Actuarial mathematics 1

Lecture 1. Introduction.

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Insurance is thus a *hedge* against financial contingent losses.

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Where does the maths come into all these?

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Consider the following:

Example 1.1 (A very simple example.)

A person rolls a fair die with the possible outcomes being $1, 2, \ldots, 6$. If the die shows a number greater than 3, the person pays 1\$. He/she is interested to transfer the risk to an insurer. What an actuary would do?

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$$\pi[\mathcal{A}] = 1\$ \cdot 0.5 = 0.5\$,$$

which is the premium the insured would be required to pay. You expected it, didn't you?

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How would our actuary determine the probability for any one of the above being equal to a specific value? And then the price? Let's go back to Example 1.1.

Solution of Example 1.1 (cont.)

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$$\mathbf{E}[X]=\mathbf{P}[X=1]=0.5,$$

and the price for the insurance becomes $\pi[X]=0.5\$$ as before,

At home.

Check that if $\mathbf{1}\{\mathcal{A}\}(\omega)$ is an indicator r.v., i.e., it is defined as $\mathbf{1}\{\mathcal{A}\}(\omega) = 1$ if $\omega \in \mathcal{A}$, and $\mathbf{1}\{\mathcal{A}\}(\omega) = 0$ if $\omega \in \mathcal{A}^c$, then

$$\mathbf{E}[\mathbf{1}\{\mathcal{A}\}] = \mathbf{P}[\{\mathcal{A}\}],$$

$$\mathbf{Var}[\mathbf{1}\{\mathcal{A}\}] = \mathbf{P}[\{\mathcal{A}\}](1 - \mathbf{P}[\{\mathcal{A}\}]),$$

and

$$\mathbf{Cov}[\mathbf{1}\{\mathcal{A}\}], [\mathbf{1}\{\mathcal{B}\}] = \mathbf{P}[\mathbf{1}\{\mathcal{A}\} \cap \mathbf{1}\{\mathcal{B}\}] - \mathbf{P}[\mathbf{1}\{\mathcal{A}\}] \cdot \mathbf{P}[\mathbf{1}\{\mathcal{B}\}].$$

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To conclude the example, we shall augment it with the time value of money. To this end, let $v = (1 + i)^{-1}$ be the discounting factor. Thus the present value (p.v.) of 1\$ payed at time 1 is

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$$\pi[X] = \mathbf{E}[X \cdot v^Y]$$

We shall call present values a la $1\$ \cdot v^Y$, actuarial present values (a.p.v.). This completes Example 1.

- The premium is nothing else but the a.p.v. of the losses.

Example 1.2

A couple wishes to insure the life of its new born child. What would an actuary do to compute the premium for the contract if the insurance amount of $10,000$\$ is repayed upon child's death, and the interest rate is fixed to i throughout?

Solution.

If the length of child's life (=time of his/her death) is described by the continuous r.v. $T(0) : \Omega \to \mathcal{A} \subset \mathbf{R}_+$, with $\Omega = \{t \in \mathbf{R}_+ : 0 < t \leq 123\}$, then the price to be payed is

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End of Example 1.2.

- This course will mostly deal with various characteristics of the r.v.'s a la $T(0)$.

Key points.

MATH 3280 3.00 F.

We shall in the sequel assume that life times are r.v.'s, and we shall thus explore:

- Various life formations (statuses) and their future life times as well as sources of their deaths (decrements),

and the corresponding

- cumulative and decumulative distribution functions (c.d.f's) and (d.d.f's);
- probability density functions (p.d.f's) and the force of mortality for continuous r.v.'s;
- probability mass functions (p.m.f's) and life tables for curtuate r.v.'s;
- percentiles, expectations, variances, covariances.