Lecture 16 and 17
Application to Evaluation of Real Integrals

Theorem 1 Residue theorem: *Let Ω be a simply connected domain and A be an isolated subset of Ω. Suppose $f : \Omega \setminus A \to \mathbb{C}$ is a holomorphic function. The for any simple closed curve γ in $\Omega \setminus A$, we have*

$$\int_\gamma f(z)dz = 2\pi\imath \sum_{a \in A} R_a(f)\eta(\gamma; a)$$

where $\eta(\gamma; a)$ denotes the winding number of γ around a.

Proof: If R is the region in Ω enclosed by γ note that $R \cup \gamma$ is a closed and bounded subset of $\mathbb{C}$. Since A is an isolated set, it follows that $A \cap R = A \cap (R \cup \gamma)$ has only finitely many elements say $a_1, \ldots, a_k$. Choose $r > 0$ so that the disc of radius r with center a_j are all contained in Ω and are all mutually disjoint and disjoint form γ. Let C_j denote the positively oriented boundary of this disc. Then by II version of Cauchy's theorem, we have

$$\int_\gamma f(z)dz = \sum_{j=1}^{k} \int_{C_j} f(z)dz = 2\pi\imath \sum_{j=1}^{k} R_{a_j}(f).$$

Since $\eta(\gamma, a) = 0$ if $a \neq a_j$ and is equal to 1 if $a = a_j$, the conclusion of the theorem follows: ♠

We shall now demonstrate the usefulness of the complex integration theory in computing definite real integrals. This should not surprise you since after all, complex integration is nothing but two real integrals which make up its real and imaginary parts. Thus given a real integral to be evaluated if we are successful in associating a complex integration and also evaluate it, then all that we have to do is to take real or(the imaginary) part of the complex integral so obtained. However, this itself does not seem to be always possible. Moreover, as we think about it, we perceive several obstacles in this approach. For instance, the complex integration theory is always about integration over closed paths whereas, a real definite integral is always over an interval, finite or infinite. So, by adding suitable curves, we somehow form a closed curve, on which the complex integration is performed and then we would like either to get rid of the value of the integration on the additional paths that we have introduced or we look for other sources and methods to evaluate them. The entire process is called *'the method of complexes'* or *residue method.* Each problem calls for a certain amount of ingenuity. Thus we see that the method has its limitations and as Ahlfors puts it "— *but even complete mastery does not guarantee success.*" However, when it works it works like magic. We shall only consider two important cases.

Trigonometric Integrals

Example 1 Let us show that

$$\int_0^{2\pi} \frac{d\theta}{1 + a\sin\theta} = \frac{2\pi}{\sqrt{1-a^2}}, \ -1 \ < \ a \ < \ 1.$$

Observe that for $a = 0$, there is nothing to prove. So let us assume that $a \neq 0$. We want to convert the integrand into a function of a complex variable and then set $z = e^{\imath\theta}$, $0 \leq \ \theta \ \leq 2\pi$, so that the integral is over the unit circle C. Since, $z = e^{\imath\theta} = \cos\theta + \imath\sin\theta$, we have, $\sin\theta = (z - z^{-1})/2\imath$, and $dz = \imath e^{\imath\theta} d\theta$, i.e., $d\theta = dz/\imath z$. Therefore,

$$\begin{aligned} I &= \int_C \frac{dz}{\imath z(1 + a(z - z^{-1})/2\imath))} \\ &= \int_C \frac{2dz}{az^2 + 2\imath z - a} = \frac{2}{a}\int_C \frac{dz}{(z - z_1)(z - z_2)}, \end{aligned}$$

where, z_1, z_2 are the two roots of the polynomial $z^2 + \frac{2\imath}{a}z - 1$. Note that

$$z_1 = \frac{(-1 + \sqrt{1-a^2})\imath}{a}, \ z_2 = \frac{(-1 - \sqrt{1-a^2})\imath}{a.}$$

It is easily seen that $|z_2| > 1$. Since $z_1 z_2 = -1$, it follows that $|z_1| \ < 1$. Therefore on the unit circle C, the integrand has no singularities and the only singularity inside the circle is a simple pole at $z = z_1$. The residue at this point is given by

$$R_{z_1} = 2/a(z_1 - z_2) = 1/\imath\sqrt{1-a^2}.$$

Hence by the Residue Theorem, we have:

$$I = 2\pi\imath R_{z_1} = 2\pi/\sqrt{1-a^2}.$$

We summerise the theme that we have gone through in the previous example as a theorem:

Theorem 2 Trigonometric integrals : *Let* $\phi(x,y) = p(x,y)/q(x,y)$ *be a rational function in two variables such that* $q(x,y) \neq 0$ *on the unit circle. Then*

$$\begin{aligned} I_\phi &:= \int_0^{2\pi} \phi(\cos\theta, \sin\theta) d\theta \\ &= 2\pi \left(\sum_{|z|<1} R_z(\tilde{\phi}) \right), \end{aligned}$$

where, $\tilde{\phi}(z) = \dfrac{1}{z}\phi\left(\dfrac{z+z^{-1}}{2}, \dfrac{z-z^{-1}}{2\imath}\right)$.

Proof: Substitute

$$z = e^{\imath\theta}, \sin\theta = (z - z^{-1})2/\imath; \cos\theta = (z+z^{-1})/2; d\theta = dz/\imath z.$$

Then $\phi(\cos\theta, \sin\theta)d\theta = \frac{1}{i}\hat{\phi}(z)dz$. Since $\hat{\phi}$ is a rational function the rest of the conclusion follows by Residue theorem.

Remark 1 This is not the only method for evaluating the above integral. The above integral is also equal to

$$\int_{-\pi}^{\pi} \frac{d\theta}{1 + a\sin\theta}.$$

Now substitute $u = \tan(\theta/2)$ to get an anti-derivative:

$$\begin{aligned} I &= \int_\infty^\infty \tfrac{2du}{1+u^2+2au} \\ &= \int_{-\infty}^\infty \tfrac{2du}{(u+a)^2+(1-a)^2)} \\ &= \tfrac{2}{\sqrt{1-a^2}} \tan^{-1}\left(\tfrac{u+a}{\sqrt{1-a^2}}\right)\Big|_{-\infty}^{\infty} \\ &= \tfrac{2\pi}{\sqrt{1-a^2}}. \end{aligned}$$

This method can be used to to compute even $\int_0^\pi \frac{d\theta}{1+a\sin\theta}$ also. The method of residues cannot be employed to evaluate this latter integral.

Improper Integrals

We shall begin with a brief introduction to the theory of improper integrals. Chiefly there are two types of them. One type arises due to the infiniteness of the interval on which the integration is being taken. The other type arises due to the fact that the integrand is not defined (shoots to infinity) at one or both end point of the interval.

Definition 1 *When* $\int_a^b f(x)dx$ *is defined for all* $d > c > R$ *we define*

$$\int_a^\infty f(x)dx := \lim_{b \longrightarrow \infty} \int_a^b f(x)dx, \tag{1}$$

if this limit exists. Similarly we define

$$\int_{-\infty}^b f(x)dx := \lim_{a \longrightarrow -\infty} \int_a^b f(x)dx, \tag{2}$$

if this limit exists. Also, we define

$$\int_{-\infty}^\infty f(x)dx := \int_0^\infty f(x)dx + \int_{-\infty}^0 f(x)dx, \tag{3}$$

provided both the integrals on the right exist.

Recall the Cauchy's criterion for the limit. It follows that the limit (1) exists iff *given* $\epsilon > 0$ *there exists* $R > 0$ *such that for all* $b > a > R$ *we have,*

$$\left|\int_a^b f(x)dx\right| < \epsilon. \tag{4}$$

In many practical situations the following theorem and statements which can be easily derived out of it come handy in ensuring the existence of the improper integral of this type.

Theorem 3 Existence of Improper Integrals : *Suppose* f *is a continuous function defined on* $[0, \infty)$ *and there exists* $\alpha > 1$ *such that* $x^\alpha f(x)$ *is bounded. Then* $\int_0^\infty f(x)\,dx$ *exists.*

Remark 2 However, the condition in the above theorem is not always necessary. For instance, the function $f(x) = \dfrac{\sin x}{x}$ does not satisfy this condition. Nevertheless $\int_0^\infty \dfrac{\sin x}{x}\,dx$ exists as will be seen soon.

Observe that there is yet another legitimate way of taking limits in (3), i.e., to take the limit of $\int_{-a}^{a} f(x)dx$, as $a \longrightarrow \infty$. However, this limit, even if it exists, is, in general, not equal to the improper integral defined in 3, above. This is called the *Cauchy's Principal Value* of the improper integral and is denoted by,

$$PV\left(\int_{-\infty}^{\infty} f(x)dx\right) := \lim_{a\longrightarrow\infty} \int_{-a}^{a} f(x)dx. \tag{5}$$

As an example consider $f(x) = x$. Then the Cauchy's PV exists but the improper integral does not. However, if the improper integral exists, then it is also equal to its principle value. This observation is going to play a very important role in the following application.

Example 2 Let us consider the problem of evaluating

$$I = \int_0^\infty \frac{2x^2 - 1}{x^4 + 5x^2 + 4} dx$$

Denoting the integrand by f, we first observe that f is an even function and hence

$$I = \frac{1}{2}\int_{-\infty}^{\infty} f(x)dx$$

which in turn is equal to its PV. Thus we can hope to compute this by first evaluating

$$I_R = \int_{-R}^{R} f(x)dx$$

and then taking the limit as $R \longrightarrow \infty$. First, we extend the rational function into a function of a complex variable so that the given function is its restriction to the real axis. This is easy here, viz., consider $f(z)$. Next we join the two end points R and $-R$ by an arc in the upper-half space, (no harm if you choose the lower half-space). What could be a better way than choosing this arc to be the semi-circle! So let C_R denote the semi-circle running from R to $-R$ in the upper-half space. Let γ_R denote the closed contour obtained by tracing the line segment from $-R$ to R and then tracing C_R. We shall compute

$$J_R = \int_{\gamma_R} f(z)dz$$

for large R using residue computation. When the number of singular points of the integrand is finite, J_R is a constant for all large R. This is the crux of the matter. We then hope that in the limit, the integral on the unwanted portions tends to zero, so that $\lim_{R\longrightarrow\infty} J_R$ itself is equal to I.

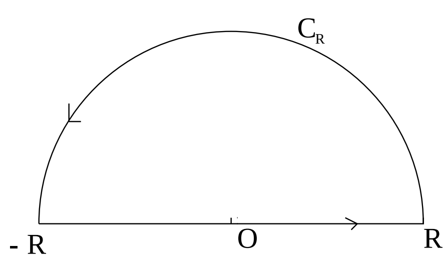

Fig. 15

The first step is precisely where we use the residue theorem. The zeros of the denominator $q(z) = z^4+5z^2+4$ are $z = \pm\imath,\ \pm 2\imath$ and *luckily* they do not lie on the real axis.(This is important.)

They are also different from the roots of the numerator. Also, for $R > 2$, two of them lie inside γ_R. (We do not care about those in the lower half-space.) Therefore by the RT, we have, $J_R = 2\pi\imath(R_\imath + R_{2\imath})$. The residue computation easily shows that $J_R = \pi/2$.

Observe that $f(z) = p(z)/q(z)$, where $|p(z)| = |z^2 - 1| \leq R^2+1$, and similarly $|q(z)| = |(z^2+1)(z^2+4)| \geq (R^2-1)(R^2-4)$. Therefore

$$|f(z)| \leq \frac{R^2+1}{(R^2-1)(R^2-4)} =: M_R.$$

This is another lucky break that we have got. Note that M_R is a rational function of R of degree -2. For, now we see that

$$|\int_{C_R} f(z)dz| \leq M_R|\int_{C_R} dz| = M_R R\pi.$$

Since M_R is of degree -2, it follows that $M_R R\pi \longrightarrow 0$ as $R \longrightarrow \infty$. Thus, we have successfully shown that the limit of $\int_{C_R} f(z)dz$ vanishes at infinity. To sum up, we have,

$$I = \frac{1}{2}\int_{-\infty}^{\infty} f(x)dx = \frac{1}{2}\lim_{R\longrightarrow\infty}\int_{-R}^{R} f(x)dx = \frac{1}{2}\lim_{R\longrightarrow\infty} J_R = \frac{\pi}{4}.$$

Indeed, we have seen enough to write down a proof of the following theorem.

Theorem 4 *Let f be a rational function without any poles on the real axis and of degree ≤ -2. Then*

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi\imath\sum_{w\in\boldsymbol{H}} R_w(f).$$

Example 3 Let us consider another example which is somewhat similar but not exactly same as the earlier example:

$$\int_{-\infty}^{\infty} f(x)dx$$

where $f(x) = (\cos 3x)(x^2+1)^{-2}$.

Except that now the integrand is a rational function of a trigonometric quantity and the variable x, this does not seem to cause any trouble as compared to the example above. For we can consider

$$F(z) = e^{3\imath z}(z^2+1)^{-2}$$

to go with and later take only the real part of whatever we get. The denominator has poles at $z = \pm\imath$ which are double poles but that need not cause any concern. When $R > 1$ the contour γ_R encloses $z = \imath$ and we find the residue at this point of the integrand, and see that $J_R = 2\pi/e^3$. Yes, the bound that we can find for the integrand now has different nature! Putting $z = x+\imath y$ we know that $|e^{3\imath z}| = |e^{-3y}|$. Therefore,

$$|f(z)| = \left|\frac{e^{3\imath z}}{(z^2+1)^2}\right| \leq \left|\frac{e^{-3y}}{(R^2-1)^2}\right|.$$

Since, e^{-3y} remains bounded by 1 for all $y > 0$ we are done. Thus, it follows that the given integral is equal to $2\pi/e^3$.

Example 4 Let us evaluate $\int_{-\infty}^{\infty} \frac{dx}{x^2+a^2}$. We can directly take the anti-derivative $\frac{1}{a}tan^{-1}\left(\frac{x}{a}\right)$ and see that the integral is equal to π/a.

On the other hand the method of complexes can be applied: (i) The complex function $1/z^2+a^2$ has no poles on the real axis, (ii) is a rational function of degree ≤ -2. Also $z = a$ is the only pole in the upper half plane and hence the intergal is equal to $2\pi i R_a = \pi/a$.

Example 5 Consider the problem of evaluating the Cauchy's Principal Value of

$$I = \int_{-\infty}^{\infty} f(x)dx, \qquad \text{where,} \quad f(x) = (x \sin x)/(x^2 + 2x + 2).$$

Writing $f(x) = g(x)\sin x$ and taking $F(z) = g(z)e^{\imath z}$, we see that, for $z = x$, we see that $f(x) = \Im(F(x))$. Also, write, $g(z) = z/(z^2+2z+2) = z/(z-z_1)(z-z_2)$ where, $z_1 = \imath - 1$ and $z_2 = -\imath - 1$, to see that $|g(z)| \le R/(R-\sqrt{2})^2 =: M_R, \ R > 2$, say. And of course, this implies that $\int_{C_R} F(z)dz$ is bounded by $\pi R M_R$, which does not tend to zero as $R \longrightarrow \infty$. Hence, this is of no use! Thus, we are now forced to consider the following stronger estimate:

Lemma 1 Jordan's Inequality

$$J := \int_0^{\pi} e^{-R\sin\theta} d\theta \ < \ \pi/R, \quad R \ > \ 0.$$

Proof: Draw the graph of $y = \sin\theta$ and $y = 2\theta/\pi$. Conclude that $\sin\theta \ > \ 2\theta/\pi$, for $0 \ < \theta < \pi/2$. Hence obtain the inequality,

$$e^{-R\sin\theta} \ < \ e^{-2R\theta/\pi}$$

Use this to obtain,

$$J := 2\int_0^{\pi/2} e^{-R\sin\theta} d\theta \ < \ 2\int_0^{\pi/2} e^{-R\theta/\pi} d\theta = 2\pi(1-e^{-R})/2R \ < \ \pi/R, \ \ R \ > \ 0.$$

♠

Let us now use this in the computation of the integral I above. We have

$$\left|\int_{C_R} F(z)dz\right| = \left|\int_0^{\pi} g(Re^{\imath\theta})e^{\imath Re^{\imath\theta}} \imath Re^{\imath\theta} d\theta\right| \ < \ M_R R \int_0^{\pi} e^{-R\sin\theta} d\theta \ < \ M_R \pi.$$

Since $M_R\pi \longrightarrow 0$ as $R \longrightarrow \infty$, we get

$$\Im(\lim_{R\longrightarrow\infty} J_R) = I,$$

as required. We leave the calculation of the residue to the reader. [Answer: $\frac{\pi}{e}(\cos 1 + \sin 1)$.]

We now have enough ideas to prove:

Theorem 5 *Let f be a holomorphic function in $\mathbb{C}$ except possibly at finitely many singularities none of which is on the real line. Suppose that $\lim_{z\to\infty} f(z) = 0$. Then for any non zero real a,*

$$PV\left(\int_{-\infty}^{\infty} f(x)e^{\imath ax}\,dx\right) = \pm 2\pi\imath \sum_{\pm w\in \boldsymbol{H}} R_w[f(z)e^{\imath az}],$$

where, the sign $\pm$ has to be chosen (in both places), according as a is positive or negative.

Remark 3 *We should also add that the conditions of the theorem are met if f is a rational function of degree ≤ -1 having no real poles.*

Bypassing a Pole

Here we shall attempt to evaluate $\int_0^\infty \frac{\sin x}{x} dx$.

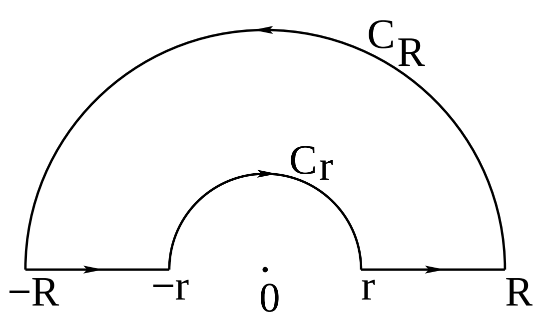

Fig. 16

First of all observe that $\dfrac{\sin x}{x}$ is an even function and hence,

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{1}{2}PV\left(\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx\right).$$

The associated complex function $F(z) = e^{\imath z}/z$ has a singularity on the x-axis and that is going to cause trouble if we try to proceed the way we did so far. Common sense tells us that, since 0 is the point at which we are facing trouble, we should simply avoid this point by going around it via a small semi-circle around 0 in the upper half-plane. Thus consider the closed contour $\gamma_{r,R}$ as shown in the figure.

Given any meromorphic function $F(z)$, with a simple pole at 0, and finitely many poles in the upper half space, in order to compute $\int_{-\infty}^{\infty} F(z)dz$, the idea is to
(i) compute $I(r,R) := \int_{\gamma_{r,R}} F(z)dz$ for large R,
(ii) take the limit as $r \longrightarrow 0$ and $R \longrightarrow \infty$, and hope that the integral on the larger circular portion tends to zero
(iii) Compute $\lim_{r\to 0}\int_{C_r} F(z)dz$.
Since 0 is a simple pole of $F(z)$ we can write $zF(z) = g(z)$ with $g(0) \neq 0$. Again using Taylor's theorem, write $g(z) = g(0) + zg_1(z)$ where g_1 is holomorphic in a neighborhood of 0. It follows that $F(z) = g(0)/z + g_1(z)$. Therefore,

$$\int_{C_r} F(z)dz = g(0)\int_0^\pi \imath d\theta + \int_{C_r} g_1(z)dz = -g(0)\pi\imath + (G_1(r) - G_1(-r))$$

where , G_1 is a primitive of g_1 in a disc around 0. By continuity of G_1, the last term tends to zero as $r \longrightarrow 0$. Therefore

$$\lim_{r\to 0}\int_{C_r} F(z)dz = -R_0(F)\pi i.$$

So it remains only to compute $g(0)$ which is nothing but the residue of $e^{\imath z}/z$ at $z = 0$.

(iv) Finally, set

$$\int_{-\infty}^{\infty} F(z)dz = \pi i R_0(F) + I(r, R), R >> 0.$$

In this particular case, Since $F(z) = e^{iz}/z$ is holomorphic inside of $\gamma_{r,R}$, it follows that the integral is zero for all $R > r > 0$. Step (ii) can be carried out using Jordan inequality. We leave this to you as an exercise. Since $R_0(F(z)) = 1$ we conclude that

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}. \tag{6}$$

Remark 4 However, note that the simplisitic approach of choosing $F(z) = \sin z/z$ which is holomorphic everywhere is bound to fail. Why? (Examine Step (ii). Indeed, having computed $\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$, it follows that $\lim_{R\to\infty} \int_{C_R} \frac{\sin z}{z} dz = -\frac{\pi}{2}$, but this cannot be used the other way round.)

Branch Cuts

Consider the problem of evaluating the integral

$$I = \int_0^{\infty} \frac{x^{-\alpha}}{x+1} dx, \quad 0 < \alpha < 1.$$

This integral is important in the theory of Gamma functions $\Gamma(a) = \int_0^{\infty} x^{a-1} e^{-x} dx$. Observe that the integral converges because in $[0, 1]$, we can compare it with $\int_0^1 x^{-\alpha} dx$, whereas, in $[1, \infty)$, we can compare it with $\int_1^{\infty} x^{-\alpha-1} dx$. The problem that we face here is that the corresponding complex function $f(z) = z^{-\alpha}$ does not have any single valued branch in any neighborhood of 0. So, an idea is to cut the plane along the positive real axis, take a well defined branch of $z^{-\alpha}$, perform the integration along a contour as shown in the figure below and then let the cuts in the circles tend to zero. The crux of the matter lies in the following observation:

Let $f(z)$ be a branch of z^{α} in $\mathbb{C} \setminus \{x \ : \ x \geq 0\}$. Suppose for any $x_0 > 0$, the limit of $f(z)$ as $z \longrightarrow x_0$ through upper-half plane is equal to $x_0^{-\alpha}$. Then the limit of $f(z)$ as $z \longrightarrow x_0$ through lower-half plane is equal to $x_0^{-\alpha} e^{-2\pi\imath\alpha}$.

This easily follows from the periodic property of the exponential. Now, let us choose such a branch $f(z)$ of $z^{-\alpha}$ and integrate $g(z) = \dfrac{f(z)}{z+1}$ along the closed contour as shown in the figure.

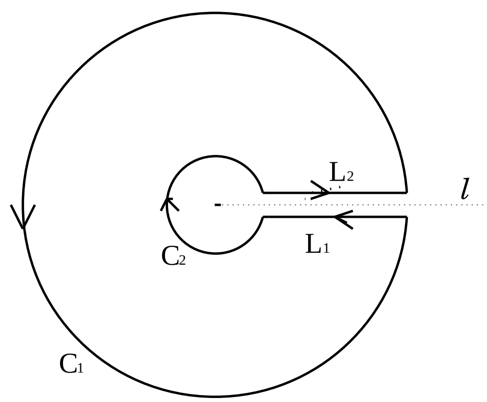

Fig. 17

When the radius r of the inner circle is smaller than 1 and radius R of the outer one is bigger that 1, this contour goes around the only singularity of $g(z)$ exactly once, in the counter clockwise sense. Hence,

$$\int_{\gamma} \frac{f(z)}{z+1}\, dz = 2\pi\imath e^{-\pi\imath\alpha} \tag{7}$$

We now let the two segmets L_1, L_2 approach the interval $[r, R]$. This is valid, since in a neighborhood of $[r, R]$, there exist continuous extensions f_1 and f_2 of g_1 and g_2 where g_1 and g_2 are restrictions of g to upper half plane and lower half plane respectively. The RHS of the above equation remains unaffected where as on the LHS, we get,

$$\int_r^R \frac{x^{-\alpha}}{x+1}\,dx + \int_{|z=R|} \frac{f(z)}{z+1}\,dz - \int_r^R \frac{x^{-\alpha}e^{-2\pi\imath\alpha}}{x+1}\,dx - \int_{|z|=r} \frac{f(z)}{z+1}\,dz = 2\pi\imath e^{-\pi\imath\alpha}.$$

Now we let $r \longrightarrow 0$ and $R \longrightarrow \infty$. It is easily checked that the two integrals on the two circles are respectively bounded by the quantities $2\pi R^{1-\alpha}/(R+1)$ and $2\pi r^{1-\alpha}/(r+1)$. Hence the limits of these integrals are both 0. Therefore,

$$(1 - e^{-2\pi\imath\alpha}) \int_0^\infty \frac{x^{-\alpha}}{x+1}\,dx = 2\pi\imath e^{-\pi\imath\alpha}.$$

Hence,

$$\int_0^\infty \frac{x^{-\alpha}}{x+1}\,dx = \frac{\pi}{\sin \pi\alpha}, \quad 0 < \alpha < 1.$$

There are different ways of carrying out the branch cut. See for example the book by Churchill and Brown, for one such. We shall cut out all this and describe yet another method here.

Theorem 6 *Let ϕ be a meromorphic function on $\mathbb{C}$ having finitely many poles none of which belongs to $[0, \infty)$. Let $a \in \mathbb{C} \setminus \mathbb{Z}$ be such that $\lim_{z\to 0} z^a\phi(z) = 0 = \lim_{z\to\infty} z^a\phi(z)$. Then the following integral exists and*

$$I_a := \int_0^\infty x^{a-1}\phi(x)\,dx = \frac{2\pi\imath}{1 - e^{2\pi\imath a}} \sum_{w\in\mathbb{C}} R_w(z^{a-1}\phi(z)). \qquad (8)$$

Proof: First substitute $x = t^2$ and see that

$$I_a = \int_0^\infty x^{a-1}\phi(x)dx = 2\int_0^\infty t^{2a-1}\phi(t^2)dt. \qquad (9)$$

Next choose a branch $g(z)$ of z^{2a-1} in $-\pi/2 < arg z < 3\pi/2$. Observe that $g(-x) = (-1)^{2a-1}g(x) = -e^{2\pi\imath a}g(x)$, for $x > 0$. Hence,

$$
\begin{aligned}
\int_{-\infty}^{\infty} z^{2a-1}\phi(z^2)dz &= \int_0^{\infty} g(x)\phi(x^2)dx + \int_{-\infty}^{0} g(x)\phi(x^2)dx \\
&= \int_0^{\infty} g(x)\phi(x^2)dx - \int_0^{\infty} e^{2\pi\imath a} g(x)\phi(x^2)dx \\
&= (1-e^{2\pi\imath a})\int_0^{\infty} z^{2a-1}\phi(z^2)dz
\end{aligned}
$$

Therefore, the integral I_a is given by

$$
\frac{2}{1-e^{2\pi\imath a}}\int_{-\infty}^{\infty} z^{2a-1}\phi(z^2)dz = \frac{4\pi\imath}{1-e^{2\pi\imath a}}\sum_{z\in \boldsymbol{H}} R_z(z^{2a-1}\phi(z^2)). \quad (10)
$$

If we set $f(z) = z^{a-1}\phi(z)$ then $zf(z^2) = z^{2a-1}\phi(z^2)$. Observe that $zf(z^2)$ has no poles on the real axis. Therefore, the sum of the residues of $zf(z^2)$ in $\boldsymbol{H}$ is equal to half the sum of the residues in the entire plane. Finally, we have seen, in exercise 12 of Tut 6 that the sum of the residues of $zf(z^2)$ and that of $f(z)$ are the same. The formula (8) follows. ♠

It may be noted that the assignment $a \mapsto I_a$ is called *Mellin's transform* corresponding to ϕ. Coming back to the special case when $\phi(z) = \dfrac{1}{z+1}$, we have $R_{-1}\dfrac{z^{a-1}}{z+1} = (-1)^{a-1} = -e^{\pi\imath a}$. Hence,

$$
\int_0^{\infty} \frac{x^{(a-1)}dx}{x+1} = \frac{\pi}{\sin \pi a}, \quad 0 < a < 1. \quad (11)
$$

Observe that the condition that a is not an integer is crucial for the non existence of the branch of z^{α} throughout a neighborhood of 0. On the other hand, that is what guarantees the existence of the integral.