.00
Total	6,100.86	1,100.86	5,000.00	

The details of the computation in Table 5.2 are given as follows:

^{a} Interest incurred in the first period: $5,000 \times 0.06 = \$300$.

^{b} Principal reduction with the first installment: $1,016.81 - 300 = \$716.81$.

[c] Outstanding balance at the end of the first year: $5,000 - 716.81 = \$4,283.19$.

[d] Interest incurred in the second period: $4,283.19 \times 0.06 = \$256.99$.

The rest of the computations follows similarly. $\qquad\square$

5.3 Sinking Fund

- A loan may also be served by payment of interest on a periodic basis, with a lump sum equal to the original principal paid at the end of the loan period.

- The borrower deposits an amount periodically into a **sinking fund** so as to accumulate to the principal.

- Consider a loan of amount $a_{\overline{n}|i}$.

- By the sinking fund method, the borrower pays an amount $i\,a_{\overline{n}|i} = 1 - v^n$ each period to serve the interest.

- He deposits installments into a sinking fund. Suppose the sinking fund accumulates interest at the rate of interest i.

- To accumulate to $a_{\overline{n}|i}$ at time n, each sinking fund installment is $a_{\overline{n}|i}/s_{\overline{n}|i} = v^n$.

- The total amount the borrower has to pay each period is

$$(1 - v^n) + v^n = 1,$$

which is the same as the amount paid in amortization.

- Table 5.3 presents the sinking fund schedule based on a loan of $a_{\overline{n}|i}$.

Table 5.3: Sinking fund schedule of a loan of $a_{\overline{n}|i}$ by a n-payment unit annuity-immediate at effective interest rate of i per period

Time	Installment	Interest payment	Sinking fund deposit	Sinking fund balance
1	1	$i\,a_{\overline{n}|} = 1 - v^n$	v^n	$v^n\,s_{\overline{1}|} = v^n$
2	1	$i\,a_{\overline{n}|} = 1 - v^n$	v^n	$v^n\,s_{\overline{2}|}$
.	.	.	.	.
.	.	.	.	.
t	1	$i\,a_{\overline{n}|} = 1 - v^n$	v^n	$v^n\,s_{\overline{t}|}$
.	.	.	.	.
.	.	.	.	.
n	1	$i\,a_{\overline{n}|} = 1 - v^n$	v^n	$v^n\,s_{\overline{n}|} = a_{\overline{n}|}$
Total	n	$n - nv^n$	nv^n	

- The interest paid in Table 5.3 remains constant at $1 - v^n$, while the interest paid in Table 5.1 using the amortization method decreases with the time period t.

- This is due to the interest earned in the sinking fund. At time t, the interest earned in the sinking fund over the payment period is
$$iv^n s_{\overline{t-1}|} = v^n[(1+i)^{t-1} - 1] = v^{n-t+1} - v^n.$$

- We subtract this amount from the interest served to obtain the net interest paid as
$$1 - v^n - \left(v^{n-t+1} - v^n\right) = 1 - v^{n-t+1},$$

which is the same as the value in Table 5.1.

Example 5.5: Construct a sinking fund schedule for the loan in Example 5.4.

Solution: The interest payment is $5{,}000 \times 0.06 = \$300$, and the sinking fund deposit is
$$\frac{5{,}000}{s_{\overline{6}|0.06}} = \$716.81.$$

Hence, the total annual installment is 300 + 716.81 = $1,016.81. The sinking fund schedule is presented in Table 5.4.

Table 5.4:   Sinking fund schedule of a loan of $5,000 by a 6-payment annuity-immediate at effective interest rate of 6% per year

Year	Installment	Interest payment	Sinking fund deposit	Sinking fund interest	Sinking fund balance
1	1,016.81	300.00	716.81	0.00	716.81
2	1,016.81	300.00	716.81	43.01	1,476.63
3	1,016.81	300.00	716.81	88.60	2,282.04
4	1,016.81	300.00	716.81	136.92	3,135.77
5	1,016.81	300.00	716.81	188.15	4,040.73
6	1,016.81	300.00	716.81	242.44	5,000.00
Total	6,100.86	1,800.00	4,300.86	699.14	

- Throughout the 6 years the sinking fund accumulates interest of $699.14, which is the difference between the interest paid under

the sinking fund method and the interest paid using amortization, namely, $1,800 - 1,100.86$. □

- We now consider the case where the sinking fund earns a rate of interest different from that charged to the loan.

- Suppose the sinking fund earns interest at the rate of j per payment period. For a loan of amount $a_{\overline{n}|i}$, to accumulate to the principal in the sinking fund over n periods the periodic installment is

$$\frac{a_{\overline{n}|i}}{s_{\overline{n}|j}}.$$

- Hence, the total installment is

$$i\, a_{\overline{n}|i} + \frac{a_{\overline{n}|i}}{s_{\overline{n}|j}} = a_{\overline{n}|i}\left(i + \frac{1}{s_{\overline{n}|j}}\right). \tag{5.4}$$

- When $i = j$, the sinking fund method and the amortization method are the same, so that the monthly payment in (5.4) is equal to 1. Hence, we have

$$i + \frac{1}{s_{\overline{n}|i}} = \frac{1}{a_{\overline{n}|i}}.$$ (5.5)

- In (5.5), the right-hand side is the n-payment annuity-immediate to pay back a loan of 1 by amortization, and the left-hand side states that the annuity consists of payment of i for the interest and $1/s_{\overline{n}|i}$ for the sinking fund.

- From (5.4) we can see that a loan of unit amount can be redeemed over n periods by a sinking fund that charges rate of interest i to the loan and credits rate of interest j to the sinking fund by periodic payment of amount

$$i + \frac{1}{s_{\overline{n}|j}}.$$

- If this amount is used to redeem a loan of unit amount by the amortization method at the rate of interest i^*, then we have

$$\frac{1}{a_{\overline{n}|i^*}} = i + \frac{1}{s_{\overline{n}|j}}. \qquad (5.6)$$

- We interpret i^* as the interest rate for the amortization method that is equivalent to the sinking fund method charging interest rate i to the loan and crediting interest rate j to the sinking fund.

- If $i > j$, then $s_{\overline{n}|i} > s_{\overline{n}|j}$ and we have

$$\begin{aligned}
1 &= a_{\overline{n}|i^*}\left(i + \frac{1}{s_{\overline{n}|j}}\right) \\
&> a_{\overline{n}|i^*}\left(i + \frac{1}{s_{\overline{n}|i}}\right) \\
&= \frac{a_{\overline{n}|i^*}}{a_{\overline{n}|i}},
\end{aligned}$$

which implies $a_{\overline{n}|i} > a_{\overline{n}|i^*}$. From this we conclude $i^* > i$.

- Thus, if the interest charged to the loan is higher than the interest credited to the sinking fund, the equivalent interest rate of the loan redeemed by amortization is higher.

Example 5.6: A 5-year loan of \$500 is to be repaid by a 5-payment annuity-immediate using the sinking fund method. The loan charges 6% interest and the sinking fund credits 4% interest. What is the annual installment? If this installment is used to pay back a loan of the same amount by amortization, what is the rate of interest?

Solution: The installment is

$$500 \times 0.06 + \frac{500}{s_{\overline{5}|0.04}} = \$122.31.$$

We solve for i^* in the equation

$$122.31 a_{\overline{5}|i^*} = 500$$

to obtain $i^* \approx 7.11\%$. □

Example 5.7: A loan is to be repaid by an 8-payment annual annuity-immediate of \$200 at interest of 5% for both the loan charge and the sinking fund credit. What is the interest component of the 4th payment in the amortization schedule? What is the interest credited to the sinking fund at the end of year 5?

Solution: In the amortization schedule, the loan balance after the third payment is

$$200 a_{\overline{5}|0.05} = \$865.90.$$

Hence, in amortization the interest in the 4th payment is

$$865.90 \times 0.05 = \$43.30.$$

For the sinking fund method, we first calculate the loan amount, which is

$$200a_{\overline{8}|0.05} = \$1{,}292.64.$$

Thus, the sinking fund payment is

$$\frac{1{,}292.64}{s_{\overline{8}|0.05}} = \$135.37.$$

In the beginning of year 5, the sinking fund balance is

$$135.37s_{\overline{4}|0.05} = \$583.46,$$

so that the interest credited to the sinking fund at the end of year 5 is

$$583.46 \times 0.05 = \$29.17.$$

□

- We now consider a fund in which the interest on the sinking fund deposits earns a **reinvestment rate of interest** different from the sinking fund deposits.

- We assume the installments paid to the sinking fund generate interest at the rate j, while the interest received is reinvested at the rate of interest j'.

- To calculate the accumulated amount, we first consider the case of a unit payment to the sinking fund at time 0. Interest of amount j is paid at time 1, 2, $\cdots$, n. These interest payments earn interest at the rate j'.

- The time diagram of the fund is given in Figure 5.1.

- The accumulated amount at time n is made up of the unit-payment
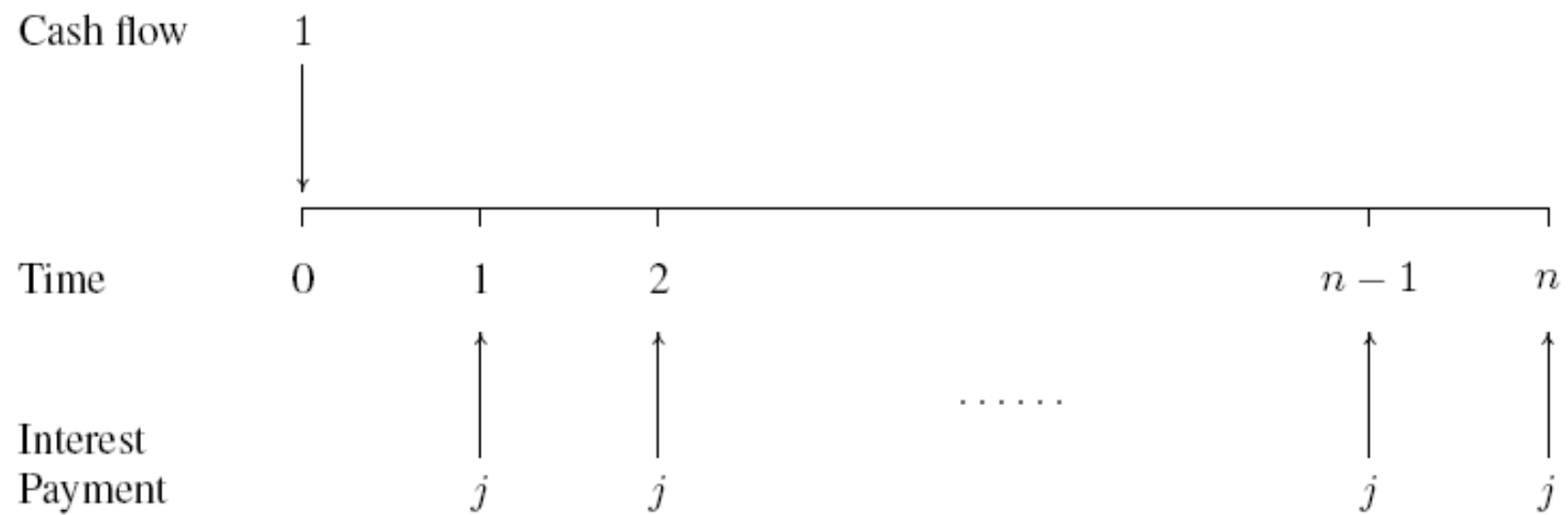
Figure 5.1:   Single payment earning interest at the rate j for n periods

at time 0 and the n-payment annuity-immediate of amount j which earns interest at rate j'. Thus, the accumulated value is $1 + js_{\overline{n}|j'}$.

- If $j = j'$, this expression reduces to $(1 + j)^n$.

- If an n-payment annuity-immediate of unit amount is paid to the sinking fund, the accumulated value of this annuity is

$$n + j(Is)_{\overline{n-1}|j'} = n + j \left(\frac{s_{\overline{n}|j'} - n}{j'} \right). \qquad (5.9)$$

Example 5.8: A loan of \$1,000 at effective rate of interest of 6% per year is to be redeemed over 5 years by a sinking fund that credits interest at 5.5% with reinvestment rate of interest of 5%. Calculate the annual installments of the annuity-immediate required to redeem the loan. What is the equivalent rate of interest using the amortization method?
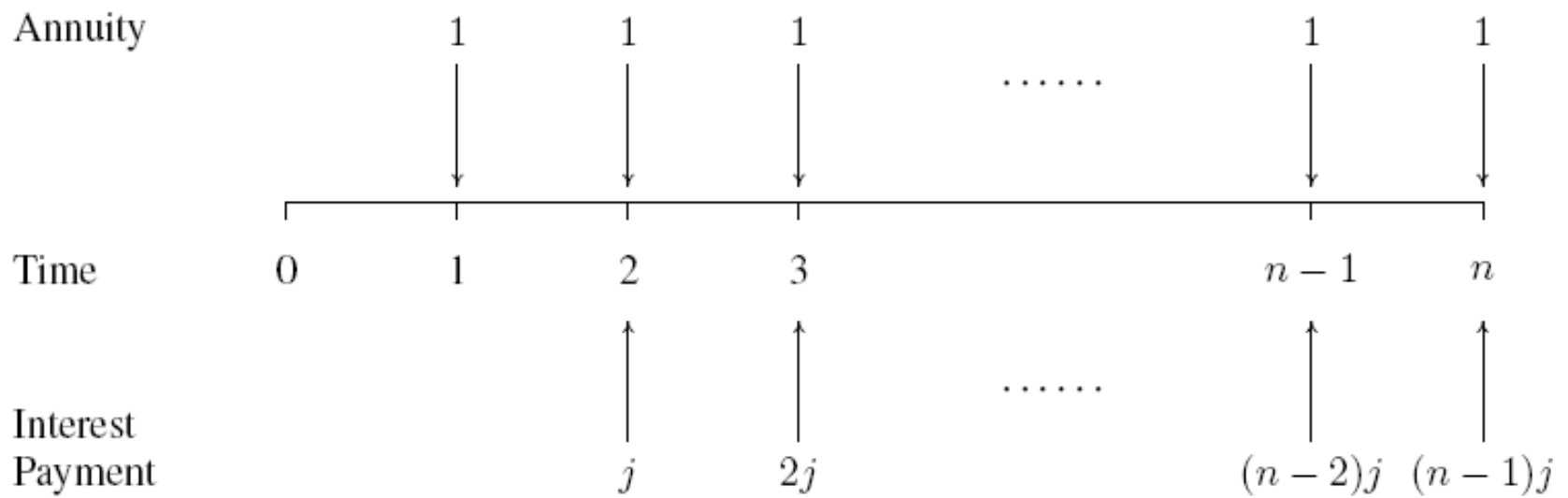
Figure 5.2: A unit-payment annuity generating interests at rate j

Solution: The interest payment per year is $1{,}000 \times 0.06 = \$60$. Let X be the annuity payments to the sinking fund, which, after 5 years accumulate to

$$X \left[5 + 0.055 (Is)_{\overline{4}|0.05} \right] \;=\; X \left[5 + 0.055 \left(\frac{s_{\overline{5}|0.05} - 5}{0.05} \right) \right]$$

$$= \; 5.5782 X.$$

Setting this to 1,000, we obtain $X = 179.27$. Hence, the annual installment is $60 + 179.27 = \$239.27$. If this payment redeems the load by amortization, the rate of interest i^* solves the equation

$$239.27 a_{\overline{5}|i^*} = 1{,}000,$$

from which we obtain $i^* \approx 6.29\%$. $\qquad\qquad\qquad \square$

5.4 Varying Installments and Varying Interest Rates

- Installments to repay a loan need not be level. Suppose a loan over n periods at rate of interest i is to be paid back by a varying n-payment annuity-immediate $A_1, \cdots, A_n$. The loan amount L is given by

$$L = \sum_{t=1}^{n} A_t v^t. \qquad (5.10)$$

- The balance of the loan B_{k-1} after the $(k-1)$th payment is

$$B_{k-1} = \sum_{t=k}^{n} A_t v^{t-k+1}.$$

- Thus, the interest in the kth payment is iB_{k-1} and the principal is reduced by the amount $A_k - iB_{k-1}$.

Example 5.9: A 20-year loan is to be repaid by installments of $100 at the end of year 1, $200 at the end of year 2, and so on, increasing by $100 each year, up to $600 at the end of the sixth year. From then onwards, a level installment of $600 will be paid. The rate of interest charged is 6%. Calculate the loan amount. What is the interest and the principal reduction in the 3rd and the 12th payments?

Solution: The time diagram of the repayment is shown in Figure 5.3. The loan amount L is

$$
100(Ia)_{\overline{6}|} + 600v^6 a_{\overline{14}|} \;=\; 100 \times \left[\frac{5.2124 - 6(1.06)^{-6}}{0.06} \right] + 3{,}931.56
$$

$$
=\; \$5{,}569.22.
$$

To calculate the balance after the second installment, we use the retrospective method to obtain

$$
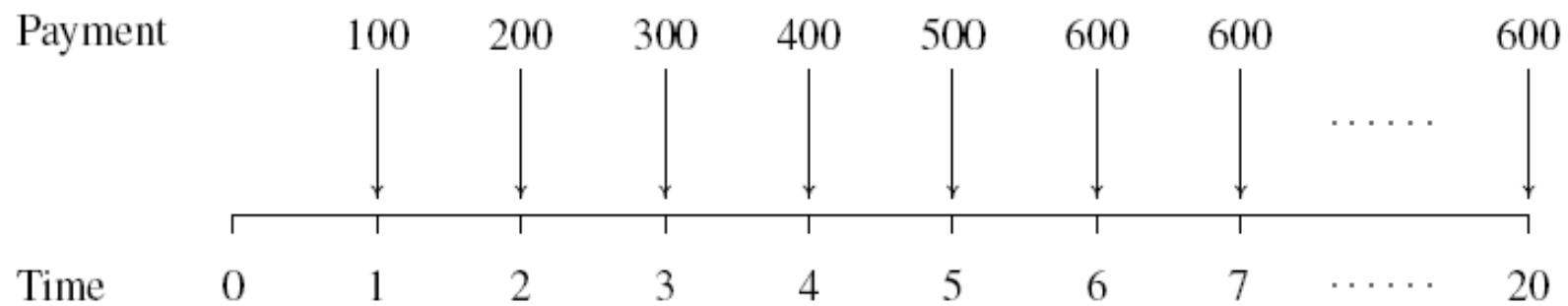B_2 = 5{,}569.22(1.06)^2 - 100(1.06) - 200 = \$5{,}951.57.
$$

Figure 5.3:   Illustration of payments in Example 5.9

Note that the loan amount has actually increased, as the initial payments are not sufficient to cover the interest. This is the case of a *negative amortization*. The interest in the third payment is

$$0.06 \times 5{,}951.57 = \$357.09.$$

The third installment of $300 is not sufficient to pay for the interest, and there is an increase of loan balance by $57.09 to $6,008.66.

To split the 12th installment into interest and principal, we first calculate the loan balance after the 11th payment using the prospective method, which is

$$
\begin{aligned}
B_{11} &= 600 a_{\overline{9}|} \\
&= \$4{,}081.02.
\end{aligned}
$$

Thus, the interest in the 12th payment is $4{,}081.02 \times 0.06 = \$244.86$, so that the principal reduction is $600 - 244.86 = \$355.14$. $\square$

- For the sinking fund method in which the loan charges rate of interest i and the sinking fund credits interest at the rate j, the interest in each installment is iL.

- If the sinking fund contribution at time t is S_t for $t = 1, \cdots, n$, then

$$\sum_{t=1}^{n} S_t(1+j)^{n-t} = L, \tag{5.11}$$

and the tth installment is

$$A_t = S_t + iL.$$

Example 5.10: Suppose a 20-year loan of \$1,000 at rate of interest 6% is to be redeemed by the sinking fund method which credits sinking fund deposits at 5.5%. If the sinking fund deposit increases by 5% in each

payment, what is the amount of the sixth installment of this loan? What is the total amount of installments to repay this loan?

Solution: Let S be the first sinking fund deposit. The tth sinking fund deposit is $S(1.05)^{t-1}$ for $t = 1, \cdots, 20$. As the deposits accumulate to \$1,000 in 20 years, we have

$$
\begin{aligned}
1,000 &= \sum_{t=1}^{20} S(1.05)^{t-1}(1.055)^{20-t} \\
&= \frac{S(1.055)^{20}}{1.05} \sum_{t=1}^{20} \left(\frac{1.05}{1.055}\right)^t \\
&= 2.7788S \sum_{t=1}^{20} (0.9953)^t \\
&= 52.91S,
\end{aligned}
$$

from which we solve for $S = \$18.90$. Thus, the 6th installment is

$$1{,}000 \times 0.06 + 18.90(1.05)^5 = \$84.12.$$

The total amount paid is

$$1{,}000 \times 0.06 \times 20 + 18.90 \sum_{t=0}^{19}(1.05)^t = \$1{,}824.95.$$

For comparison, the total payments using the amortization method is

$$20 \times \frac{1{,}000}{a_{\overline{20}|0.06}} = \$1{,}743.69.$$

$\square$

Example 5.11: A man borrows a housing loan of $300,000 from a bank for 20 years, to be paid back by monthly installments. In the first year the bank charges an interest rate of 2.5%, followed by the second year of

3%. From then onwards, the interest rate will be 5%. All rates quoted are nominal per year. The bank calculates installments *as if* the loan is to be repaid over the remaining term at the ongoing rate. Thus, the first year installment assumes rate of interest of 2.5% for 20 years.

(a) Calculate the installments in the first, second and third year.
(b) What are the interests paid in the first and the second year?

Solution: (a) The first-year payment is

$$\frac{300{,}000}{a_{\overline{240}|0.025/12}} = \frac{300{,}000}{188.71}$$
$$= \$1{,}589.71.$$

Using the prospective method, the loan balance after 12 payments is

$$1{,}589.71 a_{\overline{228}|0.025/12} = \$288{,}290$$

so that the revised installment is

$$\frac{288{,}290}{a_{\overline{228}|0.03/12}} = \frac{288{,}290}{173.63}$$

$$= \$1{,}660.38.$$

Repeating the same calculations, the balance after 24 payments is $1{,}660.38\, a_{\overline{216}|0.03/12} = \$276{,}858$. The revised installment is

$$\frac{276{,}858}{a_{\overline{216}|0.05/12}} = \$1{,}946.41.$$

For (b), the total amount of payments in the first year is $12 \times 1{,}589.71 = \$19{,}076.52$, and the reduction in principal is $300{,}000 - 288{,}290 = \$11{,}710$. Thus, the interest served is $19{,}076 - 11{,}710 = \$7{,}366$. Likewise, the interest served in the second year is $12 \times 1{,}660.38 - (288{,}290 - 276{,}858) = \$8{,}492.56$. □

5.5 Comparison of Borrowing Costs

- The cost of a loan depends on the **quoted rate of interest** and the way the interest is calculated.

- For example, the balance of a loan may be computed on a **periodic rest basis**, where the period is specified in the loan agreement.

- This means that interest is charged on the outstanding balance at the beginning of the period for the whole period, although installments may be made prior to the end of the period.

- To illustrate, we consider an annual-rest versus a monthly-rest loan.

- An annual-rest loan charges interest for the whole year, even though installments may be made monthly throughout the year.

Example 5.12: A man is borrowing a housing loan of $500,000 from a bank. The quoted rate of interest is 6.5% per annum. Calculate the monthly installments in arrears if the loan is based on (a) annual rest, and (b) monthly rest. Compare the results for a 15-year loan and a 20-year loan.

Solution: For the annual-rest loan, we first calculate the annual payment, which, for the 15-year loan, is

$$\frac{500{,}000}{a_{\overline{15}|0.065}} = \frac{500{,}000}{9.4027}$$
$$= \$53{,}176.22.$$

The monthly installment is

$$\frac{53{,}176.22}{12} = \$4{,}431.35.$$

For the monthly-rest loan, the monthly installment is

$$\frac{500{,}000}{a_{\overline{180}|0.065/12}} = \frac{500{,}000}{114.796}$$

$$= \$4{,}355.55.$$

For the 20-year loan, the monthly installments for the annual-rest and monthly-rest loans are, respectively, $3,781.52 and $3,727.87. □

Example 5.13: A bank offers two options of housing loans. Option A is annual-rest at a quoted rate of 6.25% per annum. Option B is monthly-rest at 6.5% per annum. Both options require repayment by monthly installments in arrears. If a 10-year loan is required, which option should be preferred? What is your answer for a 20-year loan?

Solution: We consider an arbitrary loan amount of $500,000. For a 10-year loan, the monthly installments of the annual-rest and monthly-rest

loans are, respectively, \$5,728.40 and \$5,677.40. Thus, the monthly-rest loan is preferred. For a 20-year loan, the installments for the annual-rest and monthly-rest loans are, respectively, \$3,706.75 and 3,727.85. Thus, the annual rest loan is preferred. □

- When loans are paid back by installments of the same payment interval, they can be compared by converting the quoted rates to an equivalent nominal rate with compounding frequency equal to the payment frequency.

- We can convert the quoted rate of interest for the loan on annual rest to an **equivalent nominal rate (ENR) of interest based on monthly rest**.

- Consider a principal of amount L. For a n-year annual-rest loan at quoted rate r to be repaid by monthly installments, the installment

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amount is

$$\frac{L}{12 a_{\overline{n}|r}}.$$ (5.12)

- We denote the annualized ENR on monthly-rest by r^*, so that the monthly installment

$$\frac{L}{a_{\overline{12n}|r^*/12}}$$ (5.23)

is equal to (5.12). This implies

$$12 a_{\overline{n}|r} = a_{\overline{12n}|r^*/12},$$ (5.14)

from which we can solve for r^* numerically.

Example 5.14: For the loans in Example 5.13, calculate the ENR of the annual-rest loan for a 10-year and 20-year loan.

Solution: For the 10-year loan, the ENR is 6.70%. This can be checked by verifying the equation

$$12a_{\overline{10}|0.0625} = a_{\overline{120}|0.0670/12}.$$

Similarly, we can verify that the ENR of the 20-year annual-rest loan is 6.43%. Thus, the ENR of the 10-year annual-rest loan is higher than the quoted rate of the monthly-rest loan, which shows that the monthly-rest loan is preferred. On the other hand, the ENR of the 20-year annual-rest loan is lower than the quoted rate of the monthly rest loan, which means that the annual-rest loan should be preferred. This is consistent with the answer in Example 5.13. □

- It should be noted that the ENR r^* defined in (5.14) is actually $r^{(12)}$ as defined in Chapter 1.

- If the installment is made m times a year, we can modify (5.14) to define the ENR accordingly, and we shall denote it simply by $r^{(m)}$.

- Note that the ENR is a *nominal* rate; it is not an *effective* rate as defined in Chapter 1.

5.6 Flat Rate Loan and Flat Rate Discount Loan

- A **flat-rate loan** charges interest on the original principal for the whole term of the loan, even though the loan is progressively repaid by periodic installments.

- For a n-year loan of amount L charged on a quoted flat rate of interest of r per annum, the monthly installment A is computed by

$$A = \frac{L(1 + rn)}{12n}.\qquad(5.15)$$

- To compute the ENR of this loan, we solve for $r^{(12)}$ from the equation

$$A a_{\overline{12n}| r^{(12)}/12} = L,$$

or, equivalently,

$$a_{\overline{12n}| r^{(12)}/12} = \frac{12n}{1 + rn}.\qquad(5.16)$$

Example 5.15: A bank charges a car loan at 3% per annum flat. If the loan period is 5 years, and the repayments are by monthly installments in arrears, what is the ENR? What is the effective rate of interest?

Solution: Using (5.16), we solve for $r^{(12)}$ in

$$a_{\overline{60}|r^{(12)}/12} = \frac{12 \times 5}{1 + 0.03 \times 5} = 52.1739$$

to obtain $r^{(12)} = 5.64\%$. The effective rate of interest is

$$\left(1 + \frac{0.0564}{12}\right)^{12} - 1 = 5.79\%.$$

$\square$

- If a loan is to be repaid by monthly installments in *advance*, i.e., by an annuity-due, to solve for the ENR, we modify (5.16) to

$$\ddot{a}_{\overline{12n}|r^{(12)}/12} = \frac{12n}{1 + rn}. \tag{5.17}$$

- A loan with such repayments is called a **flat-rate discount loan**.

Example 5.16: Solve the problem in Example 5.15, assuming now the monthly installments are to be paid in advance.

Solution: We solve for $r^{(12)}/12$ from

$$\ddot{a}_{\overline{60}|r^{(12)}/12} = 52.1739,$$

to obtain $r^{(12)}/12 = 0.004876$. Thus the ENR is $r^{(12)} = 5.84\%$ and the effective rate of interest is $(1.004876)^{12} - 1 = 6.00\%$. Hence we can see that the effective cost of the payments-in-advance loan is about 21 basis points higher than that of the payments-in-arrears loan, although the quoted rates of interest are the same. □

Example 5.17: A car buyer is offered two options of car loan. Option A charges interest at 4% flat with monthly installments paid in arrears.

Option B charges 3.75% flat with monthly installments paid in advance. A loan of $50,000 is required, and the man is considering whether to borrow for 2 years or 5 years. Which option should he prefer?

Solution: For a 2-year loan, Option A requires installments of

$$\frac{50{,}000(1 + 2 \times 0.04)}{2 \times 12} = \$2{,}250.00,$$

while Option B requires

$$\frac{50{,}000(1 + 2 \times 0.0375)}{2 \times 12} = \$2{,}239.58.$$

Similarly, the installments for the 5-year loans of Option A and Option B are, respectively, $1,000.00 and $989.58. As Option B has a lower quoted rate of interest, its monthly repayment is always lower than that of Option A with the same loan period.

We solve the ENR of the loans using (5.16) and (5.17). The ENR of Option A for the 2- and 5-year loans are, respectively, 7.5% and 7.42%. Similarly, the ENR of Option B for the 2- and 5-year loans are, respectively, 7.68% and 7.23%. Thus, based on the ENR, Option A is preferred if a 2-year loan is required, while Option B is preferred if a 5-year loan is required. In contrast, if we compare the loans by the monthly installments, we will erroneously conclude that Option B is preferred due to its lower monthly payment. □

- When the payment frequencies of different loans are not the same, the ENR is not a valid measure to compare the loans' actual costs. The appropriate measure is the effective rate of interest computed from the ENR. Below is an example.

Example 5.18: A bank offers a loan of $100,000 for 2 years under three different options. Option A charges 6.75% per annum on monthly rest, with repayment installments made monthly. Option B charges 5.50% per annum on annual rest, with repayment installments made quarterly. Option C charges interest flat at 3.5% in year 1 and flat at 4.00% in year 2. The principal is repaid by equal monthly installments, and the interest in each year is repaid by equal monthly installments in that year. If all payments are made in arrears, which option is preferred?

Solution: Option A is straightforward. The effective rate of interest is

$$\left[1 + \frac{0.0675}{12}\right]^{12} - 1 = 6.96\%.$$

For Option B, the quarterly installment is

$$\frac{100,000}{4a_{\overline{2}|0.055}} = \$13,540.59,$$

and we solve for r^* from the equation

$$13{,}540.59 \, a_{\overline{8}|r^*} = 100{,}000$$

to obtain $r^* = 0.01812$. Hence, the ENR in quarterly rest is $4 \times 0.01812 = 7.25\%$. The effective rate of interest for Option B is

$$(1.01812)^4 - 1 = 7.45\%.$$

For Option C, the monthly installment for the principal is

$$\frac{100{,}000}{24} = \$4{,}166.67.$$

Therefore, the total monthly installment for the first year is

$$4{,}166.67 + \frac{100{,}000 \times 0.035}{12} = \$4{,}458.34$$

while that for the second year is

$$4{,}166.67 + \frac{100{,}000 \times 0.040}{12} = \$4{,}500.00.$$

We solve for r^* from the equation

$$4{,}458.34\, a_{\overline{12}|r^*} + 4{,}500.00\, a_{\overline{12}|r^*}(1+r^*)^{-12} = 100{,}000$$

to obtain $r^* = 0.005856$, so that the ENR in monthly rest is $12 \times 0.005856 = 7.03\%$. The effective rate of interest of Option C is

$$(1.005856)^{12} - 1 = 7.26\%.$$

Hence, Option A has the lowest effective rate of interest, and Option B has the highest effective rate of interest. □

5.7 Borrowing Costs and Reference Rates

- The reference rates are required by regulators in communicating the borrowing costs to the clients.

- The term **Annual Percentage Rate (APR)** is commonly used in Canada, Hong Kong, UK and US.

- The equivalence of the APR in the academic literature is $r^{(m)}$, and in particular, for $m = 12$. Australia uses the term **Comparison Rate**, which takes account of most fee costs.

- In Singapore, the term used is the **Effective Interest Rate** (EIR), which is actually the Equivalent Nominal Rate defined in Section 5.5. The loan balance in early exit is determined by the so-called **Rule of 78.**

- For a n-year loan of amount L with k installments made, the Rule of 78 determines the balance B after the kth payment as

$$B = \text{Principal} + \text{Interest} - \text{Installments Paid} - \text{Interest Rebate},$$

where Interest is the total interest charged over the whole term of the loan, and the Interest Rebate, denoted by R, is

$$R = \frac{(12n - k)(12n - k + 1)Lrn}{12n(12n + 1)}. \qquad (5.18)$$

- Hence, B is given by

$$B = L(1 + rn) - Ak - R, \qquad (5.19)$$

where A is the monthly installment given in (5.15).

Example 5.19: A 6-year car loan charges flat rate at 2.5% per annum, to be repaid by a monthly annuity-due. Calculate the ENR in monthly rest of the loan. If the borrower decides to redeem early after 12, 24, 36, 48 and 60 payments, calculate the ENR in monthly rest charged over these periods.

Solution: The ENR in monthly rest is the value of r^* that satisfies the equation (see (5.17)):

$$\ddot{a}_{\overline{12n}|r^{(12)}/12} = \frac{72}{1 + 6 \times 0.025} = 62.6087,$$

from which we obtain $r^* = 4.853\%$. If the loan is redeemed after k payments, the ENR in monthly rest is r^* that solves

$$L = \left[\sum_{t=0}^{k-1} \frac{A}{\left(1 + \dfrac{r^*}{12}\right)^t} \right] + \frac{B}{\left(1 + \dfrac{r^*}{12}\right)^{k-1}},$$

where A and B are given in (5.15) and (5.19), respectively. Solving the above equation for various values of k we obtain the following results:

k	12	24	36	48	60
r^* (%)	5.369	5.085	4.972	4.906	4.867

If the loan is redeemed at maturity, the annual effective rate of interest is $(1 + 0.04853/12)^{12} - 1 = 4.9624\%$. However, if the loan is redeemed after 1 year, the effective rate of interest is $(1 + 0.05369/12)^{12} - 1 = 5.5503\%$. In the case the loan involves a further penalty for early redemption, the difference in the borrowing costs may be even higher. $\square$