Stress-strain relation

Stress-strain relation

- The relation between stress and strain in general is described by the tensor of elastic constants C_{ijkl}

- Since both stress and strain are 3×3 tensor quantities, the most general possible elastic medium has a constitutive equation of the form:

$$\sigma_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} C_{ijkl} e_{kl} = C_{ijkl} e_{kl}$$

- C, or the elastic tensor, is a 4th order tensor (3×3×3×3) and could conceivably have 34 = 81 independent elements.

Stress-strain relation (Hooke's law)

- Hook's law states that at sufficiently small strains ($\leq 10^{-6}$), the strain is directly proportional to the stress producing it.
- The strains produced by the passage of seismic waves in earth material are such that Hooke's law is always satisfied.
- Mathematically, Hooke's law can be expressed as:

$$\sigma_{ij} = c_{ijkl} e_{kl}$$

- where σ_{ij} is the stress matrix, e_{kl} is the strain matrix, and C_{ijkl} is the elastic-constants tensor, which is a fourth-order tensor consisting of 81 elastic constants (*C_{xxxx} to C_{zzzz}).*
- Symmetry of stress and strain tensors requires that:

$$C_{ijkl} = C_{jikl} = C_{ijlk}$$

- which reduces the number of independent components to 36 (six independent components of stress and strain).

Stress-strain relation (Hooke's law)

$$\bar{\bar{C}} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix}$$

is the elastic-constant matrix (36 components).

Stress-strain relation (Hooke's law)

$$\overline{\overline{C}} = \begin{bmatrix} C_{11} & & & & & \\ C_{21} & C_{22} & & & & \\ C_{31} & C_{32} & C_{33} & & & \\ C_{41} & C_{42} & C_{43} & C_{44} & & \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix}$$

A final symmetry, $\boldsymbol{C_{ijkl}} = \boldsymbol{C_{klij}}$, can be derived from thermodynamics to prove that there are, at most, 21 independent elastic constants.

Stress-strain relation

- The constants C_{ijkl} are often referred to as the elastic moduli, and describe the elastic properties of the medium.
- In an isotropic body, where the properties do not depend on direction the relation reduces to

$$c_{ijkl} = \lambda \theta \delta_{ij} + 2\mu e_{ij}$$

- where λ and μ are the Lame's parameters. μ is also called the rigidity or shear modulus.
- Based on Hook's law, each component of the stress field can be written in terms of strain as follows (noting the implied summation)

$$\sigma_{ij} = \lambda \theta \delta_{ij} + 2\mu e_{ij}$$

Stress-strain relation

- Each of the six independent components of the stress field can then be explicitly written

$$\sigma_{xx} = \lambda\theta + 2\mu e_{xx}$$
$$\sigma_{yy} = \lambda\theta + 2\mu e_{yy}$$
$$\sigma_{zz} = \lambda\theta + 2\mu e_{zz}$$
$$\sigma_{xy} = \mu(e_{xy} + e_{yx}) = 2\mu e_{xy}$$
$$\sigma_{xz} = 2\mu e_{xz}$$
$$\sigma_{yz} = 2\mu e_{yz}$$

Note that, the angle of shear is equal to twice shear component of strain. These equations are very useful to derive the wave equation for P and S waves.

- Where θ is the change in volume per unit volume of a material, which is known as **cubic dilatation, is given** by:

$$\theta = \frac{\Delta V}{V_0} = e_{ii} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = div u = \nabla \bullet u$$

Stress-strain relation

- The complete stress tensor looks like

$$\sigma_{ij} = \begin{pmatrix} (\lambda+2\mu)\varepsilon_{xx} + \lambda(\varepsilon_{yy}+\varepsilon_{zz}) & 2\mu\varepsilon_{xy} & 2\mu\varepsilon_{xz} \\ 2\mu\varepsilon_{yx} & (\lambda+2\mu)\varepsilon_{yy} + \lambda(\varepsilon_{xx}+\varepsilon_{zz}) & 2\mu\varepsilon_{yz} \\ 2\mu\varepsilon_{zx} & 2\mu\varepsilon_{zy} & (\lambda+2\mu)\varepsilon_{zz} + \lambda(\varepsilon_{xx}+\varepsilon_{yy}) \end{pmatrix}$$

- The constants of proportionality are known as the **elastic constants** and are different for different kinds of stress (twisting, compressing, stretching) and for different materials. So based on the relationships between elastic moduli and Lamé coefficients (λ and μ), the elasticity can be quantified by various elastic moduli:
  - The bulk modulus (incompressibility)
  - The shear modulus (rigidity)
  - The Young’s modulus (stretch modulus)
  - The Poisson’s ratio

Problem 2

- Suppose the elastic moduli of an isotropic medium are $\lambda = \mu = 3\times10^{11} dyn/cm^2$, and the longitudinal strain components are $2x10^{-6}$, with all other strain components equal zero, compute the stress tensor and interpret in terms of seismic wave radiation.

Scientific terms related to Elasticity

- ***Elastic*** *means that the stress and strain satisfy Hooke's law.*

- ***Homogeneous*** *means that layer properties (e.g., velocity) are constant across the whole layer.*

- ***Isotropic*** *means that wave properties (e.g., velocity) are independent of propagation direction.*

Elastic constants in isotropic media

- Lame's constants λ and μ are defined through the following forms of Hooke's law in an isotropic medium

$$\sigma_{ij} = \lambda \theta \delta_{ij} + 2\mu e_{ij}$$

(i = x,y,z)

$$\sigma_{ij} = 2\mu e_{ij}$$

(i ≠ j, i,j = x,y,z)

Elastic Constants

- **Bulk modulus *k*:**
  - This is defined as the ratio of hydrostatic pressure to the resulting volume change.

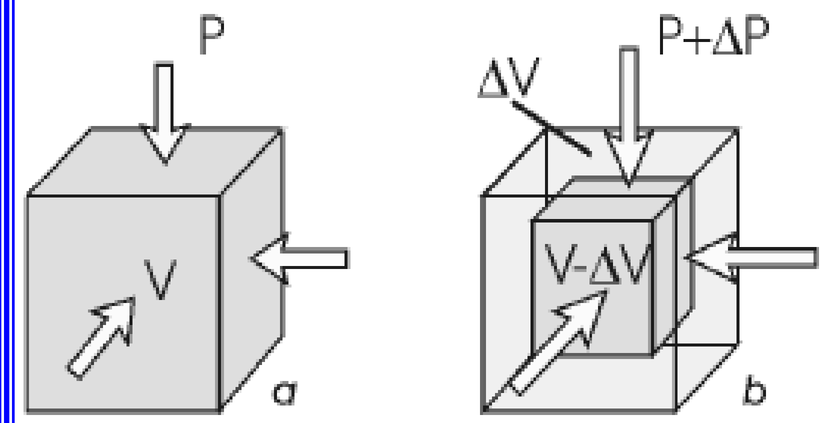
$$k = \frac{stress}{strain} = \frac{\Delta P}{\Delta V / V}$$

  - It is a measure of the incompressibility of the material

$$\kappa = \frac{p}{\theta} = \lambda + \frac{2}{3}\mu$$

  - where $P = 1/3\sigma_{ii}$

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma$$



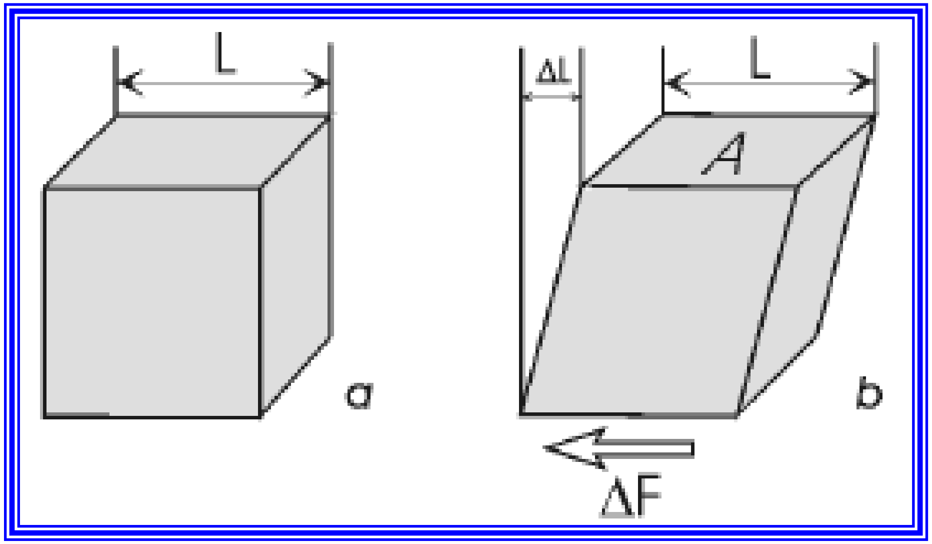
$$k = \frac{\text{stress}}{\text{strain}} = \frac{\Delta P}{\Delta V/V}$$

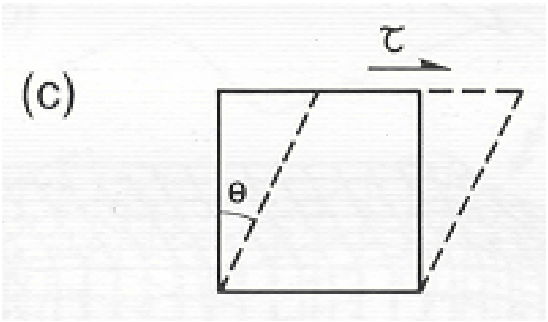
where:

$\Delta P = P' - P$ = pressure change (applied stress)
P = original confining pressure
P' = confining pressure under the applied stress
$\Delta V = V - V'$ = change in volume caused by ΔP
V = original volume
V' = volume under the applied stress.

Elastic Constants

- **Shear modulus (rigidity), μ:**
  - The shear modulus refers to the ability of a material to resist shearing. ($\mu=\infty$, $\Delta l=0$ & $\mu=0$, $\Delta l=\infty$)



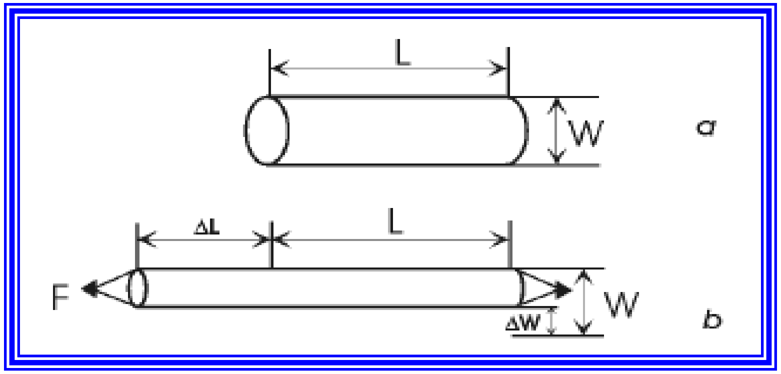


$\mu = \tau / \tan\ \theta$

$$\mu = \frac{stress}{strain} = \frac{\Delta F / A}{\Delta L / L}$$

Elastic Constants

- **Young's modulus Y:**
  - This is defined as the ratio of extensional stress to the resulting extensional strain for a cylinder being pulled apart at both ends



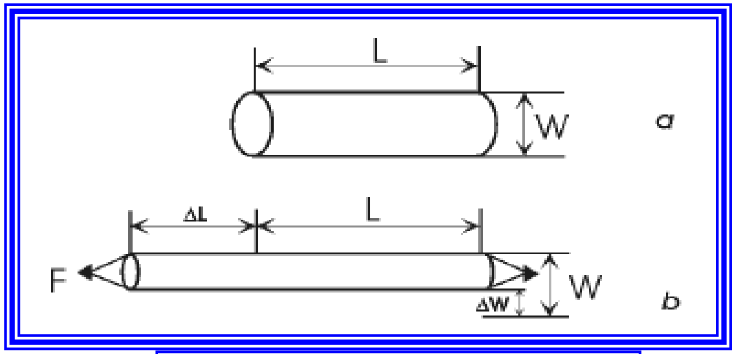
$$Y = \frac{Stress}{Strain} = \frac{F/A}{\Delta L/L}$$

$$= \sigma_{xx} / e_{xx}$$

  - This constant is given by

Elastic Constants

- **Poisson’s ratio σ:**
  - is defined as the ratio of transverse contraction strain to longitudinal extension strain in the direction of stretching force.
  - This is given by $\sigma = \lambda / 2(\lambda + \mu)$
  - Tensile deformation is considered positive and compressive deformation is considered negative. The definition of Poisson's ratio contains a minus sign so that normal materials have a positive ratio $\sigma = - e_{transverse} / e_{longitudinal}$
  - ***Poisson solid***: material for which $\lambda = \mu$, giving s = **0.25**



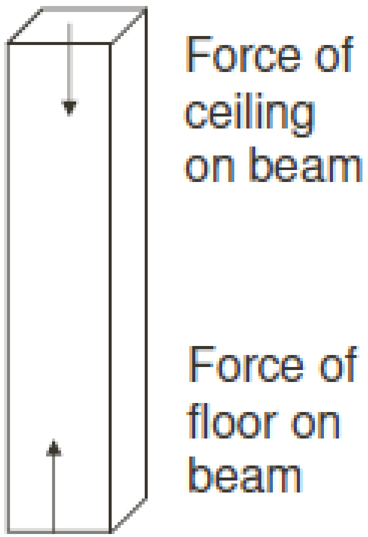
$$\sigma = (\Delta W / W) / (\Delta L / L)$$

Homework Problem (3)

- **Please Solve the following problems:**
  - A steel beam is placed vertically in the basement of a building to keep the floor above from sagging. The load on the beam is 5.8 $x10^4$ N and the length of the beam is 2.5 m, and the cross-sectional area of the beam is 7.5 x 10^{-3} m^2. Find the vertical compression of the beam. Notice that young modulus equals $200x10^9$ Pa for steel.
  - A 0.50 m long string, of cross- sectional area 1.0 $x10^{-6}$ m^2, has a Young's modulus of 2.0 x 10^9 Pa. By how much must you stretch a string to obtain a tension of 20.0 N?
  - The upper surface of a cube of gelatin, 5.0 cm on a side, is displaced by 0.64 cm by a tangential force. If the shear modulus of the gelatin is 940 Pa, what is the magnitude of the tangential force?
  - An anchor, made of cast iron of bulk modulus 60.0 x 10^9 Pa and a volume of 0.230 m^3, is lowered over the side of a ship to the bottom of the harbor where the pressure is greater than sea level pressure by 1.75 x 10^6 Pa. Find the change in the volume of the anchor.

Answers

Exercise 1



$$\frac{F}{A} = Y\frac{\Delta L}{L}$$

$$\Delta L = \left(\frac{F}{A}\right)\left(\frac{L}{Y}\right)$$

For steel Y=200×10^{9} Pa.

$$\Delta L = \left(\frac{F}{A}\right)\left(\frac{L}{Y}\right) = \left(\frac{5.8\times10^{4}\ \text{N}}{7.5\times10^{-3}\ \text{m}^2}\right)\left(\frac{2.5\ \text{m}}{200\times10^{9}\ \text{N/m}^2}\right) = 1.0\times10^{-4}\ \text{m}$$

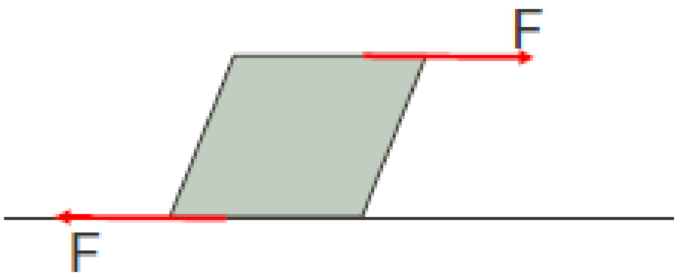
Exercise 2

$$\frac{F}{A} = Y\frac{\Delta L}{L}$$

$$\Delta L = \left(\frac{F}{A}\right)\left(\frac{L}{Y}\right) = \left(\frac{20.0\ \text{N}}{1.0\times10^{-6}\ \text{m}^2}\right)\left(\frac{0.5\ \text{m}}{2.0\times10^{9}\ \text{N/m}^2}\right)$$

$$= 5.0\times10^{-3}\ \text{m} = 5.0\ \text{mm}$$

Exercise 3



$$\mu = \frac{stress}{strain} = \frac{\Delta F / A}{\Delta L / L}$$

From Hooke's Law:

$$\Delta F = \mu \cdot \frac{A \cdot \Delta L}{L}$$

$$= \left(940\ \text{N/m}^2\right)\left(0.0025\ \text{m}^2\right)\left(\frac{0.64\ \text{cm}}{5.0\ \text{cm}}\right) = 0.30\ \text{N}$$

Exercise 4

$$\Delta P = -\boldsymbol{k}\,\frac{\Delta V}{V}$$

$$\Delta V = -\frac{V\Delta P}{\boldsymbol{k}} = -\frac{\left(0.23\ \text{m}^3\right)\left(1.75\times10^6\ \text{Pa}\right)}{60.0\times10^9\ \text{Pa}}$$

$$= -6.7\times10^{-6}\ \text{m}^3$$