Inner Product space

Def: Inner Product, $\langle \, , \rangle$, is a mapping : $V \times V \longrightarrow \mathbb{R}$ where V is a vector space such that satisfies :

(1) $\langle v_1, v_2 \rangle = \langle v_2, v_1 \rangle$

(2) $\langle v_1 + v_2, v_3 \rangle = \langle v_1, v_3 \rangle + \langle v_2, v_3 \rangle$

(3) $\langle \lambda v_1, v_2 \rangle = \lambda \langle v_1, v_2 \rangle$ where $\lambda \in \mathbb{R}$.

(4) $\langle v_1, v_1 \rangle \geq 0$

(5) $\langle v_1, v_1 \rangle = 0$ iff $v_1 = 0$

for example :

(1) Let $V = \mathbb{R}^2$ and define $\langle v_1, v_2 \rangle = v_1 \cdot v_2$ (dot. Product)

(i.e. $\langle (a,b), (c,d) \rangle = (a,b) \cdot (c,d) = ac + bd$)

It is easy to check that all five conditions are satisfied. So, $\langle v_1, v_2 \rangle$ is inner Product.

(2) In general, $V = \mathbb{R}^n$ and the dot. Product are constructing innear product called normal inner Product.

in this case, $\langle (a_1, \dots, a_n), (b_1, \dots, b_n) \rangle = a_1 b_1 + \dots + a_n b_n$

(3) Let V be a vector space of all continous functions on $\mathbb{R}$. Define $\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx$. Then $\langle f, g \rangle$ is inner Product.

(4) Let $V = \mathbb{R}^2$ and define $\langle (a,b), (c,d) \rangle = ac - b$.

Notice that $\langle (1,2), (5,3) \rangle = 5 - 2 = 3$

$$\langle (5,3), (1,2) \rangle = 5 - 3 = 2$$

So, $\langle \, , \rangle$ is not inner product.

(5) Let $V = \mathbb{R}^2$ and $\langle (a,b), (c,d) \rangle = ac + bd + 1$.

Notice that, $\langle \underbrace{(0,0)}_{v=0}, \underbrace{(0,0)}_{v=0} \rangle = 0 + 0 + 1 = 1 \neq 0$

So, $\langle , \rangle$ is not inner product.

Some Properties : If $\langle , \rangle$ is inner product on V.

① $\langle 0, u \rangle = 0$ and $\langle u, 0 \rangle$ for every $u \in V$.

② $\langle \lambda u, \lambda' v \rangle = \lambda(\lambda') \langle u, v \rangle$ where $\lambda, \lambda' \in \mathbb{R}$.

③ $\langle u_1, u_2 + u_3 \rangle = \langle u_1, u_2 \rangle + \langle u_1, u_3 \rangle$

Def The vector space with inner product $\langle , \rangle$ is Called inner vector space.

up to now, all vector spaces in this chapter are inner product space.

Remark: If we say $\mathbb{R}^n$ is inner product vector space without mention the definition of $\langle , \rangle$, we will consider $\langle , \rangle$ is a dot. product

Def : The norm $\|v\|$:- $\|v\| = \sqrt{\langle v, v \rangle}$.

example: Let $V = \mathbb{R}^2$. $\|\underbrace{(a,b)}_{v}\| = \sqrt{\langle (a,b), (a,b) \rangle}$

$= \sqrt{a^2 + b^2}$

example : For any vector space V,

$\|0\| = \sqrt{\langle 0, 0 \rangle} = \sqrt{0} = 0$

example : Let $V = \mathbb{R}^3$ and $\langle (a_1, a_2, a_3), (b_1, b_2, b_3) \rangle = \sqrt{a_1 b_1 + a_2 b_2 + a_3 b_3}$

then $\|(1,1,2)\| = \sqrt{\langle (1,1,2), (1,1,2) \rangle} = \sqrt{\sqrt{1+1+4}} = \left[(6)^{1/2}\right]^{1/2} = 6^{1/4}$

$= \sqrt[4]{6}$.

Properties

(1) $\|v\| = 0$ iff $v = 0$

(2) $\|\lambda v\| = |\lambda| \, \|v\|$ where $\lambda \in \mathbb{R}$.

Remark

$\boxed{\|v\|^2 = \langle v, v\rangle} \longrightarrow$ Rule.

Example : If $\|v\| = 4$ $\|u\| = 3$ and $\langle u, v\rangle = -2$

find $\langle u+v , 3(u+v)\rangle$?

Solution

$$\begin{aligned}
\langle u+v, 3(u+v)\rangle &= 3\langle u+v, u+v\rangle \\
&= 3\left[\langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle\right] \\
&= 3\left[\|u\|^2 + 2\langle u,v\rangle + \|v\|^2\right] \\
&= 3\left[(3)^2 + 2(-2) + (4)^2\right] \\
&= \text{------}
\end{aligned}$$

Def : Orthognality : Tow vectors u and v are orthogonal if $\langle u, v\rangle = 0$.

① clear that the zero vector 0 is orthogonal with any vector v because $\langle v, 0\rangle = 0$.

② Let $V = \mathbb{R}^2$. Then $(-1, 1)$ and $(2, -2)$ are orthogonal because $\langle (-1,1), (2,-2)\rangle = (-1,1)\cdot(2,-2) = -2 + 2 = 0$

③ The only vector which is orthogonal with itself is the zero vector. Because if $v \neq 0$ then $\langle v, v\rangle > 0$ (By definition of inner product).

** Pythagorean Theorem :

If u and v are <u>orthogonal</u> then $\|u+v\|^2 = \|u\|^2 + \|v\|^2$

<u>Proof</u> : $L.H.S = \|u+v\|^2$

$$= \langle u+v, u+v \rangle$$
$$= \langle u,u \rangle + \langle u,v \rangle + \langle v,u \rangle + \langle v,v \rangle$$
$$= \langle u,u \rangle + 2\langle u,v \rangle + \langle v,v \rangle$$
$$= \underbrace{\|u\|^2}_{} + \underbrace{0}_{\{u \text{ and } v \text{ orthogonal}\}} + \|v\|^2 = \|u\|^2 + \|v\|^2 = R.H.S$$

** Orthogonal decomposition:

Let $u, v \in V$ where $v \neq 0$.

① Put $c = \dfrac{\langle u,v \rangle}{\|v\|^2}$ and $w = u - cv$ (Notice that c is scalar)

② Notice that (i) $\langle w, v \rangle = 0$

(2) $u = cv + w$

<u>Practise</u> : Let $V = \mathbb{R}^2$, $u = (3,2)$ and $v = (-1,3)$.
Find the orthogonal decomposition of u and v?

Solution

Notice that $v \neq 0$.

Let $C = \frac{\langle u, v\rangle}{\|v\|^2} = \frac{\langle u, v\rangle}{\langle v, v\rangle} = \frac{\langle (3,2), (-1,3)\rangle}{\langle (-1,3), (-1,3)\rangle}$

$= \frac{(3)(-1) + (2)(3)}{(-1)(-1) + (3)(3)} = \frac{3}{10}$

Now, Put $w = u - cv$

$= (3,2) - \frac{3}{10}(-1,3) = (3,2) + \left(\frac{3}{10}, \frac{-9}{10}\right)$

$= \left(\frac{33}{10}, \frac{11}{10}\right)$

Now, we should to examine :

(1) $\langle w, v\rangle = \left\langle \left(\frac{33}{10}, \frac{11}{10}\right), (-1,3)\right\rangle = \frac{33}{10}(-1) + \left(\frac{11}{10}\right)3 = 0$

(2) $cv + w = \frac{3}{10}(-1,3) + \left(\frac{33}{10}, \frac{11}{10}\right) = \left(\frac{-3}{10}, \frac{9}{10}\right) + \left(\frac{33}{10} + \frac{11}{10}\right)$

$= (3,2) = u$ ☒☒

Remark :

By above method, we can get two orthogonal vectors v, w if we are given two vectors v, u where $\boxed{w = u - cv}$.

★★ Cauchy-Schwarz inequality :

$|\langle u, v\rangle| \leq \|u\| \, \|v\|$

for example : In $\mathbb{R}^2$, let $u = (1,2)$ $v = (3,-1)$ then

$\|u\| = \sqrt{\langle u, u\rangle} = \sqrt{\langle (1,2), (1,2)\rangle} = \sqrt{1+4} = \sqrt{5}$

$\|v\| = \sqrt{\langle v, v\rangle} = \sqrt{\langle (3,-1), (3,-1)\rangle} = \sqrt{10}$

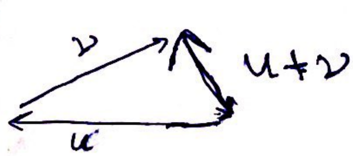
$\langle u, v\rangle = \langle (1,2), (3,-1)\rangle = 3 - 2 = 1$

$|\langle u, v\rangle| = |1| = 1$

clear that $|\langle u, v\rangle| \leq \|u\| \cdot \|v\|$.

★★ Tringle Inequality :

$\|u + v\| \leq \|u\| + \|v\|$



Example Let u and v be two vectors in inner Product vector Space. Prove that

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 - 2\|v\|^2$$

solution

$$\begin{aligned}
L.H.S &= \|u+v\|^2 + \|u-v\|^2 \\
&= \langle u+v, u+v\rangle + \langle u-v, u-v\rangle \\
&= \langle u, u\rangle + 2\langle u, v\rangle + \langle v, v\rangle + \langle u, u\rangle - 2\langle u, v\rangle + \langle v, v\rangle \\
&= 2\langle u, u\rangle + 2\langle v, v\rangle \\
&= 2\|u\|^2 + 2\|v\|^2 = R.H.S
\end{aligned}$$

Example Let $\langle (a,b), (c,d)\rangle = |ac| + |bd|$ where (a,b) and $(c,d) \in \mathbb{R}^2$. Show that $\langle\, ,\rangle$ is not inner product?

solution:

Let $u = (a,b)$ $v_1 = (c,d)$ $v_2 = (c', d')$

The condition

$\langle u, v_1+v_2\rangle = \langle u, v_1\rangle + \langle u, v_2\rangle$ is <u>not</u> satified

because $|n+m| \neq |n| + |m|$

(write the full answer!!)

Example suppose that u and v has the same norm. Prove that $u+v$ and $u-v$ are orthogonal?

Solution

$$\begin{aligned}
\langle u+v, u-v\rangle &= \langle u, u\rangle + \langle u, v\rangle - \langle v, u\rangle - \langle v, v\rangle \\
&= \langle u, u\rangle - \langle v, v\rangle \\
&= \|u\|^2 - \|v\|^2 \\
&= 0 \quad (\text{because } \|u\| = \|v\|)
\end{aligned}$$

Hence $u+v$ and $u-v$ are orthogonal!!

Ex: suppose that $u, v \in V$. Prove that

$\langle u, v \rangle = 0$ (i.e u, v are orthogonal) $\iff$ $\|u\| \leq \|u + av\|$ for all $a \in \mathbb{R}$.

solution

($\Rightarrow$) suppose that $\langle u, v \rangle = 0$

Hence, $\langle u, av \rangle = a \langle u, v \rangle = 0$

Now,

$$\|u + av\|^2 = \langle u + av, u + av \rangle$$

$$= \langle u, u \rangle + \langle av, av \rangle \quad \text{.... why ?}$$

$$= \|u\|^2 + \underbrace{a^2 \|v\|^2}_{(+)\text{ value}}.$$

So,

$$\|u\|^2 \leq \|u + av\|^2$$

There fore

$$\|u\| \leq \|u + av\|$$

($\Leftarrow$) suppose that $\|u\| \leq \|u + av\|$ for every $a \in \mathbb{R}$

Now, $\|u + av\|^2 = \langle u + av, u + av \rangle$

$$= \|u\|^2 + 2a \langle u, v \rangle + \|v\|^2$$

By assumption:

$$\|u\| \leq \|u + av\| = \sqrt{\|u\|^2 + 2a\langle u, v \rangle + \|v\|^2}$$

+ value

To ansure the statment will be true, we should Put $2a \langle u, v \rangle = 0$

$$\Rightarrow \langle u, v \rangle = 0 \quad \text{---} \ \blacksquare$$

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Def (orthonormal) :

Let $\{v_1, \dots, v_n\}$ be the set of vectors in V. It is called an orthonormal set if

① $\|v_i\| = 1 \quad \forall i$

② $\langle v_i, v_j \rangle = 0 \quad \forall i \neq j$

for example : $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ and $(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ are orthonormal in $\mathbb{R}^3$.

Notice that

① $\|(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})\| = \sqrt{\langle (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) \rangle}$

$= \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$

② $\|(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)\| = 1$

③ $\langle (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0) \rangle = \frac{-1}{\sqrt{6}} + \frac{1}{\sqrt{6}} + 0 = 0$

Rule ① Let $\{v_1, \dots, v_n\}$ be orthonormal then

$\|a_1 v_1 + \dots + a_n v_n\| = |a_1|^2 + \dots + |a_n|^2$

where $a_1, \dots, a_n \in \mathbb{R}$

Rule ② Every orthonormal set is independent (the converse is not always true!!)

Remark : Let $S = \{v_1, \dots, v_n\}$ be a basis of V which is orthonormal. Then S is called orthonormal basis.

Rule ③ Let S be orthonormal set of vectors in V such that $|S| = \text{Dim}(V)$. then S is an orthonormal basis.

Remark : Let $S = \{e_1, \cdots, e_n\}$ be orthonormal basis. Then S is a spannig set. So, every vector $v \in V$ is a Linear Combination of S.

So,

$$v = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$$

$$\Rightarrow \|v\| = |a_1|^2 + |a_2|^2 + \cdots + |a_n|^2$$

in fact

$$\boxed{a_1 = \langle v, e_1 \rangle} \quad \boxed{a_2 = \langle v, e_n \rangle} \quad \boxed{a_n = \langle v, e_n \rangle}$$

for example: Let $S = \{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\}$

1. Prove that S is orthonormal basis ?
2. write $v = (1,2)$ as Linear combination of S.

<u>solution</u>

[1] $\|(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\| = \langle (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \rangle = \frac{1}{2} + \frac{1}{2} = 1$

$\|(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\| = \frac{1}{2} + \frac{1}{2} = 1$

$\langle (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \rangle = \frac{-1}{2} + \frac{1}{2} = 0$

$|S| = 2 = \mathrm{Dim}(\mathbb{R}^2)$

Hence, S is orthonormal basis !

[2] $v = a_1 (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) + a_2 (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

where $a_1 = \langle (1,2), (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \rangle = \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{3}{\sqrt{2}}$

$a_2 = \langle (1,2), (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \rangle = \frac{-1}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

Notice that

$$a_1^2 + a_2^2 = \frac{9}{2} + \frac{1}{2} = 5 = \|v\|$$

Gram - Schmidt Prosedure :

It aims to convert the basis B of V to orthonormal basis. through the following steps :

suppose that $B = \{v_1, v_2, \dots, v_n\}$

STEP1 $e_1 = \frac{1}{\|v_1\|} \cdot v_1$

STEP2 $e_2 = \frac{v_2 - \langle v_2, e_1\rangle e_1}{\| v_2 - \langle v_2, e_1\rangle e_1\|}$

STEP3 $e_3 = \frac{v_3 - \langle v_3, e_1\rangle e_1 - \langle v_3, e_2\rangle e_2}{\| v_3 - \langle v_3, e_1\rangle e_1 - \langle v_3, e_2\rangle e_2\|}$

and so on till e_n !!

the orthonormal basis $= \{e_1, e_2, \dots, e_n\}$

(ex) suppose that $\{(1,1,0), (2,2,3)\}$ is the basis of W which is a subspace of $\mathbb{R}^3$. Find the orthonormal basis of W ?

Solution

I will use Gram - Schmidt Prosedure.

Let $v_1 = (1,1,0)$ $v_2 = (2,2,3)$

Now, $e_1 = \frac{1}{\|v_1\|} \cdot v_1 = \frac{1}{\sqrt{\langle v_1, v_1\rangle}} \cdot v_1$

$= \frac{1}{\sqrt{1+1}} (1,1,0) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$

$e_2 = \frac{v_2 - \langle v_2, e_1\rangle e_1}{\| \cdots\cdots \|} = \cancel{(2,2,3) - \langle (2,2,3), (\frac{1}{\sqrt{2}}}$

$= \frac{(2,2,3) - \langle (2,2,3), (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)\rangle (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)}{\| \cdots\cdots \|}$

$= \frac{(2,2,3) - \frac{4}{\sqrt{2}} (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)}{\| \cdots \|} = \frac{(2,2,3) - (2, 2, 0)}{\| \quad \|}$ ← H

$= \frac{(0,0,3)}{\|(0,0,3)\|} = (0,0,1)$?