MATHEMATICS

Improper Integral Residue Theorem

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I'm stuck on a question involving evaluating improper integrals using the residue theorem. Here's what I'm trying to evaluate:
$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^3} dx$

To begin with, we can define a contour integral over the the semi-circle disc above the Real axis called C which can be broken up in two parts involving the Real number line, and the curve of the semi-circular disc
$\int_C \frac{1}{(1+z^2)^3} dz = \int_{-R}^{R} \frac{1}{(1+x^2)^3} dx + \int_{Cr} \frac{1}{(1+z^2)^3} dz$

We can see that the left-hand side of the equation has poles at i and $-i$ with multiplicity 3 respectively. Using the Residue Theorem
$\int_C \frac{1}{(1+z^2)^3} dz = 2\pi i(3 \frac{1}{((i)+i)^3})$

Because there are 3 instances of the i being the pole inside the region defined.
Also:

$\int_{Cr} \frac{1}{(1+z^2)^3} dz = 0$
Because parameterizing $z = Re^{it}$ and setting the limit as R goes to infinity will show the integrand to be zero.

That leaves us with the final answer of
6π

The correct answer is $\frac{3\pi}{8}$ and I'm not quite sure how that is formed.

Any help would be appreciated.
Thanks.

complex-analysis indefinite-integrals residue-calculus

asked Dec 13 '18 at 17:01
HoopsMcCann
803 4 9

2 Answers

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The residue of $\dfrac{1}{(1+z^2)^3}$ at i is $-\dfrac{3i}{16}$ and therefore your integral is equal to

$$2\pi i \times \frac{-3i}{16} = \frac{3\pi}{8}$$

indeed.

In order to compute that residue, you can use the fact that

$$\frac{1}{(1+z^2)^3} = \frac{\frac{1}{(z+i)^3}}{(z-i)^3}$$

and that therefore the residue that we are interested in is $\dfrac{\varphi''(i)}{2}$, where $\varphi(z) = \dfrac{1}{(z+i)^3}$.

And, as I wrote, $\dfrac{\varphi''(i)}{2} = -\dfrac{3i}{16}$.

edited Dec 13 '18 at 17:13

answered Dec 13 '18 at 17:06
José Carlos Santos
258k 30 182 322

Why can't we find the residue by multiplying the original function by $(z-1)^3$, isn't that how you find a residue usually? – HoopsMcCann Dec 13 '18 at 17:39

Why $(z-1)^3$? The number 1 has nothing to do with this problem. It should be $(z-i)^3$, and that is exactly what I did. – José Carlos Santos Dec 13 '18 at 17:51

Typo on my end. So the idea is, if we have a higher order pole, we differentiate it until the order of it is 1, and divide by how many times we differentiated factorial? – HoopsMcCann Dec 13 '18 at 17:56

I would not put it like that, but that's the right idea. – José Carlos Santos Dec 13 '18 at 18:08

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By the Cauchy integral formula

if a is inside the contour, and $f(z)$ is holomorphic inside the contour.

and

$\int_\gamma \frac{f(z)}{(z-a)^n}dz = 2\pi i\frac{1}{(n-1)!}f^{(n-1)}(a)$

in this case let $a = i$

and $f(z) = \frac{1}{(z+i)^3}$

answered Dec 13 '18 at 17:18

Doug M
49.5k 3 24 59