CH.8
Functions of Random Variables

- **Method of Distribution Functions**
- **One-to-One Transformations**
- **Two-to Two Transformations**

  (Joint distribution of Functions of Random Variables **)**
- **Method of Moment-Generating Functions**

Method of Distribution Functions (The CDF technique)

- $X_1,\ldots,X_n \sim f(x_1,\ldots,x_n)$
- $U=g(X_1,\ldots,X_n)$ – Want to obtain $f_U(u)$
- Find values in $(x_1,\ldots,x_n)$ space where $U=u$
- Find region where $U \leq u$
- Obtain $F_U(u)=P(U \leq u)$ by integrating $f(x_1,\ldots,x_n)$ over the region where $U \leq u$
- $f_U(u) = dF_U(u)/du$

Example:

EXAMPLE 2: LET X BE ANY CONTINUOUS RANDOM VARIABLE.

LET $Y = X^2$

THEN $F_Y(y) = P[Y \le y] = P[X^2 \le y]$

$$= P[-\sqrt{y} \le X \le \sqrt{y}] = \begin{cases} F_X(\sqrt{y}) - F_X(-\sqrt{y}) & y > 0 \\ 0 & \text{OTHERWISE} \end{cases}$$

CDF METHOD

EXAMPLE 1: $X \sim EXP(\theta)$

$f_X(x) = \frac{1}{\theta} e^{-x/\theta}$ (pdf) — annotation: $f_X(x)$

$F_X(x) = 1 - e^{-x/\theta}$ (CDF)

$x > 0 \quad \theta > 0$

LET $Y = e^X$. FIND CDF OF Y IN TERMS OF THE CDF OF X.

BY DEFINITION: $F_Y(y) = P[Y \le y]$ NOTE: $1 < Y < \infty$

$$= P[e^X \le y] = P[X \le \ln y]$$

$$= F_X[\ln y] = 1 - e^{-(\ln y)/\theta}$$

$$= 1 - e^{\ln(y^{-1/\theta})} = \begin{cases} 1 - y^{-1/\theta} & 1 < y < \infty \\ 0 & \text{OTHERWISE} \end{cases}$$

Example:

EXAMPLE 3: LET $X \sim N(\mu, \sigma^2)$ LET $Z = \frac{X-\mu}{\sigma}$

$$F_Z(z) = P[Z \le z] = P\left[\frac{X-\mu}{\sigma} \le z\right]$$

$$= P[X \le z\sigma + \mu]$$

$$= F_X[z\sigma + \mu]$$

~~THEOREM 6.3.3~~ $U = F(X)$, X CONTINUOUS , $F(x)$ IS 1-TO-1 }
F(x) INCREASING FUNCTION }

$$F_U(u) = P[U \le u] = P[F(X) \le u]$$

$$= P[F^{-1}(F(X)) \le F^{-1}(u)]$$

$$= P[X \le F^{-1}(u)]$$

$$= F[F^{-1}(u)] = u$$

NOTE: CDF $F_U(u) = u$ FOR $0 \le u \le 1$ $\Rightarrow U \sim \text{UNIF}(0,1)$

$\Rightarrow F(X) \sim \text{UNIF}(0,1)$

EXAMPLE 5 LET $X \sim BIN(n, p)$. LET $Y = n - X$ $\rightarrow u(x) = n - x$

$w(y) = n - y$

$$f_X(x) = \binom{n}{x} p^x q^{n-x} \qquad x = 0, 1, \ldots, n$$

CDF METHOD

$$f_Y(y) = P[Y = y] = P[n - X = y] = P[X = n - y]$$

$$= f_X(n-y) = \binom{n}{n-y} p^{n-y} q^{n-(n-y)} \qquad y = 0, 1, \ldots, n$$

$$= \binom{n}{y} q^y p^{n-y}$$

NOTE: $Y \sim BIN(n, q)$

TRANSFORMATION METHOD

$$f_Y(y) = f_X(w(y)) = f_X(n-y) = \binom{n}{y} q^y p^{n-y}$$

One-to-One Transformations

-Assume:

-A random variable X has CDF FX(x) and pdf f(x)

- We are interested in some function of X, say Y = g(X).

-If the equation y = g(x) can be solved uniquely, say $x = g^{-1}(y)$, then we say the transformation is one-to-one.

- To find the pdf of Y=g(x) ,(if inverse of g(x) exist) from the given pdf of X,

we use the formula

$$f_Y(y) = f_X(g^{-1}(y)).\ \left|\frac{d}{dy} g^{-1}(y)\right|$$

Example:

If $X \sim Uniform(2,5)$. Find the dist. of $Y = \frac{X}{1+X}$

Solution:

- $f_X(x) = \frac{1}{3} \quad 2 < x < 5$;
- $X = g^{-1}(y) = \frac{y}{1-y}, \quad \frac{2}{3} < y < \frac{5}{6}$
- $f_X(g^{-1}(y)) = f_X\left(\frac{y}{1-y}\right) = \frac{1}{3}$
- $\frac{d}{dy} g^{-1}(y) = \frac{1}{(1-y)^2}$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| = \frac{1}{3(1-y)^2} \quad \frac{2}{3} < y < \frac{5}{6}$$

Example:

$X \sim EXP(\lambda)$, *Find the distribution of* $y = x^{1/\beta}$

Solution

$f_X(x) = \lambda e^{-\lambda x}$, $x > 0$,

$Y = g(X) = x^{\frac{1}{\beta}}$, *then* $g^{-1}(y) = y^{\beta}$, $y > 0$

$$\frac{d}{dy} g^{-1}(y) = \frac{d}{dy} y^{\beta} = \beta y^{\beta-1}$$

$f_X(y^{\beta}) = \lambda e^{-\lambda y^{\beta}}$, the pdf of Y is given by

$$f_Y(y) = f_X(g^{-1}(y)). \left|\frac{d}{dy} g^{-1}(y)\right| = \lambda e^{-\lambda y^{\beta}} \beta y^{\beta-1} = \lambda\beta y^{\beta-1} e^{-\lambda y^{\beta}}, \quad y > 0$$

Note : this *Weibull distribution form*

Method of Transformations

Joint distribution of Functions of Random Variables

TRANSFORMATION (two from two)

$(1): f_{X_1,X_2}(x_1,x_2)$ joint distribution of two random variables X_1 and X_2

$(2):$ given two functions $Y_1 = g_1(x_1,x_2)$ and $Y_2 = g_2(x_1,x_2)$

Use the formula for joint pdf of Y_1 and Y_2:

$$f_{Y_1,Y_2}(y_1,y_2) = |J(x_1,x_2)|^{-1} f_{X_1,X_2}(x_1,x_2)$$

where X_1 and X_2 are replaced by by their inverse fuctions:

$$X_1 = h_1(y_1,y_2) \text{ and } X_2 = h_2(y_1,y_2)$$

both in jacobian and joint distribution.

Example (transformation technique for joint distributions)

- If X1 and X2 are two independent random variables having exponential distributions with λ_1 and λ_2 respectively. Find the distribution of y_1 and y_2

where $\quad Y_1 = X_1 + X_2, \quad Y_2 = X_1 - X_2.$

Sol : $f_{X_1} = \lambda_1 e^{-\lambda_1 x_1}, X_1 > 0; f_{X2} = \lambda_2 e^{-\lambda_2 x_2}, X_2 > 0$ are pdf of X_1, X_2

The joint distribution is; $f_{X_1, X_2}(x_1, x_2) = \lambda_1 \lambda_2 e^{-\lambda_1 x_1} \times e^{-\lambda_2 x_2}$ (as X_1, X_2 are independen)

The inverse functions are : $X_1 = \frac{1}{2}(y_1 + y_2),\ \mathrm{X}_1 = \frac{1}{2}(y_1 - y_2)$.........(1)

The Jacobian is : $\mathrm{J} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \Rightarrow |J|^{-1} = \frac{1}{2}$

Therefore the joint $distribution\ of$ Y_1, Y_2 is :

$$f_{Y_1, Y_2}(y_1, y_2) = \begin{cases} \frac{1}{2} \lambda_1 \lambda_2 e^{-\frac{\lambda_1}{2}(y_1 + y_2)} \times e^{-\frac{\lambda_1}{2}(y_1 - y_2)}, & (y_1 + y_2) > 0, (y_1 - y_2) > 0 \\ 0, & \text{otherwise} \end{cases}$$

Example

Let $X_1, X_2 \sim \text{GAMMA}(1,2)$ INDEPENDENT (SEE EXAMPLE 4)

$$f_X(x_1,x_2) = \begin{cases} x_1 e^{-x_1} x_2 e^{-x_2} & x_1 > 0 \quad x_2 > 0 \\ 0 & \text{OTHERWISE} \end{cases}$$

GOAL: USE u_1, u_2 TO FIND DISTRIBUTION OF $X_1 + X_2$

LET $Y1(x_1,x_2) = x_1$ $\quad Y2(x_1,x_2) = x_1 + x_2$

SOLVE $\begin{cases} y_1 = x_1 \\ y_2 = x_1 + x_2 \end{cases}$ FOR x_1, x_2 IN TERMS OF y_1, y_2.

WE KNOW $x_1 = y_1$ $\Rightarrow y_2 = y_1 + x_2 \Rightarrow x_2 = y_2 - y_1$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$$

. FIND $f_Y(y_1,y_2) = f_X(x_1,x_2)\,|J|$

$$= f_X(y_1, y_2 - y_1) = y_1 e^{-y_1} (y_2 - y_1) e^{-(y_2 - y_1)}$$

$$= \begin{cases} y_1 (y_2 - y_1) e^{-y_2} & 0 < y_1 < y_2 < \infty \\ 0 & \text{OTHERWISE} \end{cases}$$

- FIND MARGINAL PDF $f_2(y_2) = \int f(y_1, y_2)\, dy_1$

$$= \int_0^{y_2} y_1(y_2 - y_1) e^{-y_2}\, dy_1 = e^{-y_2} \int_0^{y_2} y_1 y_2 - y_1^2\, dy_1$$

$$= e^{-y_2} \left[\frac{y_1^2 y_2}{2} - \frac{y_1^3}{3} \right] \Bigg|_0^{y_2} = \begin{cases} \frac{y_2^3}{6} e^{-y_2} & 0 < y_2 < \infty \\ 0 & \text{OTHERWISE} \end{cases}$$

← GAMMA(1,4)

NOTE $Y_2 = X_1 + X_2 \sim$ GAMMA(1,4)

Method of Moment-Generating Functions

- X,Y are two random variables
- CDF's: $F_X(x)$ and $F_Y(y)$
- MGF's: $M_X(t)$ and $M_Y(t)$ exist and equal for $|t|<h, h>0$
- Then the CDF's $F_X(x)$ and $F_Y(y)$ are equal
- Three Properties:
  - $Y=aX+b \Rightarrow M_Y(t)=E(e^{tY})=E(e^{t(aX+b)})=e^{bt}E(e^{(at)X})=e^{bt}M_X(at)$
  - X,Y independent $\Rightarrow M_{X+Y}(t)=M_X(t)M_Y(t)$
  - $M_{X1,X2}(t_1,t_2) = E[e^{t1X1+t2X2}] = M_{X1}(t_1)M_{X2}(t_2)$ if X_1,X_2 are indep.

Sum of Independent Gammas

$$X_i \sim Gamma(\alpha_i, \beta) \quad i = 1, \ldots, n \quad \text{(independent)}$$

$$M_{X_i}(t) = (1 - \beta t)^{-\alpha_i} \quad i = 1, \ldots, n$$

$$Y = \sum_{i=1}^{n} X_i$$

$$M_Y(t) = E\left(e^{tY}\right) = E\left(e^{t(X_1 + \ldots + X_n)}\right) = E\left(e^{tX_1} \cdots e^{tX_n}\right) = M_{X_1}(t) \cdots M_{X_n}(t)$$

$$(1 - \beta t)^{-\alpha_1} \cdots (1 - \beta t)^{-\alpha_n} = (1 - \beta t)^{-\sum_{i=1}^{n} \alpha_i}$$

$$\Rightarrow Y = \sum_{i=1}^{n} X_i \sim Gamma\left(\sum_{i=1}^{n} \alpha_i, \beta\right)$$

Linear Function of Independent Normals

$$X_i \sim Normal(\mu_i, \sigma_i^2) \quad i = 1, \ldots, n \quad \text{(independent)}$$

$$M_{X_i}(t) = \exp\left\{ \mu_i t + \frac{\sigma_i^2 t^2}{2} \right\} \quad i = 1, \ldots, n$$

$$Y = \sum_{i=1}^{n} a_i X_i \quad \{a_i\} \equiv \text{fixed constants}$$

$$M_Y(t) = E\left(e^{tY}\right) = E\left(e^{t(a_1 X_1 + \ldots + a_n X_n)}\right) = E\left(e^{t a_1 X_1} \cdots e^{t a_n X_n}\right) = M_{X_1}(a_1 t) \cdots M_{X_n}(a_n t)$$

$$\exp\left\{ \mu_1 a_1 t + \frac{\sigma_1^2 a_1 t^2}{2} \right\} \cdots \exp\left\{ \mu_n a_n t + \frac{\sigma_n^2 a_n t^2}{2} \right\} = \exp\left\{ \left(\sum_{i=1}^{n} a_i \mu_i\right) t + \frac{\left(\sum_{i=1}^{n} a_i^2 \sigma_i^2\right) t^2}{2} \right\}$$

$$\Rightarrow Y = \sum_{i=1}^{n} a_i X_i \sim Normal\left(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2 \right)$$

Distribution of Z^2 (Z~N(0,1))

$$Z \sim N(0,1) \quad \Rightarrow \quad f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad -\infty < z < \infty$$

$$M_{Z^2}(t) = \int_{-\infty}^{\infty} e^{tz^2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2\left(\frac{1-2t}{2}\right)} dz = 2\int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2\left(\frac{1-2t}{2}\right)} dz \quad \text{(symmetric about 0)}$$

$$\text{Let } u = z^2 \quad \Rightarrow \quad z = \sqrt{u} \quad \Rightarrow \quad \frac{dz}{du} = \frac{1}{2\sqrt{u}} = 0.5u^{-1/2} \quad \Rightarrow \quad dz = 0.5u^{-1/2} du$$

$$\Rightarrow 2\int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2\left(\frac{1-2t}{2}\right)} dz = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} u^{-1/2} e^{-u/\left[\frac{2}{1-2t}\right]} du = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} u^{1/2-1} e^{-u/\left[\frac{2}{1-2t}\right]} du = \frac{1}{\sqrt{2\pi}} \Gamma(1/2) \left[\frac{2}{1-2t}\right]^{1/2}$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\pi}\sqrt{2}(1-2t)^{-1/2} = (1-2t)^{-1/2}$$

$$\Rightarrow Z^2 \sim Gamma(\alpha = 1/2, \beta = 2) \equiv \chi_1^2$$

Notes:

$$\int_0^{\infty} y^{\alpha-1} e^{-y/\beta} dy = \Gamma(\alpha)\beta^{\alpha}$$

$$\Gamma(1/2) = \sqrt{\pi}$$

$$Z_1, \ldots, Z_n \text{ mutually independent} \Rightarrow \sum_{i=1}^{n} Z_i^2 \sim Gamma(\alpha = n/2, \beta = 2) \equiv \chi_n^2$$

Example – Uniform X

- Stores located on a linear city with density $f(x)=0.05 \quad -10 \leq x \leq 10$, 0 otherwise
- Courier incurs a cost of $U=16X^2$ when she delivers to a store located at X (her office is located at 0)

$$U = u \quad \Rightarrow \quad 16X^2 = u \quad X = \pm\frac{\sqrt{u}}{4}$$

$$U \leq u \quad \Rightarrow \quad -\frac{\sqrt{u}}{4} \leq X \leq \frac{\sqrt{u}}{4}$$

$$F_U(u) = P(U \leq u) = \int_{-\sqrt{u}/4}^{\sqrt{u}/4} 0.05dx = 0.05\left(\frac{\sqrt{u}}{4} - \left(-\frac{\sqrt{u}}{4}\right)\right) = \frac{\sqrt{u}}{40} \quad 0 \leq u \leq 1600$$

$$f_U(u) = \frac{dF_U(u)}{du} = \frac{u^{-1/2}}{80} \quad 0 \leq u \leq 1600$$

Example – Sum of Exponentials

- X_1, X_2 independent Exponential(θ)
- $f(x_i)=\theta^{-1}e^{-xi/\theta}$ $x_i>0$, $\theta>0$, $i=1,2$
- $f(x_1,x_2)= \theta^{-2}e^{-(x1+x2)/\theta}$ $x_1,x_2>0$
- $U=X_1+X_2$

$$U=u \Rightarrow X_1+X_2=u \Rightarrow X_1=u-x_2$$

$$U \le u \Rightarrow X_1+X_2 \le u \Rightarrow X_2 \le u, X_1 \le u-X_2$$

$$P(U \le u)=\int_0^u \int_0^{u-x_2} \frac{1}{\theta^2} e^{-x_1/\theta} e^{-x_2/\theta} dx_1 dx_2 = \int_0^u \frac{1}{\theta} e^{-x_2/\theta} \left(-e^{-x_1/\theta}\right)\Big|_0^{u-x_2} dx_2$$

$$= \int_0^u \frac{1}{\theta} e^{-x_2/\theta}\left[1-e^{-(u-x_2)/\theta}\right]dx_2 = \int_0^u \frac{1}{\theta} e^{-x_2/\theta} dx_2 - \int_0^u \frac{1}{\theta} e^{-(x_2-u+x_2)/\theta} dx_2$$

$$= \left(1-e^{-u/\theta}\right) - \frac{1}{\theta} u e^{-u/\theta} \Rightarrow \quad f_U(u)=\frac{1}{\theta} e^{-u/\theta} - \left[\left(\frac{1}{\theta}\right) e^{-u/\theta} - \left(\frac{u}{\theta^2}\right) e^{-u/\theta}\right]$$

$$= \frac{1}{\theta^2} u e^{-u/\theta} \quad u>0 \quad \Rightarrow \quad U \sim Gamma(\alpha=2, \beta=\theta)$$