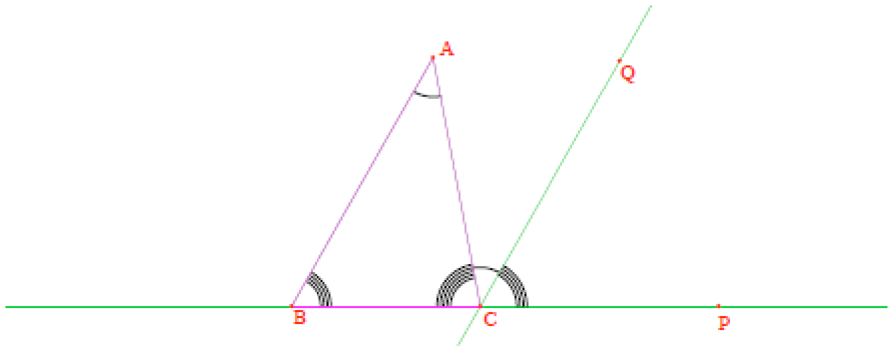
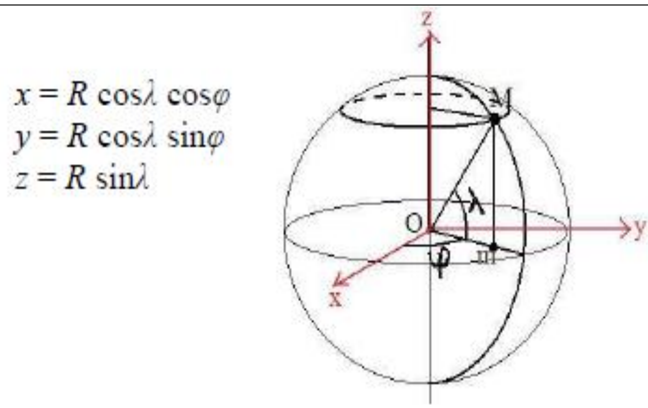
Euclidean geometry



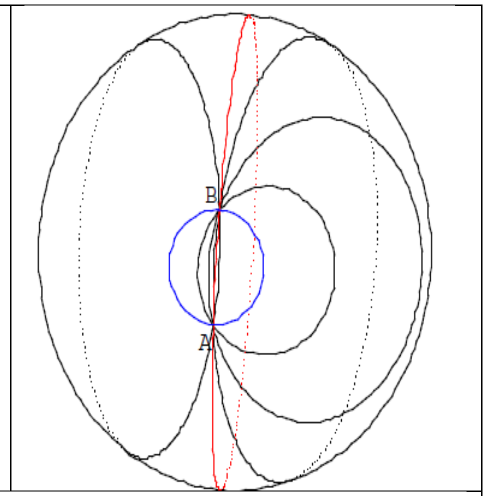
Sum of angles for triangle is $\pi = 180^{\circ}$

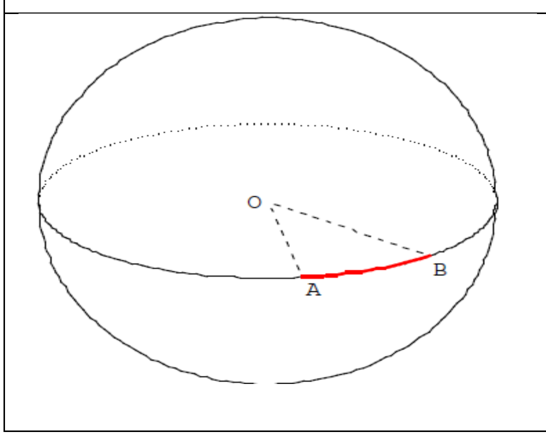
Non Euclidean geometry : Spherical & Hyperbolic (Gauss-Bolyai-Lobatchevski)

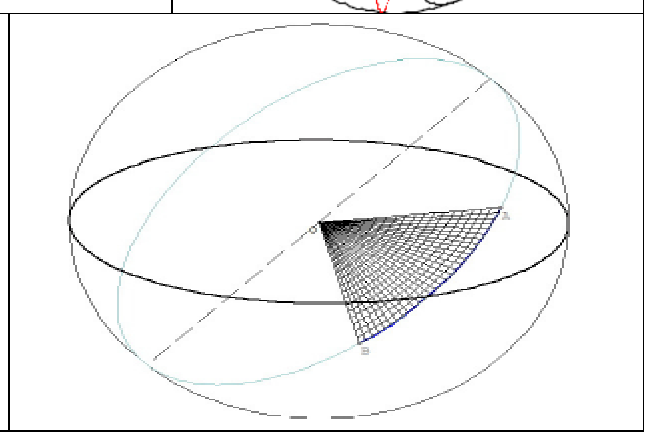
Case : Spherical Geometry:



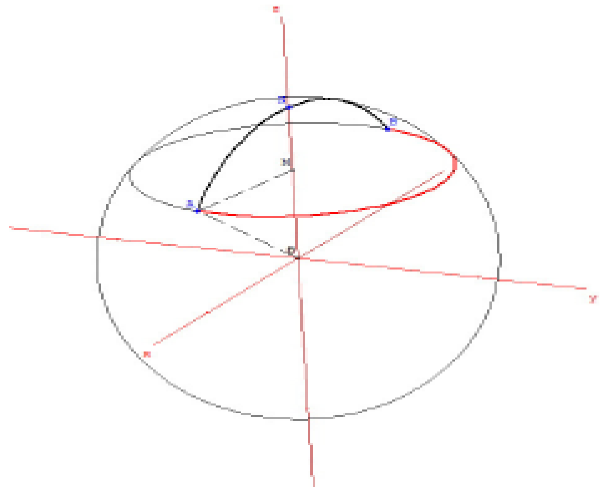
Spherical coordinates





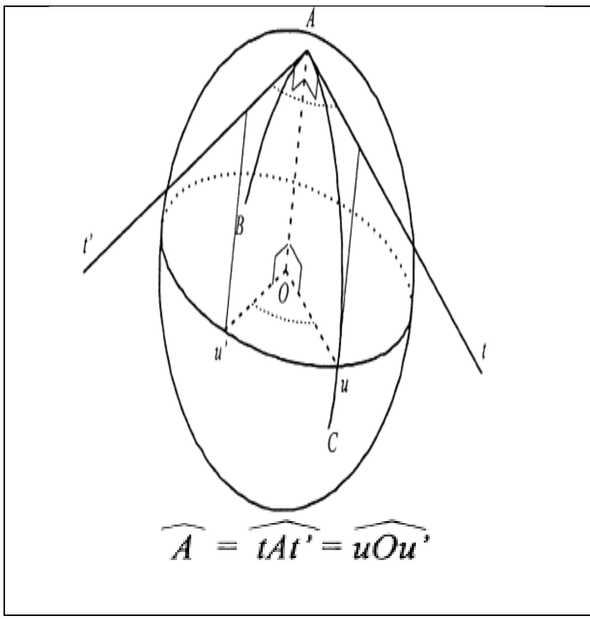


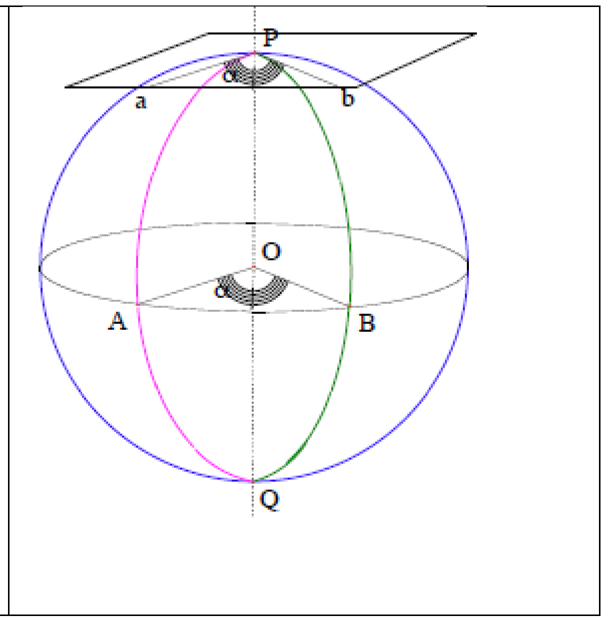
The distance between 2 points in the unit sphere is $d\left(A,B\right) = A\hat{O}B$.

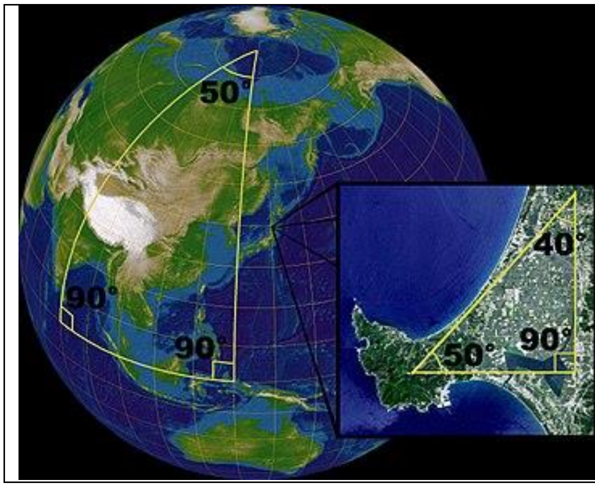


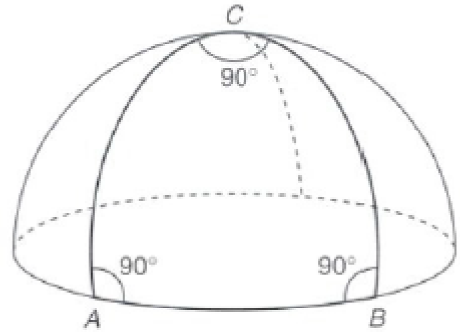
$r = R\sin\theta$ So $L_{AB} = \pi R\sin\theta$ but $\ell_{AB} = R\,2\theta$.

As $\pi\sin\theta \geq 2\theta$ on the interval $\left[0, \pi/2\right]$ then $L_{AB} \geq \ell_{AB}$.

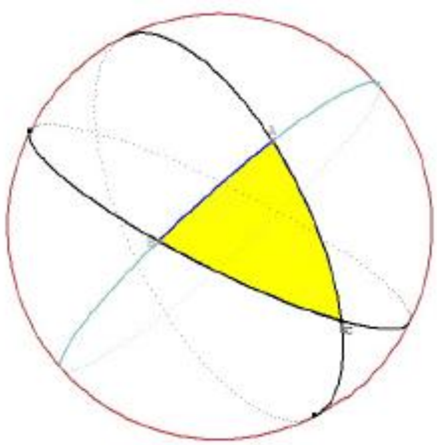


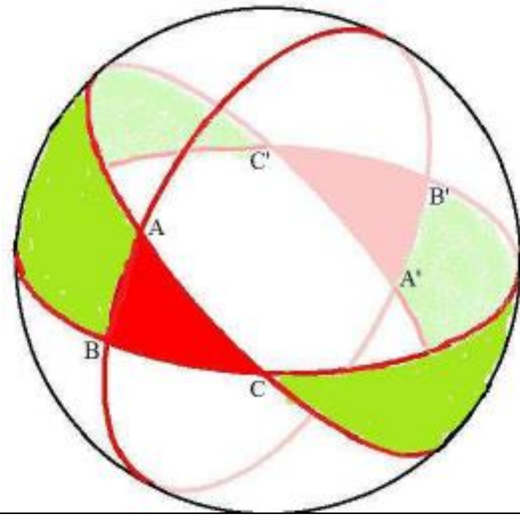




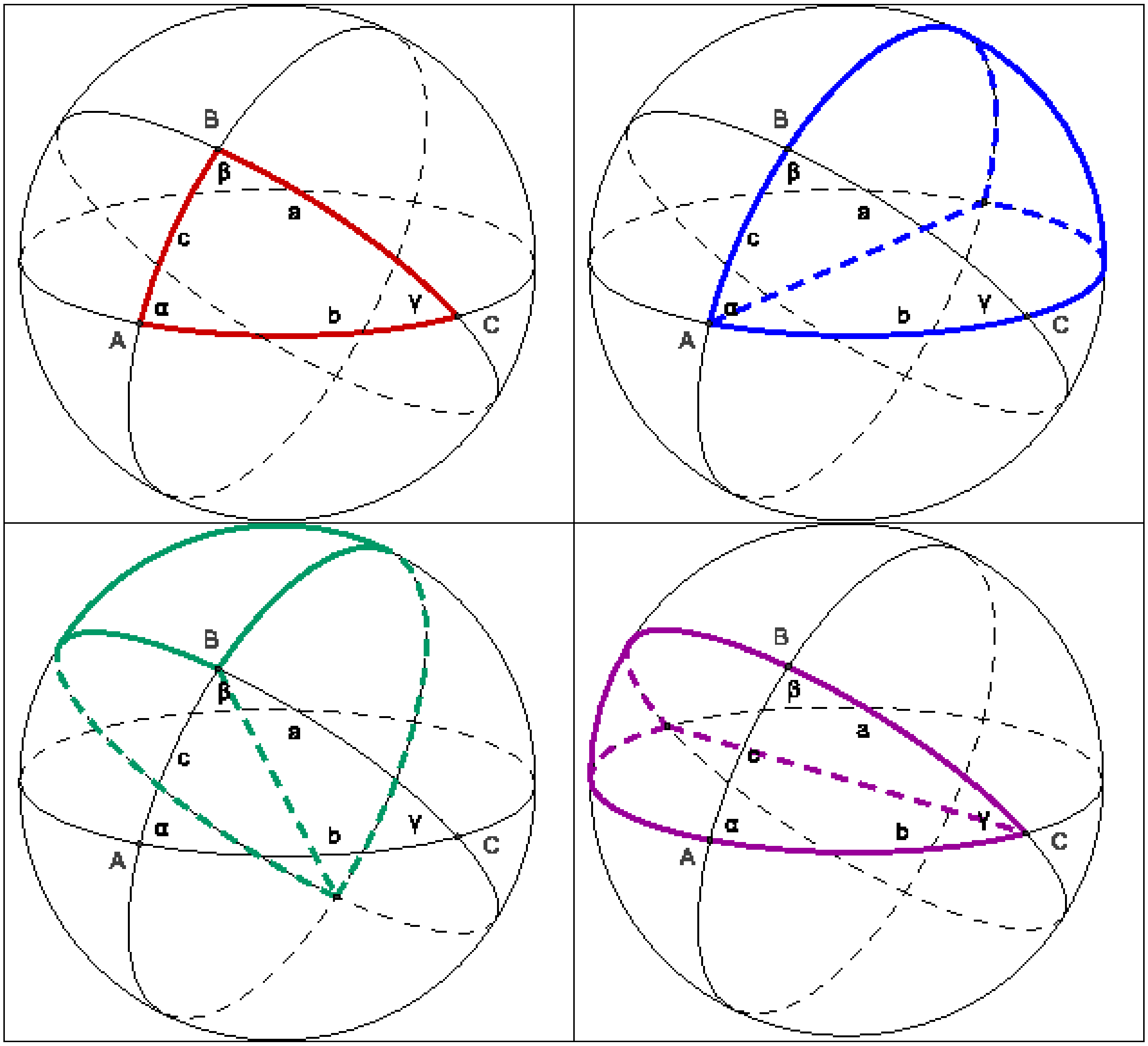


Spherical triangle





The area surface of shaded triangle is $S = R^2\left(\alpha + \beta + \gamma - \pi\right)$.



Trigonometric Properties

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha$$
$$\cos b = \cos a \cos c + \sin a \sin c \cos \beta$$
$$\cos c = \cos b \cos a + \sin b \sin a \cos \gamma$$

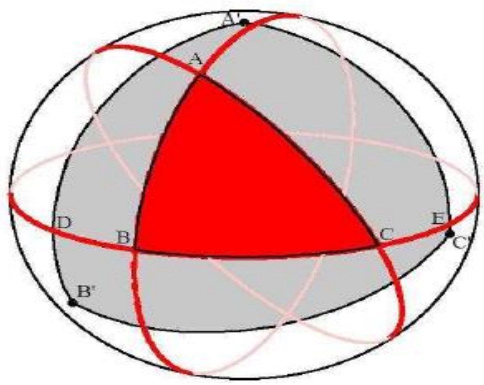
$$\cos \alpha = -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cos a$$
$$\cos \beta = -\cos \alpha \cos \gamma + \sin \alpha \sin \gamma \cos b$$
$$\cos \gamma = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cos c$$

$$\frac{\sin \alpha}{\sin a} = \frac{\sin \beta}{\sin b} = \frac{\sin \gamma}{\sin c}$$

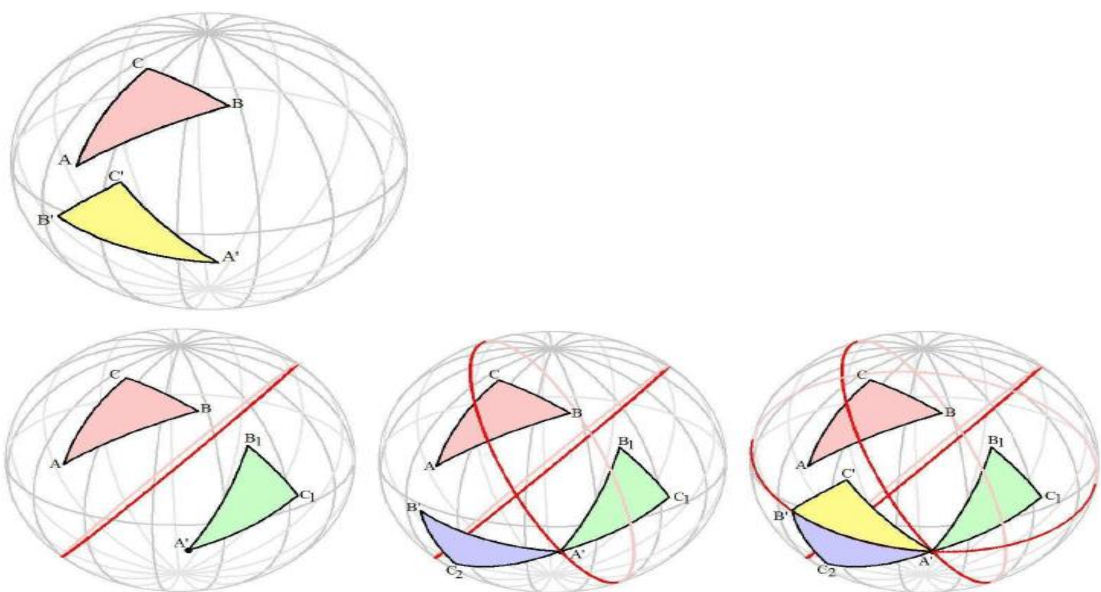
When the spherical triangle is rectangle at C we have:

$$\cos c = \cos a \cos b$$

$$\sin\alpha = \frac{\sin a}{\sin c}$$

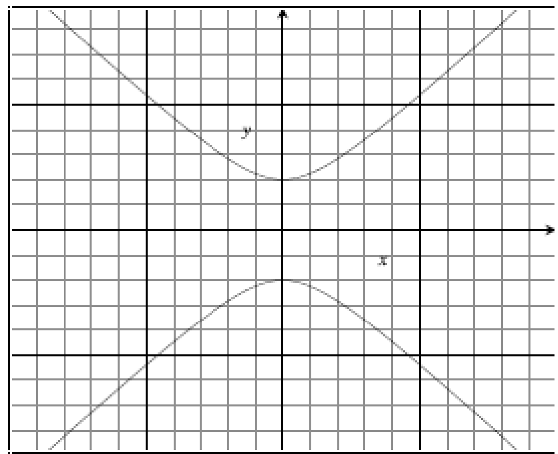


Polar spherical triangle

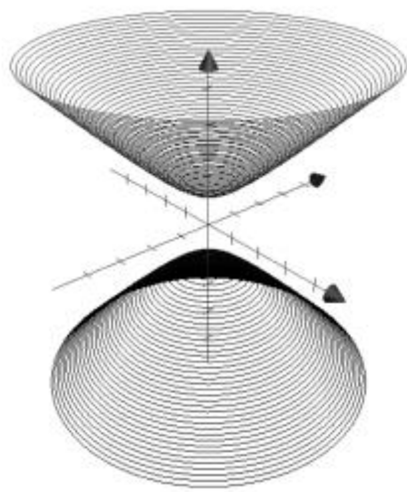


Isometry of 2 triangles by 3 reflections

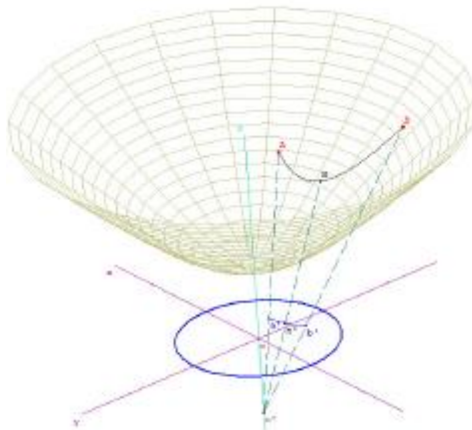
Case Hyperbolic Geometry



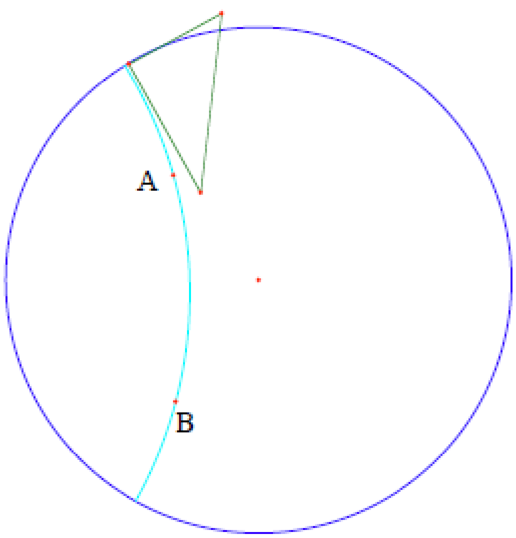
Graph of hyperbola equation: $y^2 - x^2 = 1$.

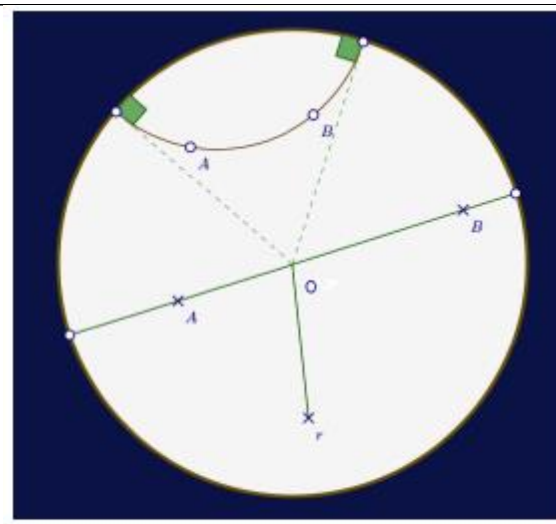


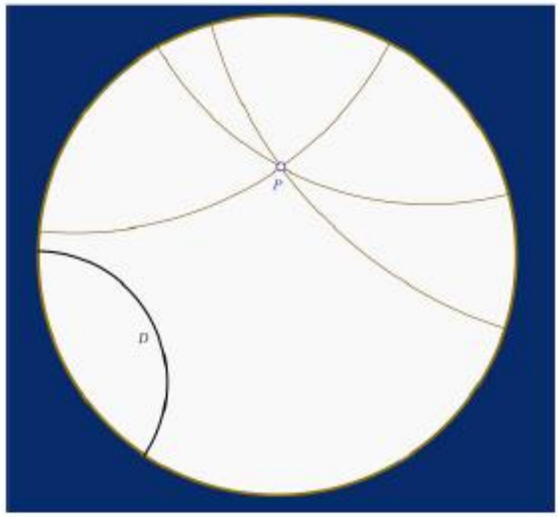
Graph of hyperboloid with 2 sheets equation: $x^2 + y^2 = z^2 - 1$.



Poincare Disk $D = \{z \in \mathbb{C}, \quad |z| < 1\}$.







Half plane Poincare H (by Mobius transform: $z \mapsto \dfrac{z+i}{z+i}$)