441 OR

handout :

#1

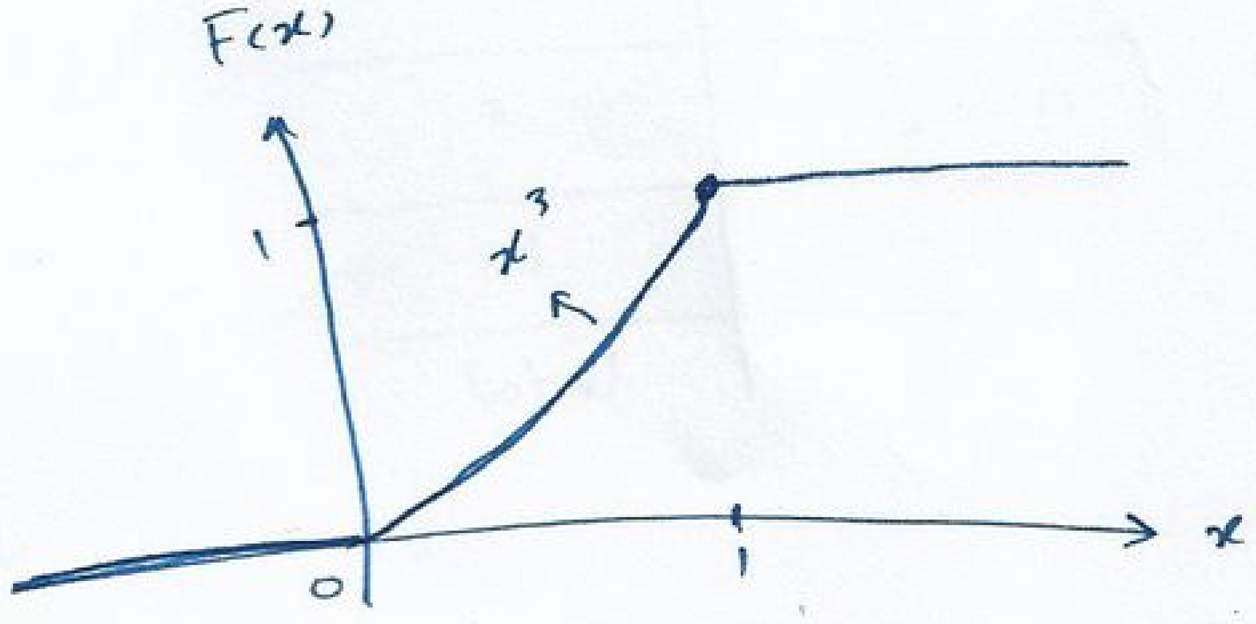
14 *

16 *

## ①

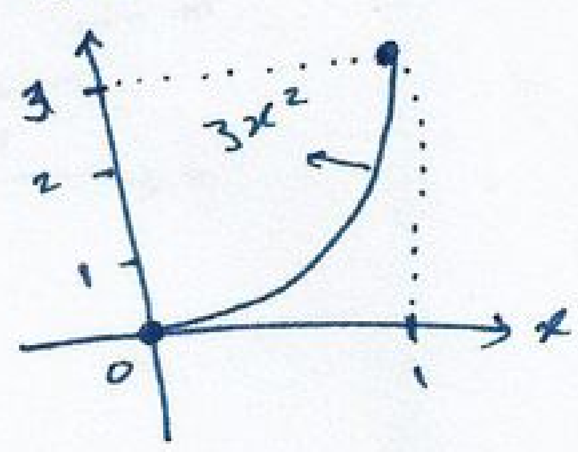
واضح أن $F(x)$ لمتغير عشوائي متصل لأنه يتقدم له $x$ وليست كلها أعداد..

ⓐ




$f(x) = x^3$

ⓑ $f(x) = \frac{d}{dx} F(x) \Rightarrow f(x) = \begin{cases} 3x^2 & , \quad 0 < x < 1 \\ 0 & , \quad o.w \end{cases}$

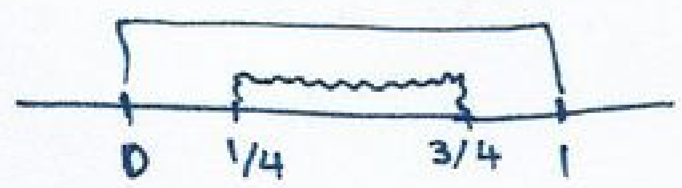



$f(x) = 3x^2$

ⓒ $E(X) = \int_0^1 x f(x)\, dx = 3\int_0^1 x^3\, dx = \frac{3}{4}$

ⓓ $P(\frac{1}{4} < X < \frac{3}{4}) \xrightarrow{\text{by cdf}} = F(\frac{3}{4}) - F(\frac{1}{4}) = (\frac{3}{4})^3 - (\frac{1}{4})^3 = \frac{13}{32} = .40625$

by pdf

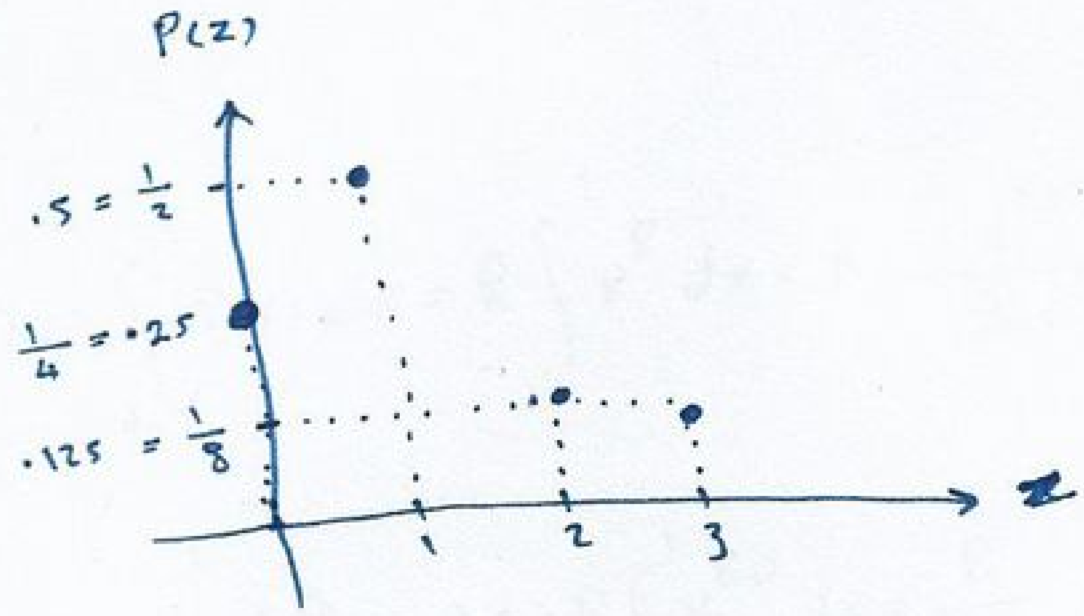


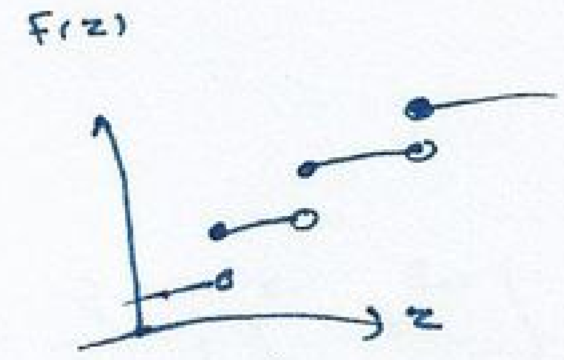

$= \int_{1/4}^{3/4} f(x)\, dx = 3\int_{1/4}^{3/4} x^2\, dx = \frac{13}{32} = .40625$

②

| $z$ | $p(z) = P(Z=z)$ |
|---|---|
| 0 | $\frac{1}{4} = .25$ |
| 1 | $\frac{1}{2} = .5$ |
| 2 | $\frac{1}{8} = .125$ |
| 3 | $\frac{1}{8}$ |
| total | 1 |

ⓐ




ⓑ $E(Z) = \sum_{\forall z} z\,p(z) = 1.125 = \frac{9}{8}$

ⓒ $E(Z^2) = \sum_{\forall z} z^2\,p(z) = 2.125$

$\therefore Var(Z) = E(Z^2) - [E(Z)]^2 = .859375$

④

$$f(x) = \begin{cases} R\,x^{R-1} & , \quad 0 < x < 1 \\ 0 & , \quad o.w. \end{cases}$$

$$R > 0$$

ⓐ

$$F(x) = \begin{cases} 0 & , \quad x < 0 \\ \int_0^x f(t)\,dt = R\int_0^x t^{R-1}\,dt = x^R & , \quad 0 \le x < 1 \\ 1 & , \quad 1 \le x \end{cases}$$

ⓑ $$E(X) = \int_0^1 x f(x)\,dx = R\int_0^1 x^R\,dx = R\,\frac{x^{R+1}}{R+1}\Big|_0^1 = \frac{R}{R+1}\left[1^{R+1} - 0^{R+1}\right] = \frac{R}{R+1}$$

ⓒ $$E(X^2) = \int_0^1 x^2 f(x)\,dx = R\int_0^1 x^{R+1}\,dx = \frac{R}{R+2}\,x^{R+2}\Big|_0^1 = \frac{R}{R+2}\left[1^{R+2} - 0^{R+2}\right] = \frac{R}{R+2}$$

$$\therefore Var(X) = E(X^2) - \left[E(X)\right]^2 = \underbrace{\frac{R}{R+2} - \left(\frac{R}{R+1}\right)^2}_{\text{ممكن تبسيطها}}$$

5

$$F(v) = \begin{cases} 0 & , \ v<0 \\ 1-(1-v)^A & , \ 0 \leq v < 1 \\ 1 & , \ 1 \leq v \end{cases} \quad , \ A>0$$

$$* \ f(v) = \frac{d}{dv} F(v) \Rightarrow f(v) = \begin{cases} -A(1-v)^{A-1}(-1) = A(1-v)^{A-1} & , \ 0<v<1 \\ 0 & , \ o.w. \end{cases}$$

$$* \ E(V) = \int_0^1 v f(v)\,dv = A\int_0^1 v(1-v)^{A-1}\,dv$$

$$= A\int_1^0 (1-u)\,u^{A-1}\,-du = A\left[\int_0^1 u^{A-1}du - \int_0^1 u^A du\right] = A\left[\frac{1}{A} - \frac{1}{A+1}\right]$$

$$= A\left[\frac{A+1-A}{A(A+1)}\right] = A\left[\frac{1}{A(A+1)}\right] = \frac{\cancel{A}}{\cancel{A}(A+1)} = \frac{1}{A+1}$$

$$* \ E(V^2) = \int_0^1 v^2 f(v)\,dv = A\int_0^1 v^2(1-v)^{A-1}\,dv$$

$$= A\int_1^0 (1-u)^2 u^{A-1}\,-du = A\int_0^1 (1-u)^2 u^{A-1}\,du \quad , \ (1-u)^2 = 1-2u+u^2$$

$$= A\left[\int_0^1 u^{A-1}du - 2\int_0^1 u^A du + \int_0^1 u^{A+1}du\right] = A\left[\frac{1}{A} - 2\frac{1}{A+1} + \frac{1}{A+2}\right]$$

$$= A\left[\frac{(A+1)(A+2) - 2A(A+2) + A(A+1)}{A(A+1)(A+2)}\right] = A\left[\frac{A^2+A+2A+2-2A^2-4A+A^2+A}{A(A+1)(A+2)}\right]$$

$$= A\left[\frac{2}{\cancel{A}(A+1)(A+2)}\right] = \frac{2}{(A+1)(A+2)}$$

$$\therefore Var(V) = E(V^2) - [E(V)]^2$$

$$= \frac{2}{(A+1)(A+2)} - \frac{1}{(A+1)^2} = \frac{2(A+1)-(A+2)}{(A+1)^2(A+2)} = \frac{2A+\cancel{2}-A-\cancel{2}}{(A+1)^2(A+2)}$$

$$= \frac{A}{(A+1)^2(A+2)}$$

التكامل بالتعويض

$$u = 1 - v \longrightarrow du = -dv \Rightarrow -du = dv$$

$$v = 1 - u$$

$$v = 0 \Rightarrow u = 1$$

$$v = 1 \Rightarrow u = 0$$

كنت ساعتبره معكوك ذو الحدين لكن $A > 0$ , لا يطلبنا عق

$$\Rightarrow A - 1 > -1$$

$\boxed{6}$

$$f(x) = \begin{cases} x & , 0 < x < 1 \\ 2-x & , 1 < x < 2 \\ 0 & , \text{o.w.} \end{cases}$$

$$* \quad F(x) = \begin{cases} 0 & , x < 0 \\ \int_0^x f(t)dt = \int_0^x t\,dt = \frac{x^2}{2} & , 0 \le x < 1 \\ \int_0^1 f(t)dt + \int_1^x f(t)dt = \int_0^1 t\,dt + \int_1^x (2-t)\,dt & , 1 \le x < 2 \\ = 2x - \frac{1}{2}x^2 - 1 & \\ 1 & , 2 \le x \end{cases}$$

$$\frac{t^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$= 2\int_1^x dt - \int_1^x t\,dt$$
$$= 2t\Big|_1^x - \frac{t^2}{2}\Big|_1^x$$
$$= 2x - \frac{1}{2}x^2 - \frac{3}{2}$$

$$* \quad E(X) = \int x f(x)dx = \int_0^1 x^2 dx + \int_1^2 x(2-x)dx = 1$$

$$* \quad E(X^2) = \int x^2 f(x)dx = \int_0^1 x^3 dx + \int_1^2 x^2(2-x)dx = \frac{7}{6} = 1.166666667$$

$$\therefore \operatorname{Var}(X) = E(X^2) - [E(X)]^2 = \frac{7}{6} - 1 = \frac{1}{6} = .166666667$$

$$V(X) = E[(X-\mu)^2] = E[(X-1)^2]$$
$$= \int_0^1 (x-1)^2 x\,dx + \int_1^2 (x-1)^2 (2-x)\,dx$$

⑦

if

$X \xrightarrow{\text{discrete}}$ r.v. with $\mu = E(X) = \sum_{\forall x} x\,p(x)$ , $p(x) = p(X=x)$

and $\sigma^2 = Var(X) = E\left((X-\mu)^2\right) = E(X^2) - [E(X)]^2$

$\left((X-E(X))\right)$

$*$ $E(Y) = \cancel{E(Y)} = E(aX+b)$ by the definition $E(X) = \sum_{\forall x} x\,p(x)$

$= \sum_{\forall x} (ax+b)\,p(x)$

$= \sum_{\forall x} ax\,p(x) + \sum_{\forall x} b\,p(x)$

$= a \underbrace{\sum_{\forall x} x\,p(x)}_{\mu = E(X)} + b \underbrace{\sum_{\forall x} p(x)}_{1}$

$= a E(X) + b$

$= a\mu + b$

من فقرة اللي قبلها

$*$

$Var(Y) = Var(aX+b) = E\left[\left((aX+b) - \underline{E(aX+b)}\right)^2\right]$ by definition $Var(X) = E\left[(X-E(X))^2\right]$

$= E\left[\left((aX+\cancel{b}) - (a\mu+\cancel{b})\right)^2\right]$

$= E\left[\left(a(X-\mu)\right)^2\right]$

$= a^2\, E\left[(X-\mu)^2\right]$

$= a^2\, Var(X)$

8

Power Sum

$$\sum_{k=1}^{n} 1 = n$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

$$* \quad E(K) = \sum_{\forall k} k\,p(k) = \sum_{k=1}^{n} k \cdot \frac{2(n-k)}{n(n-1)} = \frac{2}{n(n-1)} \sum_{k=1}^{n} k(n-k)$$

$$= \frac{2}{n(n-1)} \left[ n \sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2 \right]$$

$$= \frac{2}{n(n-1)} \left[ n \frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6} \right]$$

$$= \frac{n(n+1)}{n-1} - \frac{(n+1)(2n+1)}{3(n-1)} = \frac{3n(n+1) - (n+1)(2n+1)}{3(n-1)}$$

$$= \frac{n^2-1}{3(n-1)} = \frac{\cancel{(n-1)}(n+1)}{3\cancel{(n-1)}} = \frac{n+1}{3}$$

$$* \quad E(K^2) = \sum_{\forall k} k^2 p(k) = \sum_{k=1}^{n} k^2 \frac{2(n-k)}{n(n-1)} = \frac{2}{n(n-1)} \sum_{k=1}^{n} k^2(n-k)$$

$$= \frac{2}{n(n-1)} \left[ n \sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k^3 \right]$$

$$= \frac{2}{n(n-1)} \left[ n \frac{n(n+1)(2n+1)}{6} - \frac{n^2(n+1)^2}{4} \right]$$

$$= \frac{n(n+1)(2n+1)}{3(n-1)} - \frac{n(n+1)^2}{2(n-1)} = \frac{2n(n+1)(2n+1) - 3n(n+1)^2}{6(n-1)}$$

$$= \frac{n^3-n}{6(n-1)} = \frac{n(n^2-1)}{6(n-1)} = \frac{n\cancel{(n-1)}(n+1)}{6\cancel{(n-1)}} = \frac{n(n+1)}{6}$$

$$\therefore \operatorname{Var}(K) = E(K^2) - [E(K)]^2 = \frac{n(n+1)}{6} - \frac{(n+1)^2}{3^2} = \frac{3^2 n(n+1) - 6(n+1)^2}{6(3^2)}$$

$$= \frac{3n^2 - 3n - 6}{54}$$

9

sum of indep. r.vs $Z = X + Y$

منفصل (سهل جدا)

متصل (صعب لأنه بيحتاج الى
عملية مكاملة صعبة جدا)

$$P(Z=z) = \sum_{\substack{\forall k \\ k=-\infty}}^{\infty} P(X=k)\, P(Y=z-k)$$

$$= \sum P(X=z-k)\, P(Y=k)$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, g_Y(z-x)\, dx$$

$$= \int_{-\infty}^{\infty} f_X(z-y)\, g_Y(y)\, dy$$

X كم عدد في

Y كم عدد في

$(2)(3) = 6$

| $x$ | $P_X(x)$ | $y$ | $P_Y(y)$ | $z = x + y$ | $P_Z(z)$ |
|---|---|---|---|---|---|
| 0 | 1/2 | 1 | 1/6 | 1 | $\frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$ |
| 0 | 1/2 | 2 | 1/3 | 2 | $\frac{1}{2} \frac{1}{3} = \frac{1}{6}$ |
| 0 | 1/2 | 3 | 1/2 | 3 | $\frac{1}{2} \frac{1}{2} = \frac{1}{4}$ |
| 3 | 1/2 | 1 | 1/6 | 4 | $\frac{1}{2} \frac{1}{6} = \frac{1}{12}$ |
| 3 | 1/2 | 2 | 1/3 | 5 | $\frac{1}{2} \frac{1}{3} = \frac{1}{6}$ |
| 3 | 1/2 | 3 | 1/2 | 6 | $\frac{1}{2} \frac{1}{2} = \frac{1}{4}$ |

$\therefore$

| $z$ | 1 | 2 | 3 | 4 | 5 | 6 | total |
|---|---|---|---|---|---|---|---|
| $P_Z(z)$ | 1/12 | 1/6 | 1/4 | 1/12 | 1/6 | 1/4 | 1 |

10

عدد قيم X        عدد قيم Y

$(3)(2) = 6$

| $x$ | $P_X(x)$ | $y$ | $P_Y(y)$ | $z = x + y$ | $P_Z(z)$ |
|---|---|---|---|---|---|
| 0 | $1/3$ | 1 | $1/2$ | 1 | $1/3 \cdot 1/2 = \frac{1}{6}$ |
| 0 | $1/3$ | 2 | $1/2$ | 2 | $1/3 \cdot 1/2 = 1/6$ |
| 1 | $1/3$ | 1 | $1/2$ | 2 | $1/3 \cdot 1/2 = 1/6$ |
| 1 | $1/3$ | 2 | $1/2$ | 3 | $1/3 \cdot 1/2 = 1/6$ |
| 2 | $1/3$ | 1 | $1/2$ | 3 | $1/3 \cdot 1/2 = 1/6$ |
| 2 | $1/3$ | 2 | $1/2$ | 4 | $1/3 \cdot 1/2 = 1/6$ |

$+ = \frac{2}{6}$

$+ = \frac{2}{6}$

$\therefore$

| $z$ | 1 | 2 | 3 | 4 | total |
|---|---|---|---|---|---|
| $P_Z(z)$ | $1/6$ | $2/6$ | $2/6$ | $1/6$ | 1 |

11

$U, V, W$ indep.

$Var(U) = Var(V) = Var(W) = \sigma^2$

$$\begin{aligned} cov(X,Y) &= cov(U+W, V-W) \\ &= cov(U,V) + cov(U,-W) \\ &\quad + cov(W,V) + cov(W,-W) \\ &= \cancelto{0}{cov(U,V)} - \cancelto{0}{cov(U,W)} \\ &\quad + \cancelto{0}{cov(W,V)} - cov(W,W) = -cov(W,W) = -Var(W) = -\sigma^2 \end{aligned}$$

as $U, V, W$ indep. $\Rightarrow cov(U,W) = 0$

$cov(U,V) = 0$

$cov(V,W) = 0$

12

Fair die

$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$$

$$P(\text{even}) = P(\text{odd}) = \frac{1}{2}$$

$$P(2) + P(4) + P(6) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6}$$

$$P(1) + P(3) + P(5) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6}$$

$$\Omega = \underbrace{\{\text{even}, \text{odd}\} \times \{\text{even}, \text{odd}\} \times \cdots \times \{\text{even}, \text{odd}\}}_{10 \text{ times}}$$

$$n(\Omega) = \underbrace{2 \times 2 \times \cdots \times 2}_{10 \text{ times}} = 2^{10}$$

أول طريقة

this is a binomial dis.

X: number of show "even number" in 10 times

P: Probability of even number $\Rightarrow P = \frac{1}{2}$

$$\therefore f_X(x) = \binom{10}{x} \left(\frac{1}{2}\right)^x \left(1 - \frac{1}{2}\right)^{10-x}, \quad x = 0, 1, \cdots, 10$$

لم يظهر أي عدد زوجي من أصل 10

أي 0 عدد زوجي من أصل 10

$$f_X(0) = \binom{10}{0} \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{10} = \frac{1}{2^{10}} = \frac{1}{1024}$$

طريقة ثانية

باستخدام تعريف الاحتمال التقليدي

P( not show even number for 10 times )

$= P(A\text{: show odd number for all 10 times}) = P(\{(\underbrace{odd, \ldots, odd}_{10 \text{ مرات}})\})$

$$= \frac{1}{2^{10}} = \frac{n(A)}{n(\Omega)}$$

# "even" $\geq 1$

أو 2

أو 3

⋮

أو 10

13

$p(H) = p(T) = \frac{1}{2}$

اول طريقة

this is a binomal dis.

$X$: number of "H" in 5 times

$p$: Probability of "H" $\Longrightarrow p = \frac{1}{2}$

$\therefore f_X(x) = \binom{5}{x} \frac{1}{2}^{x} (1-\frac{1}{2})^{5-x}, \quad x = 0, 1, \ldots, 4, 5$

لدينا H واحدة و T أربعة

لدينا H أربعة T واحدة

$\therefore$ P( one different from athor four )

$= P(X=1) + P(X=4) = f_X(1) + f_X(4)$

$= \binom{5}{1} \frac{1}{2} (\frac{1}{2})^4 + \binom{5}{4} (\frac{1}{2})^4 (\frac{1}{2})^1 = \frac{5}{16} = .3125$

ثاني طريقة

تعريف الاحتمال التقليدي

$\Omega = \underbrace{\{H,T\} \times \cdots \times \{H,T\}}_{5 \text{ times}}$

$n(\Omega) = \underbrace{2 \times \cdots \times 2}_{5 \text{ times}} = 2^5$

A: one different from athor four

$A = \{$ HTTTT, THHHH,
THTTT, HTHHH,
TTHTT, HHTHH,
TTTHT, HHHTH,
TTTTH, HHHHT $\}$

$\therefore P(A) = \frac{n(A)}{n(\Omega)} = \frac{10}{2^5} = \frac{5}{16}$

14

$p = .4 \rightarrow$ get a speeding ticket on any one trip

(a) p( not get a speeding ticket in any one day)

$= p(X=0) = C_0^{10} (.4)^0 (.6)^{10} = .006$

(b) average fine $= 80\,E(X) = \$80 \times n \times p$

$= 80 \times 10 \times .4 = 320\,\$$

X ; the no. of getting speeding ticket in 10 round trips

P ; the probability of getting speeding ticket in any round trip.

$f_X(x) = P(X=x) = \binom{10}{x} (.4)^x (.6)^{10-x}$

$x = 0, 1, 2, \ldots 10$

getting 0 speeding ticket

getting 1 speeding ticket

E(X) = the mean of getting speeding ticket

15

X: number of customers that arrive at service per minute

~~traveling~~

$\lambda$: average or rate of $\Rightarrow \lambda = 4$

$$f_X(x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, \ldots$$

$$= \frac{e^{-4} 4^x}{x!}$$

P( at least one customer arrive in 30sec )

Y: number of customers that arrive at service per 30 sec.

$\lambda'$: average or rate of $\Rightarrow \lambda' = ?$

$X: \lambda = 4 \longrightarrow 1$ min. $= 60$ sec.

$Y: \lambda' = ? \longrightarrow 30$ sec.

$$\therefore \lambda' = \frac{30(4)}{60} = 2$$

$$\therefore f_Y(y) = \frac{e^{-\lambda'} (\lambda')^y}{y!}, \quad y = 0, 1, \ldots$$

$$= \frac{e^{-2} 2^y}{y!}$$

$$\therefore P(Y \geq 1) = 1 - P(Y < 1) = 1 - P(Y = 0)$$

$$= 1 - f_Y(0) = 1 - \frac{e^{-2} 2^0}{0!} = .86466$$

For the service

$\boxed{18}$

X: no. of arrival customers in 1 hour

$\lambda$ : no. of arrival customers in 1 hour

$\lambda = 20$ cust / 1 hour

$$f(x) = P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!} = \frac{e^{-20} 20^x}{x!} , \quad x = 0,1,2 \dots$$

(a) $P(X=0) = \frac{e^{-20} 20^0}{0!}$

(b) $P(X \geq 3) = P(X=3) + P(X=4) + \dots$

$= 1 - P(X < 3)$

$= 1 - [P(X=2) + P(X=1) + P(X=0)]$